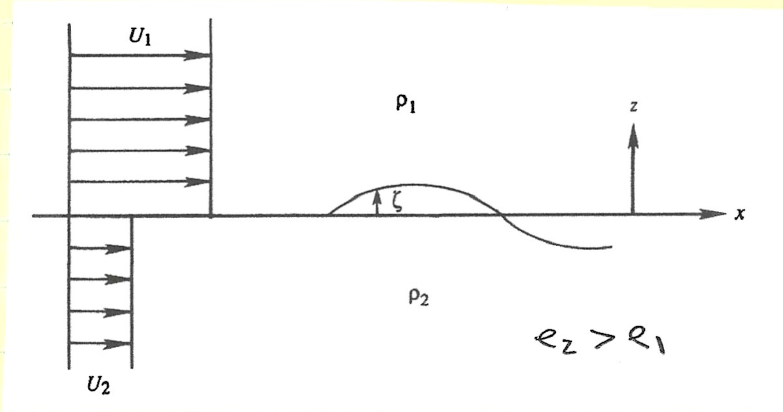
Enhanced momentum, heat, moisture transport in atmosphere mixing two or more streams.

Kelvin-Helmholtz Instability

Two horizontal parallel streams different U



infinite depth zero thickness

Assume ρ = constant, $\underline{\omega} = 0$ & $\mu = 0$

Step 1 Basic Flow

$\tilde{\Phi}_1 = U_1 x$

$\tilde{\Phi}_2 = U_2 x$

note tangential velocity & stresses are discontinuous across the interface: vortex sheet

Step 2 Add Disturbance

$\tilde{\Phi}_1 = U_1 x + \Phi_1$ $\Phi_1 \to 0$ $z \to \infty$

$\tilde{\Phi}_2 = U_2 x + \Phi_2$ $\Phi_2 \to 0$ $z \to -\infty$

$\nabla^2 \Phi_1 = 0$ $\nabla^2 \Phi_2 = 0$ $z = \zeta(x,t)$

Step 3 and 4 subtract 1 and linearize

Kinematic and dynamic interface conditions

Kinematic condition is that the vertical velocity is continuous across the interface and the interface moves with that velocity in the normal direction

$F = z - \zeta \qquad \frac{DF}{Dt} = 0 = \frac{\partial F}{\partial t} + \underline{V}\cdot\nabla F = -\zeta_t - (U_1+u_1)\zeta_x - v_1\zeta_y + w_1 = 0$

$\nabla F = -\zeta_x\hat{i} - \zeta_y\hat{j} + \hat{k}$; or $\zeta_t + (U_1+u_1)\zeta_x + v_1\zeta_y = w_1 = \phi_{1z}$

$\nabla\phi_1 = u_1\hat{i} + v_1\hat{j} + w_1\hat{k}$

$$\frac{\partial\phi_1}{\partial z} = \frac{D\zeta}{Dt} = \frac{\partial\zeta}{\partial t} + (U_1+u_1)\frac{\partial\zeta}{\partial x} + v_1\frac{\partial\zeta}{\partial y}$$

on $z=\zeta$

$$\frac{\partial\phi_2}{\partial z} = \frac{\partial\zeta}{\partial t} + (U_2+u_2)\frac{\partial\zeta}{\partial x} + v_2\frac{\partial\zeta}{\partial y}$$

linearize:
$$\phi_{1z} = \zeta_t + U_1\zeta_x$$
$$\phi_{2z} = \zeta_t + U_2\zeta_x$$
on $z=0$

Dynamic condition is that the stress must be continuous across the interface ie $\tilde{p}_1 = \tilde{p}_2$ on $z=\zeta$ (neglecting surface tension) $\tilde{p}_1 = P_1 + p_1$, $\tilde{p}_2 = P_2 + p_2$

$$\frac{\partial\tilde{\phi}_1}{\partial t} + \frac{1}{2}(\nabla\tilde{\phi}_1)^2 + \frac{\tilde{p}_1}{\rho} + gz = c_1$$

on $z=\zeta$

$$\frac{\partial\tilde{\phi}_2}{\partial t} + \frac{1}{2}(\nabla\tilde{\phi}_2)^2 + \frac{\tilde{p}_2}{\rho} + gz = c_2$$

