Chapter 1 & 2 (3.2)-2026

Reynolds Transport Theorem

Preliminary: Leibniz integral theorem = derivative single variable integral having $f(x,t)$ integrand and limits $a(t)$and $b(t)$.

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\ dx = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\ dx + \frac{db(t)}{dt} f(b(t),t) - \frac{da(t)}{dt} f(a(t),t)$$

(1) (2) (3)

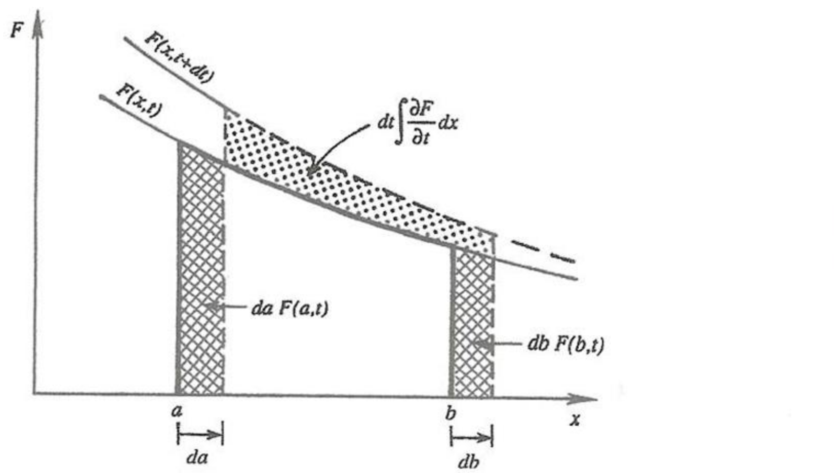

FIGURE 3.19 Graphical illustration of the Liebniz theorem. The three marked areas correspond to the three contributions shown on the right in (3.30). Here da, db, and $\partial F/\partial t$ are all shown as positive.

(1) integral of $\frac{\partial f}{\partial t}$ with lower and upper limits a(t) and b(t)

(2) gain f at upper limit moving at $\frac{db}{dt}$

(3) loss f at lower limit moving at $\frac{da}{dt}$

Total derivative LHS = integral partial derivatives with lower and upper limits a(t) and b(t) + terms that account for time dependence of a and b.

Generalization 3D: RTT

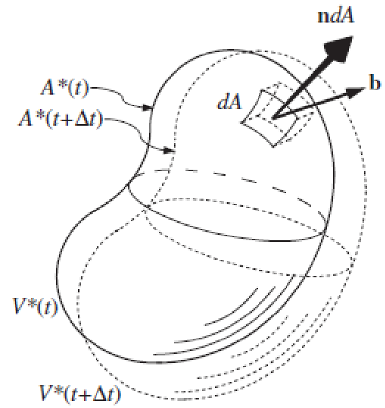

FIGURE 3.18 Geometrical depiction of a control volume $V^*(t)$ having a surface $A^*(t)$ that moves at a nonuniform velocity $\mathbf{b}$ during a small time increment Δt. When Δt is small enough, the volume increment $\Delta V = V^*(t+\Delta t) - V^*(t)$ will lie very near $A^*(t)$, so the volume-increment element adjacent to dA will be $(\mathbf{b}\Delta t) \cdot \mathbf{n} dA$ where $\mathbf{n}$ is the outward normal on $A^*(t)$.

V^* = CV bounded by A^*= CS with outward normal $\underline{n}$ and nonuniform velocity $\underline{b}$. Assume $F(\underline{x},t)$ is single valued continuous function.

$$\frac{d}{dt}\int_{V^*(t)} F(\underline{x},t)\ dV = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{V^*(t+\Delta t)} F(\underline{x},t+\Delta t)\ dV - \int_{V^*(t)} F(\underline{x},t)\, dV\right]$$

Define $\Delta V = V^*(t+\Delta t) - V^*(t)$

1^{st} order TS: $F(\underline{x},t+\Delta t) = F(\underline{x},t) + \frac{\partial F}{\partial t}\Delta t \quad$ for $\Delta t \to 0$

$$\int_{V^*(t+\Delta t)} F(\underline{x},t+\Delta t)\ dV = \int_{V^*(t)} F(\underline{x},t)\, dV + \int_{V^*}\frac{\partial F}{\partial t}\Delta t\ dV + \int_{\Delta V} F(\underline{x},t)\ dV$$

$$+\int_{\Delta V}\frac{\partial F}{\partial t}\Delta t\ dV$$

$$\frac{d}{dt}\int_{V^*} F(\underline{x},t)\ dV = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{V^*}\frac{\partial F}{\partial t}\Delta t\ dV + \int_{\Delta V} F(\underline{x},t)\ dV\right]$$

Need relationship ΔV and Δt: $\Delta V = (\underline{b}\,\Delta t)\cdot\underline{n}\, dA^*$

Therefore: $\int_{\Delta V} F(\underline{x},t)\ dV = \int_{A^*} F(\underline{x},t)(\underline{b}\,\Delta t)\cdot \underline{n}\ dA^*$

where all ΔV summed via surface integral.

