Stability and Transition

Solutions NS equations may be subject instabilities for a variety of reasons. Flows for which the departure from the original state (eg, steady state) is damped out are stable, which is often the case for flows at low Re where viscosity is strong; however, in some cases viscosity can be de-stabilizing.

Typical and important examples are transition from laminar to turbulent flow and Taylor vortices the latter of which is a transition from one laminar to another laminar flow.

Whether a flow is open or closed has important consequences on the stability.

Closed = particles travel in closed X and stay in domain of interest (eg Taylor-Couette flow).

Such flows pass through a series of flow states as a parameter (Re, Ta, or other) increases. Theory of dynamical systems has been helpful for these flows.

Open: particles enter and is convected through and out of the domain of interest. BL and free shear flows are in this category

Linear stability theory is intended to answer the question of a flow is stable or unstable when subjected to a small disturbance. It is only partially successful, e.g., Poiseuille pipe flow and plane Couette flows are linearly stable but transition to turbulence for sufficiently high Re. Or plane Poiseuille flow turbulence develops at lower Re than predicted linear stability theory. Transition occurs more rapidly for free shear than wall bounded flows.

Stability theory is mathematically very complex subject often requiring very advanced methods.

Linear Stability of normal modes or perturbations

V_i satisfies steady NS equations and $v_i' \ll V_i$ is a small perturbation.

$$v_i(x_i,t) = V_i(x_i) + v_i'(x_i,t)$$

In the sense of perturbation theory $\varepsilon v_i'$ is the first order term.

v_i is substituted into the NS and a new set of equations generated for v_i' by neglecting products of v_i' (order ε^2). The equations are linear in v_i' as a result of the small amplitude assumption. Thus it marks the start of the instability as as the disturbance grows it violates the linear assumption.

Linear theory answers the questions:

1. types of disturbances that grow
2. amplification rate
3. critical value Re or other parameter at which this will occur.

It is assumed the disturbance can be decomposed into normal modes of various wave lengths.

Say $V_i = (V_x(y), 0, 0)$ 2D shear flow with normal mode disturbance of a travel wave with amplitude $\hat{v}_i(y)$

$$v_i' = \hat{v}_i(y) \exp[i(\alpha x + \beta z - \alpha c t)] \quad \text{Real part}$$

$\hat{v}_i(y)$ = complex amplitude

α = real wave number x direction

β = real " " z "

c = complex wave speed = $c_R + i c_I$.

$$\underline{k} = (\alpha, 0, \beta) \quad |\underline{k}| = (\alpha^2 + \beta^2)^{1/2}$$

$$v_i' = \hat{v}_i(y) \exp[i(\alpha x + \beta z - \alpha c_R t)] \exp(\alpha c_I t)$$

At fixed $x_i = (x, y, z)$ the disturbance mode oscillates with frequency $\omega = 2\pi f = \alpha c_R$ as waves of wavenumber magnitude k of length $\lambda = 2\pi/k$ pass by. The phase velocity of the mode is in the α, β direction with magnitude c_α

$$c_\alpha = \frac{\alpha c_R}{k} = \omega/k = \frac{2\pi f}{2\pi/\lambda} = \lambda f$$

$$v_i' = \hat{v}_i(y)\exp[ik(\hat{e}_k \cdot \underline{x} - c_e t)]\exp(\alpha c_I t)$$

$$\hat{e}_k = \underline{k}/|\underline{k}| = \underline{k}/k = k^{-1}(\alpha, 0, \beta)$$

$$\hat{e}_k \cdot \underline{x} = (\alpha x + \beta z)/k$$

$$k c_e = \alpha c_R$$

For simplicity, assume disturbance is in the flow direction, ie, $\beta = 0 \Rightarrow k = \alpha$ and $c_e = c_R$ and $\alpha > 0$. Therefore the disturbance will grow or decay as $f(t)$ depending sign c_I

$c_I < 0$ stable

$c_I = 0$ neutral

$c_I > 0$ unstable

For $c_I = 0$, if small change stability parameter say Re move to $c_I > 0$ is called marginally stable (or neutrally). Such points important as their locus marks boundary between stable and unstable conditions

Above ie c complex = temporal stability analysis. When c = real and $\alpha = \alpha_R + i\alpha_I$ complex the perturbation grows in x

$$v_i' = \hat{v}_i(y)\exp[i(\alpha_R x + \beta z - \alpha_R c t)]\exp(\alpha_I x)$$

Spatial stability analysis