For basic flow $P_1 = P_2$ on $z=0$

$$\rho_1\left(\tfrac{1}{2}U_1^2 - c_1\right) = \rho_2\left(\tfrac{1}{2}U_2^2 - c_2\right) \quad \text{on } z=0$$

For the disturbed flow

$$\rho_1c_1 - \rho_1\frac{\partial\tilde{\phi}_1}{\partial t} - \frac{\rho_1}{2}\left[(U_1+u_1)^2 + v_1^2 + w_1^2\right] - \rho_1 g\zeta$$
$$= \rho_2c_2 - \rho_2\frac{\partial\tilde{\phi}_2}{\partial t} - \frac{\rho_2}{2}\left[(U_2+u_2)^2 + v_2^2 + w_2^2\right] - \rho_2 g\zeta$$

Subtracting disturbed flow from basic flow and linearizing

$$\rho_1\left[\phi_{1t} + U_1\phi_{1x} + g\zeta\right] = \rho_2\left[\phi_{2t} + U_2\phi_{2x} + g\zeta\right] \quad \text{on } z=0$$

ϕ_1 & ϕ_2 satisfy ∇^2 subject BC ∞ and linearized kinematic & dynamic interface conditions

step 5 Assume normal modes with $\underline{k} = (k, 0, 0)$

$$\phi_1(x,z,t) = A_1(z)\, e^{ik(x-ct)} \qquad \phi_2(x,z,t) = A_2(z)\, e^{ik(x-ct)}$$

k = real $\quad c = c_r + i c_i \quad -ikc = -ikc_r + kc_i$

so flow unstable $c_i > 0$

note σt form more convenient convective cells

Step 6 Substitution in ∇^2: $-k^2 A_1 + A_{1zz} = 0$

$-k^2 A_2 + A_{2zz} = 0$

Solutions $A_\pm e^{\pm kz}$ with $\pm$ determined by BC ∞

$$\phi_1 = A_- e^{ik(x-ct)} e^{-kz} \qquad \phi_2 = A_+ e^{ik(x-ct)} e^{kz}$$

Similarly assume $\zeta = \zeta_0\, e^{ik(x-ct)} \qquad \delta = ik(x-ct)$

$$\phi_{1z} = A_-(-k) e^{\delta} e^{-kz} \qquad \zeta_t = \zeta_0 (-ikc) e^{\delta}$$

$$\phi_{2z} = A_+(k) e^{\delta} e^{kz} \qquad \zeta_x = \zeta_0 (ik) e^{\delta}$$

$$\phi_{1z} - U_1 \zeta_x = \zeta_t = \phi_{2z} - U_2 \zeta_x \qquad \text{on } z = 0$$

$$A_-(-k) - U_1 \zeta_0 (ik) = \zeta_0(-ikc) = A_+(k) - U_2 \zeta_0 (ik)$$

$$kA_- = -\zeta_0 (ikU_1 - ikc)$$

$$k A_+ = \zeta_0 (ikU_2 - ikc)$$

$$a_{1z} = A_-(-ikc)\, e^{\delta} e^{-kz} \qquad a_{1x} = A_-(ik)\, e^{\delta} e^{-kz}$$
$$a_{2z} = A_+(-ikc)\, e^{\delta} e^{kz} \qquad a_{2x} = A_+(ik)\, e^{\delta} e^{kz}$$

$$\rho_1\left[A_-(-ikc) + U_1 A_-(ik) + g\zeta_0\right] =$$
$$\rho_2\left[A_+(-ikc) + U_2 A_+(ik) + g\zeta_0\right] \qquad \text{on } z=0$$

$$\rho_1\left[kA_-(-ic + iU_1) + g\zeta_0\right] = \rho_2\left[kA_+(-ic + iU_2) + g\zeta_0\right]$$

$$\rho_1\left[\frac{kA_-}{\zeta_0}(-ikc + ikU_1) + kg\right] = \rho_2\left[\frac{kA_+}{\zeta_0}(-ikc + ikU_2) + kg\right]$$

$$\rho_1\left[-(-ikc + ikU_1)^2 + kg\right] = \rho_2\left[(ikc + ikU_2)^2 + kg\right]$$

