Instability.

Solutions conservation laws may be unobservable due instabilities small disturbances invariably present.

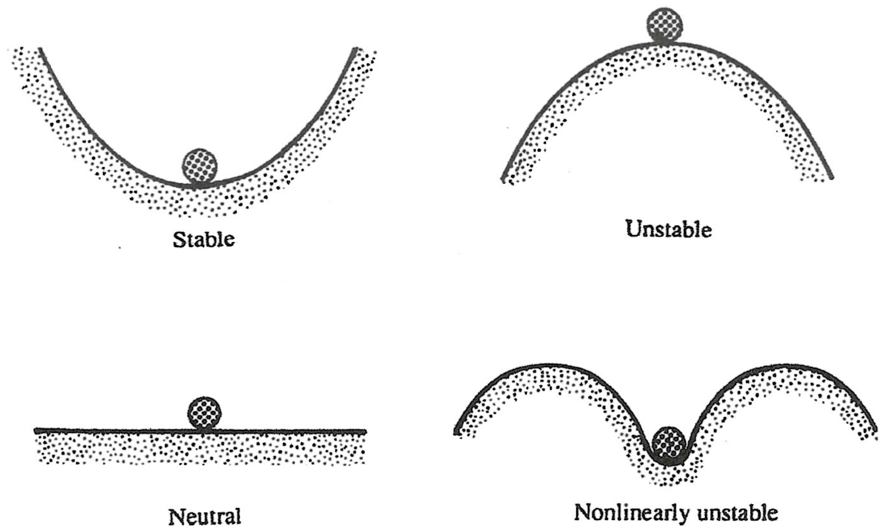

FIGURE 11.1 Stable and unstable mechanical systems. Here, gravity is presumed to act downward. In the upper-left and lower-right panels, a small displacement of the round object away from equilibrium will be opposed by the action of gravity and the object will move back toward its equilibrium location. These are linearly stable situations. In the upper right panel, a small displacement of the object will be enhanced by the action of gravity and the object will move away from its equilibrium location, an unstable situation. In the lower left, a small displacement of the object does not produce a new force, thus the situation is neutrally stable. In the lower right panel, a sufficiently large displacement of the object may place it beyond its region of stability; thus this situation is nonlinearly unstable.

Smooth laminar flows are only stable to small disturbances when certain conditions satisfied: e.g., channel flow $Re < Re_{crit}$ and stratified shear flow $Ri < Ri_{crit}$. Otherwise, disturbances grow and change character original flow. Sometimes, disturbances grow to finite amplitude and new steady state equilibrium, which may also become unstable and evolve to another steady state equilibrium condition and so on.

In the limit, flow becomes superposition of a variety of interacting disturbances with random phases i.e. turbulent flow.
Issues are: (1) process of laminar to turbulent transition; and (2) prediction transition

Linear stability theory:

(1) infinitesimal disturbances imposed on particular flow to determine growth or decay
(2) perturbations assumed small enough that linearization acceptable, i.e. quadratic and higher order terms neglected (of perturbation variables and their derivatives).
(3) limited initial behavior i.e. does not predict transition just instability
(4) real flow may be stable linear theory but unstable larger (non linear) disturbances

In spite limitations many successes:

(1) thermal convection in layer fluid
(2) Tollmien-Schlichting waves in viscous BL
(3) Taylor vortices

all confirmed experiments

Method of Normal Modes

Sinusoidal disturbances imposed on basic state (background, initial, or equilibrium state).

Assume basic state parallel x and $f(z)$ only
i.e. $\underline{U} = U(z)\hat{x}$

Real part to obtain physical quantities

$$u = (x, y, z, t) = \hat{u}(z)\exp(ilx + imy + \sigma t)$$
$$= \hat{u}(z)\exp[iK(\hat{e}_K \cdot \underline{x} - ct)]$$

$\hat{u}(z)$ = complex amplitude

$\underline{K} = (l, m, 0)$ $\quad \hat{e}_K = \underline{K}/|\underline{K}|$ $\quad K = |\underline{K}|$ = real > 0

σ = temporal growth rate = $\sigma_r + i\sigma_i$

c = complex wave speed = $c_r + ic_i$

Two forms useful for stationary and travelling waves, respectively. Behavior examined for all possible $\underline{K}$. If σ_r or $c_i > 0$ for any $\underline{K}$ the system is unstable at this $\underline{K}$ otherwise stable.

$\sigma_r < 0$ or $c_i < 0$ Stable

$\sigma_r = 0$ or $c_i = 0$ neutrally Stable

$\sigma_r > 0$ or $c_i > 0$ unstable

Each mode examined separately as linearity implies modes do not interact. The method leads to eigenvalue problem.

Boundary between stable & unstable called marginal state. Two types:

(1) $\sigma_i = c_r = 0$ $\Rightarrow$ stationary motion i.e. cellular convection / secondary flow

(2) $\sigma_i \neq 0$ or $c_r \neq 0$ $\Rightarrow$ traveling oscillations of growing amplitude

Called overstability or oscillating mode

Neutral stability & marginal state both have $\sigma_r = c_i = 0$; however, marginal state also is border line between stable & unstable.

In many cases, stability criterion found by simply setting σ_r or $c_i = 0$ without proof that represent border line condition.