Thus taking $\lim_{\Delta t \to 0} \frac{1}{\Delta t}$:

$$\frac{d}{dt}\int_{V^*(t)} F(\underline{x},t)\ dV = \int_{V^*(t)} \frac{\partial F}{\partial t}\ dV + \int_{A^*} F\ \underline{b}\cdot\underline{n}\, dA^* \qquad F = F(\underline{x},t)$$

Inflows/outflows F($\underline{x}$,t) accounted for via sign $\underline{b}\cdot\underline{n}$, which monitors whether A^*(t) is advancing $\underline{b}\cdot\underline{n} > 0$ or retreating $\underline{b}\cdot\underline{n} < 0$.

Physical Interpretation

(1) F = 1: conservation of volume

Exercise 3.33. Starting from (3.35), set $F = 1$ and derive (3.14) when $\mathbf{b} = \mathbf{u}$ and $V^*(t) = \delta V \to 0$.

Solution 3.33. With $F = 1$, $\mathbf{b} = \mathbf{u}$, and $V^*(t) = \delta V$ with surface δA, (3.35) becomes:

$$\frac{d}{dt}\int_{\delta V} dV = 0 + \int_{\delta A} \mathbf{u}\cdot\mathbf{n}dA.$$

The first integral is merely δV. Use Gauss' divergence theorem on the second term to convert it to volume integral.

$$\frac{d}{dt}(\delta V) = \int_{\delta V} \nabla\cdot\mathbf{u}dV.$$

As $\delta V \to 0$ the integral reduces to a product of δV and the integrand evaluated at the center point of δV. Divide both sides of the last equation by δV and take the limit as $\delta V \to 0$:

$$\lim_{\delta V\to 0}\frac{1}{\delta V}\frac{d}{dt}(\delta V) = \lim_{\delta V\to 0}\frac{1}{\delta V}\int_{\delta V}\nabla\cdot\mathbf{u}dV = \lim_{\delta V\to 0}\frac{1}{\delta V}\left[(\nabla\cdot\mathbf{u})\delta V + \ldots\right] = \nabla\cdot\mathbf{u} = S_{ii}.$$

and this is (3.14).

$$\frac{1}{\delta V}\frac{D(\delta V)}{Dt} = u_x + v_y + w_z = u_{i,i}$$

(2) $RTT = \frac{DF}{Dt} = \frac{\partial F}{\partial t} + \underline{u} \cdot \nabla F = \frac{\partial F}{\partial t} + u_i \frac{\partial F}{\partial x_i}$

for $V^*(t) = \delta V \rightarrow 0$ and $\underline{b} = \underline{u}$

3.35. Show that (3.35) reduces to (3.5) when $V^*(t) = \delta V \rightarrow 0$ and the control surface velocity **b** is equal to the fluid velocity $\mathbf{u}(\mathbf{x},t)$.

Solution 3.35. When $V^*(t) = \delta V$ with surface δA, δV is small, and $\mathbf{b} = \mathbf{u}$, δV represents a fluid particle. Under these conditions (3.35) becomes:

$$\frac{d}{dt}\int_{\delta V} F(\mathbf{x},t)dV = \int_{\delta V} \frac{\partial F(\mathbf{x},t)}{\partial t} dV + \int_{\delta A} F(\mathbf{x},t)\mathbf{u}\cdot\mathbf{n}dA,$$

and the time derivative is evaluated following δV. Use Gauss' divergence theorem on the final term to convert it to a volume integral,

$$\int_{\delta A} F(\mathbf{x},t)\mathbf{u}\cdot\mathbf{n}dA = \int_{\delta V} \nabla\cdot(F(\mathbf{x},t)\mathbf{u})dV,$$

so that (3.35) becomes:

$$\frac{d}{dt}\int_{\delta V} F(\mathbf{x},t)dV = \int_{\delta V}\left[\frac{\partial F(\mathbf{x},t)}{\partial t} + \nabla\cdot(F(\mathbf{x},t)\mathbf{u})\right]dV = \int_{\delta V}\left[\frac{\partial F(\mathbf{x},t)}{\partial t} + F(\mathbf{x},t)\nabla\cdot\mathbf{u} + (\mathbf{u}\cdot\nabla)F(\mathbf{x},t)\right]dV,$$

where the second equality follows from expanding the divergence of the product $F\mathbf{u}$.