$$\rho_1\left[+(kc)^2 - 2k^2cU_1 + k^2U_1^2 + kg\right] =$$
$$\rho_2\left[-(kc)^2 + 2k^2cU_2 - k^2U_2^2 + kg\right]$$

$$\rho_1\left[c^2 - 2cU_1 + U_1^2 + g/k\right] = \rho_2\left[-c^2 + 2cU_2 - U_2^2 + g/k\right]$$
$$c^2(\rho_1+\rho_2) - c(2\rho_1U_1 + 2\rho_2U_2) + \rho_1U_1^2 + \rho_2U_2^2 + g/k\,(\rho_1-\rho_2)$$

$$c^2 + c\underbrace{\left[\frac{-2(\rho_1U_1+\rho_2U_2)}{(\rho_1+\rho_2)}\right]}_{b} + \underbrace{\frac{\rho_1U_1^2+\rho_2U_2^2}{(\rho_1+\rho_2)} + g/k\,\frac{(\rho_1-\rho_2)}{(\rho_1+\rho_2)}}_{c} = 0$$

$$b^2 = 4(\rho_1U_1+\rho_2U_2)^2/(\rho_1+\rho_2)^2$$
$$-4ac = -4(\rho_1U_1^2+\rho_2U_2^2)(\rho_1+\rho_2)^{-1} + 4\frac{g}{k}\frac{(\rho_2-\rho_1)}{(\rho_1+\rho_2)}$$

$$(\rho_1+\rho_2)^{-2}\left[(\rho_1U_1+\rho_2U_2)^2 - (\rho_1U_1^2+\rho_2U_2^2)(\rho_1+\rho_2)\right]$$

$$= \frac{-\rho_1\rho_2(U_2-U_1)^2}{(\rho_1+\rho_2)^2}$$

$$c = \frac{\rho_2 U_2 + \rho_1 U_1}{\rho_2 + \rho_1} \pm \left[\underbrace{\frac{g}{k} \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}}_{1} - \underbrace{\rho_1 \rho_2 \left(\frac{U_2 - U_1}{\rho_2 + \rho_1} \right)^2}_{2} \right]^{1/2}$$

for (2) < (1) c real
for both solutions ∴ solutions stable

for (2) > (1) $c_i > 0$ ∴ + solution unstable

Step 7 $\qquad g(\rho_2^2 - \rho_1^2) < k \rho_1 \rho_2 (U_2 - U_1)^2$ $\qquad$ ΔU high enough or $\Delta\rho$ small enough or k large enough

$(\rho_2 - \rho_1)(\rho_2 + \rho_1) = \rho_2^2 - \rho_1^2$

Note for each growing solution there is a corresponding decaying solution

Limiting cases:

(1) $U_1 = U_2 = 0 \qquad c = \pm \left[\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} \right) \frac{g}{k} \right]^{1/2}$

$\rho_2 > \rho_1$ neutral stability ∴ c = dispersion relation for waves on a density interface (Kundu et al. Section 8.7)

(2) For $U_1 \neq U_2$ k exists for which flow is unstable ie flow is always unstable short waves $\lambda = 2\pi/k$