Sinusoidal traveling waves

$\eta(x,t) = a \cos\left[\frac{2\pi}{\lambda}(x - ct)\right]$ x = horizontal

z = vertical $= \eta(x,t)$

a = amplitude $\pm$ crest / trough = waveform

λ = wave length k = wave number rad/m

c = phase speed

$2\pi(x - ct)/\lambda$ = phase

λ = distance between crests $T = 2\pi/kc = \lambda/c$

= period i.e. Δt

f = cyclic frequency $= T^{-1}$ Hz between passage

$\omega = 2\pi f$ = radian frequency rad/s crests

$\eta(x,t) = a\cos(kx - \omega t)$ $\omega = 2\pi \times \frac{kc}{2\pi} = kc$

speed wave crests $\eta = a \Rightarrow$ phase $= 2n\pi$

n = integer

$\frac{2\pi}{\lambda}(x_{crest} - ct) = 2n\pi = kx_{crest} - \omega t$

$x_{crest} = (\omega/k)t + 2n\pi/k$ $\Delta x_{crest} = (\omega/k)\Delta t$

$\lambda = (\omega/k)T$

$= (\omega/k)\lambda/c$

phase speed specifies the travel speed of constant-phase wave features such as troughs or crests.

$c = \omega/k$

$= 2\pi f / \frac{2\pi}{\lambda}$

$= \lambda f$

3D generalization

$$\eta = a\cos(kx + ly + mz - \omega t) = a\cos(\underline{K}\cdot\underline{x} - \omega t)$$

$c = \omega/K$ $\quad \underline{c} = (\omega/K)\hat{e}_K$ $\quad \underline{K} = (k, l, m)$

$\hat{e}_K = \underline{K}/|\underline{K}|$ $\quad K^2 = k^2 + l^2 + m^2$

$= \underline{K}/K$ $\quad \lambda = 2\pi/K$

trace velocity:

$c_x = \omega/k$

$c_y = \omega/l$

$c_z = \omega/m$

all $> c = \omega/K$

as when $\neq 0$

all smaller K

Thus c_x, c_y, c_z

not vector

components

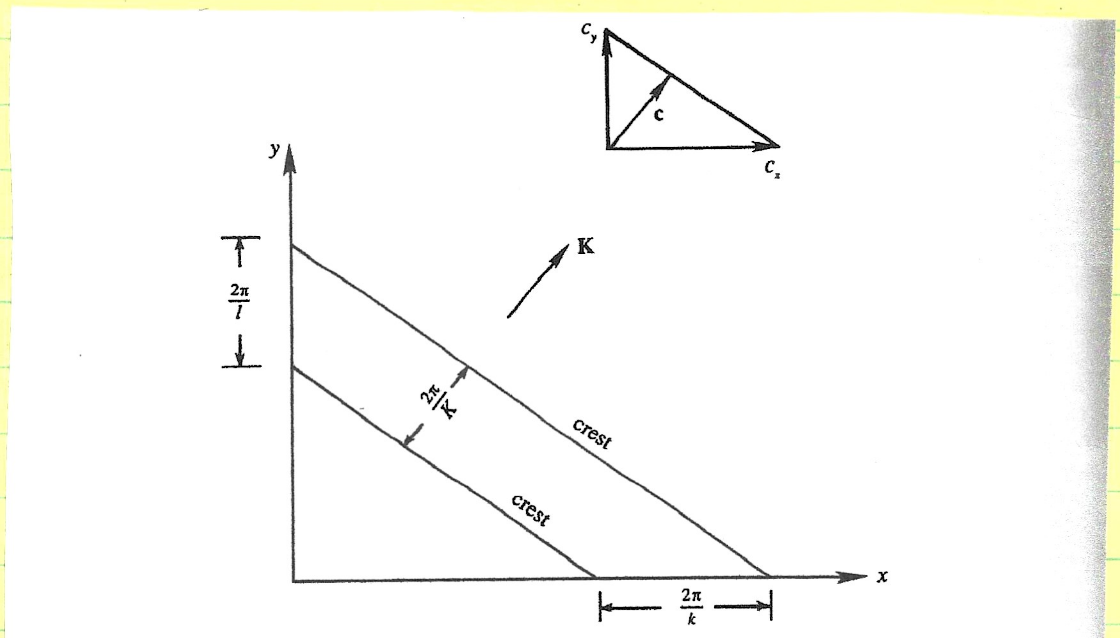

FIGURE 8.1 Wave crests propagating in the x-y plane. The crest spacing along the coordinate axes is larger than the wavelength $\lambda = 2\pi/K$. The inset shows how the trace velocities c_x and c_y are combined to give the phase velocity vector **c**.

but reflect that constant phase surfaces appear to travel faster along directions not coinciding $\underline{K}$

If waves exist in fluid moving $\underline{U} \Rightarrow \underline{c}_0 = \underline{c} + \underline{U}$ $\quad \underline{c}\cdot\underline{K} = \frac{\omega}{K}\times\frac{K^2}{K} = \omega$

$\underline{c}_0 \cdot \underline{K} \Rightarrow \omega_0 = \omega + \underbrace{\underline{U}\cdot\underline{K}}_{\text{Doppler shifted}}$ $\quad \omega_0$ = observed

ω = intrinsic (measured moving with the flow)

Say $\omega = 0$ but flow has periodicity x with $\lambda = 2\pi/k$ and moves at speed U then frequency at point is $\omega_0 = Uk$.