As $\delta V \rightarrow 0$ the various integrals reduce to a product of δV and the integrand evaluated at the center point of δV. Divide both sides of the prior equation by δV and take the limit as $\delta V \rightarrow 0$ to find:

$$\lim_{\delta V\to 0}\frac{1}{\delta V}\frac{d}{dt}\int_{\delta V} F(\mathbf{x},t)dV = \lim_{\delta V\to 0}\frac{1}{\delta V}\int_{\delta V}\left[\frac{\partial F(\mathbf{x},t)}{\partial t} + F(\mathbf{x},t)\nabla\cdot\mathbf{u} + (\mathbf{u}\cdot\nabla)F(\mathbf{x},t)\right]dV,$$

$$\lim_{\delta V\to 0}\frac{1}{\delta V}\frac{d}{dt}\left[F(\mathbf{x},t)\delta V + \ldots\right] = \lim_{\delta V\to 0}\frac{1}{\delta V}\left[\left(\frac{\partial F(\mathbf{x},t)}{\partial t} + F(\mathbf{x},t)\nabla\cdot\mathbf{u} + (\mathbf{u}\cdot\nabla)F(\mathbf{x},t)\right)\delta V + \ldots\right], \text{ or}$$

$$\frac{d}{dt}F(\mathbf{x},t) + F(\mathbf{x},t)\lim_{\delta V\to 0}\frac{1}{\delta V}\frac{d}{dt}(\delta V) = \frac{\partial F(\mathbf{x},t)}{\partial t} + F(\mathbf{x},t)\nabla\cdot\mathbf{u} + (\mathbf{u}\cdot\nabla)F(\mathbf{x},t),$$

where the product rule for derivative has been used on product $F\delta V$ in [.]-braces on the left. From (3.14) or Exercise 3.33: $\lim_{\delta V\to 0}\frac{1}{\delta V}\frac{d}{dt}(\delta V) = \nabla\cdot\mathbf{u}$, so the second terms on both sides of the last equation are equal and may be subtracted out leaving:

$$\frac{d}{dt}F(\mathbf{x},t) = \frac{\partial F(\mathbf{x},t)}{\partial t} + (\mathbf{u}\cdot\nabla)F(\mathbf{x},t),$$

and this is (3.5) when the identification $D/Dt \equiv d/dt$ is made.

Relationship CV & material volume = mv

(1) mv: $\frac{d}{dt}\int_{V(t)} F(\underline{x},t)\ dV = \int_{V(t)} \frac{\partial F}{\partial t}\ dV + \int_{A^*} F\ \underline{u}\cdot\underline{n}\,dA$

V(t) = MV A(t) = MV boundary with local $\underline{n}$ moving at nonuniform velocity $\underline{u}$(x,t)

Green's theorem: $\int_V \nabla\cdot\underline{b}\,dV = \int_S \underline{b}\cdot\underline{n}\,dA$

$$\frac{d}{dt}\int_{MV} F\ dV = \int_{MV}\left[\frac{\partial F}{\partial t} + \nabla\cdot\left(F\underline{u}\right)\right] dV$$

$$\lim_{MV\to 0}\frac{d}{dt}\int_{MV} F\ dV = \frac{\partial F}{\partial t} + \nabla\cdot\left(F\underline{u}\right)$$

$$F = \beta\rho \text{ and LHS} = \frac{dB_{sys}}{dt}$$

(2) Assume at time t MV & CV coincide

$$\frac{d}{dt}\int_{MV} F\ dV = \int_{V^*}\frac{\partial F}{\partial t}\ dV + \int_{A^*} F\ \underline{u}\cdot\underline{n}\,dA^*$$

However, from RTT: $\int_{V^*}\frac{\partial F}{\partial t}\,dV = \frac{d}{dt}\int_{V^*} F\ dV - \int_{A^*} F\ \underline{u_s}\cdot\underline{n}\ dA^*$