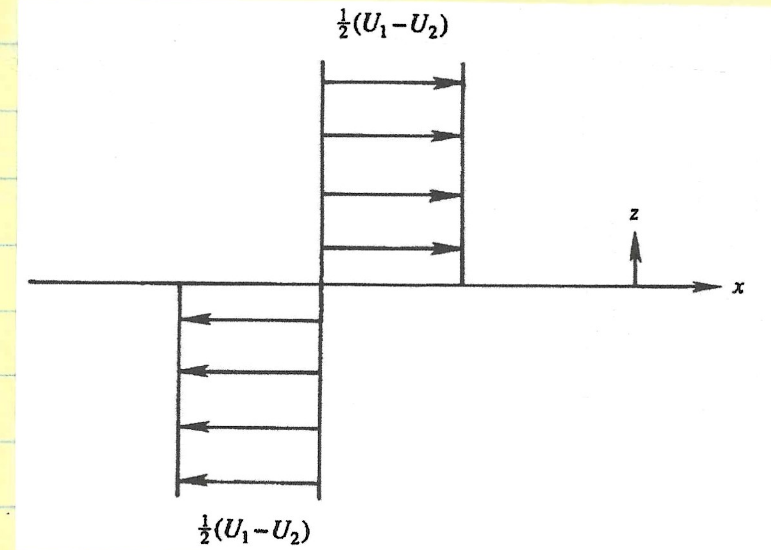
③ For $\rho_1 = \rho_2$

ΔU discontinuous

ie vortex sheet

with strength $\gamma = (U_2 - U_1)$

$$c = \tfrac{1}{2}(U_2 + U_1) \pm \tfrac{1}{2} i (U_2 - U_1)$$



∴ unstable all k

unstable waves move with phase velocity ie average velocity $= c_R$

Basic flow in moving reference frame at average velocity

Kelvin-Helmholtz caused by destabilizing effects shear overcoming stabilizing effects density stratification

Laboratory

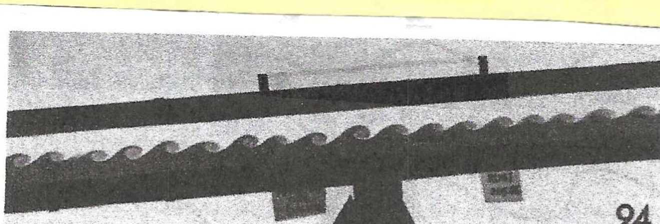


FIGURE 11.4 Kelvin-Helmholtz instability generated by tilting a horizontal channel containing two liquids of different densities. The lower layer is dyed and 18 wavelengths of the developing interfacial disturbance are shown. The mean flow in the lower layer is down the plane (to the left) and that in the upper layer is up the plane (to the right). *S. A. Thorpe,* Journal of Fluid Mechanics, *46, 299–319, 1971; reprinted with the permission of Cambridge University Press.*

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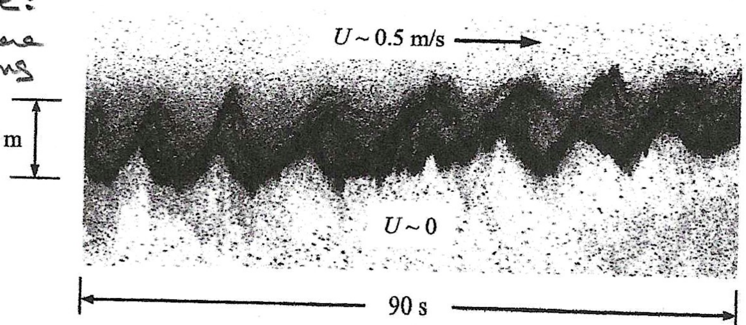

FIGURE 11.5 Natural overturning motions generated by the Kelvin-Helmholtz instability in a salt-stratified estuary at early ebb tide (see Lavery et al., 2013). The similarity of these structures to those shown in Figures 11.4 and 11.6 is striking. The image was created from a downward looking backscatter sonar system operating in the 375 kHz to 600 kHz band on a research vessel moving upstream into the river mouth. The upper fresh water layer from the Connecticut River is ~2 m deep. The lower saltwater layer is ~6 m deep. The Reynolds number of the flow based on the vertical extent of the overturning structures is ~500,000. The gradient Richardson number here is ~1/2 so the structures are not growing at this point (see Section 11.7, and Example 11.4). This image provided by, and used with the permission of, *Jonathan Fincke & Andone C. Lavery of Woods Hole Oceanographic Institution.*

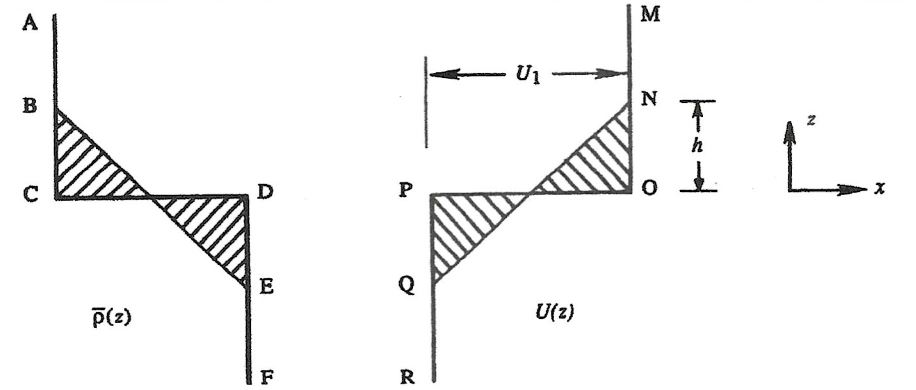

FIGURE 11.7 Smearing out of sharp density and velocity profiles, resulting in an increase of potential energy and a decrease of kinetic energy. When turbulent, the overturning eddies or structure shown in Figures 11.4 and 11.5 lead to vertical (cross stream) momentum transport and fluid mixing. The discontinuous profiles ACDF and MOPR evolve toward ABEF and MNQR as the instability develops.

Unstable waves at interface transform ACDF to ABEF for $\bar{\rho}(z)$ & MOPR to MNQR for $U(z)$. High density fluid depth DE raised upward (and mixed lower density fluid depth BC), which means PE of system increased due instability. The required energy is drawn from the KE of the basic field, as the KE of the initial profile MOPR is > than the final profile MNQR

For $U_2 = 0$, after mixing $U(z) = \frac{U_1}{2}(1 + z/h)$ $\quad -h \le z \le h$

$$\bar{\rho}(z) = \rho_2 - \frac{(\rho_2 - \rho_1)}{2}(1 + z/h)$$

ΔKE in range $-h \le z \le h$:

$$E_{initial} = \frac{1}{2}\rho_1 U_1^2 h \qquad E_{final} = \frac{1}{2}\int_{-h}^{h} \bar{\rho}(z) U^2(z) = \frac{\rho_1 U_1^2 h}{3} + \frac{(\rho_2 - \rho_1) U_1^2 h}{12}$$

For $(\rho_2 - \rho_1) \le \rho_1$, KE decreases whereas total momentum $= \int U dz$ nearly unchanged. If $\int U(z)$ does not change, then $\int U^2(z)$ decreases if gradients decrease

Fig 11.4 & 11.5 show advanced nonlinear stage of the instability in which interface is a rolled up layer of vorticity, which is replicated nonlinear numerical predictions

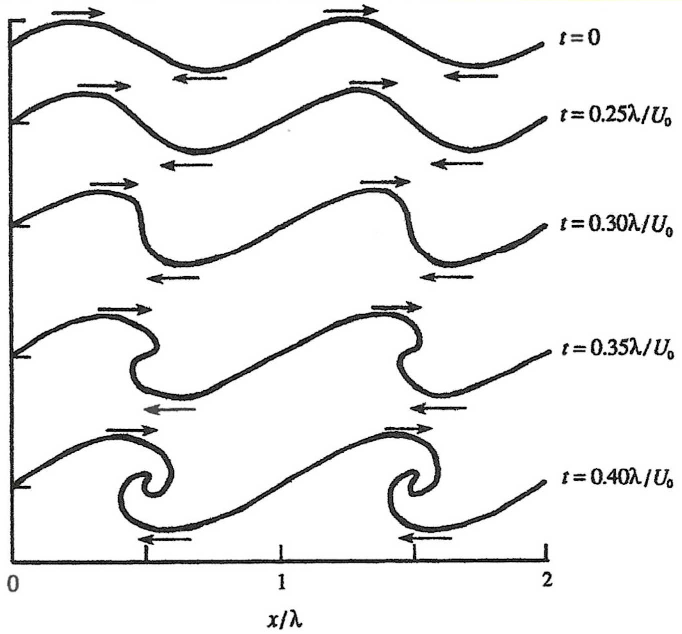

FIGURE 11.6 Nonlinear numerical calculation of the evolution of a vortex sheet that has been given a small transverse sinusoidal displacement with wavelength λ. The density difference across the interface is zero, and U_0 is the velocity difference across the vortex sheet. Here again, the similarity of the interface shape at the last time with the results shown in Figures 11.4 and 11.5 is striking. The smaller vertical displacements shown Figures 11.4 and 11.5 are consistent with the effects of stratification that are absent from the calculations shown in this figure. *J. S. Turner,* Buoyancy Effects in Fluids, *1973; reprinted with the permission of Cambridge University Press.*

Energy source of the K-H instability is the KE of the two streams

The disturbances evolve to smear out gradients until they cannot grow further