$t + \Delta t$ Therefore: $\frac{d}{dt}\int_{MV} F\ dV = \frac{d}{dt}\int_{V^*} F\ dV + \int_{A^*} F\ \underbrace{\left(\underline{u}-\underline{u_s}\right)}_{\underline{u_R}}\cdot$

Provides relationship $\frac{d}{dt}\int_{MV} F\ dV$ & RHS which represents equivalent change for CV

Application CV GDE

$F = \beta\rho \qquad \beta = \frac{dB}{dm} \qquad B = \int \beta \, dm = \int_V \beta\rho \, dV$

$\frac{dB_{sys}}{dt} = \frac{d}{dt}(m, m\underline{u}, E) = \text{RHS} = (0, \sum \underline{F}, \dot{Q} - \dot{W})$

$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{MV} \beta\rho \, dV = \frac{d}{dt}\int_{CV} \beta\rho \, dV + \int_{CS} \beta\rho \, \underline{u_R} \cdot \underline{n} \, dA$

$\beta = (1, \underline{u}, e)$ and $u_R = \underline{u} - \underline{u_{cs}}$

Specific CV cases depending on $\underline{V_s}(\underline{x_{CS}}(t), t)$.

1) Deforming CV:

   (a) $\underline{V_s} = \underline{V_s}(\underline{x_{CS}}(t), t)$ non-uniform/accelerating velocity
   (b) $\underline{V_s} = \underline{V_s}(\underline{x_{CS}}(t))$ uniform/constant velocity (steady moving)
   (c) $\int_{CS} \underline{V_s}(\underline{x_{CS}}(t)) \cdot \underline{n} dA = 0$ as a whole at rest (stationary)

2) Non deforming CV:

   (a) $\underline{V_s} = \underline{V_s}(t)$ accelerating velocity
   (b) $\underline{V_s} =$ constant velocity, i.e., relative inertial coordinates (steady moving)
   (c) $\underline{V_s} = 0$ at rest (stationary)

Next step: apply $\beta = (1, \underline{u}, e)$

I. Conservation of mass $B_{sys} = m$, $\beta = 1$

$$\frac{dm}{dt} = 0 = \frac{d}{dt}\int_{V^*} \rho \, dV + \int_{A^*} \rho \, \underline{u_R} \cdot \underline{n} \, dA$$

$$-\frac{d}{dt}\int_{CV} \rho \, dV = \int_{CS} \rho \, \underline{u_R} \cdot \underline{n} \, dA$$

rate of decrease = net outflow

(1) most general case
$\rho(\underline{x},t)$ and $\underline{u_R}$
$\underline{u}$ & $\underline{u_s} = f(\underline{x},t)$

(2) other specific cases depend on $\rho \neq f(t)$ or $\rho \neq f(\underline{x})$ and form $\underline{b}(\underline{x},t)$ as per RTT for V*

i.e., deforming or non-deforming and each case either accelerating / steady moving / or stationary

II. Conservation of momentum

$$B_{sys} = m\underline{u} \quad \beta = \underline{u} \quad CV=V^* \quad CS=A^*$$

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}(m\underline{u}) = \frac{d}{dt}\int_{V^*} \rho\underline{u}\, dV + \int_{A^*} \rho\underline{u}\, \underline{u_r}\cdot\underline{n}\, dA$$

$RHS = \sum F$

act on dV V^*

$$= \underbrace{\int_{V^*} \rho g\, dV}_{\text{body force}} + \underbrace{\int_{A^*} f(\underline{n},\underline{x},t)\, dA}_{\text{surface force}} = \text{RHS}$$

Here again: 1) most general case $\rho = \rho(\underline{x},t)$ and $\underline{u}$ and $\underline{u_s}$ all f($\underline{x}$,t) and other specific cases depend on different forms $\underline{b}(\underline{x},t)$ as per RTT for V*.

Body force $\rho\underline{g}\, dV$ acts on dV without physical contact and is conservative since (by definition) conservative body forces can be expressed as the gradient of a potential function.

g=: m/s²

e per unit m

PE gz m²/s²

KE = u²/2

internal: $\widehat{u}$

$$\underline{g} = -\nabla\Phi \quad \text{or} \quad g_i = -\frac{\partial\Phi}{\partial x_i}$$

$$\Phi = |\underline{g}|\,z = gz$$

$$\underline{g} = -g\,\hat{z}$$

Φ = force potential with units energy per unit mass

Surface forces act on fluid elements via direct contact with the CS with units of stress N/m² and normal and tangential components.

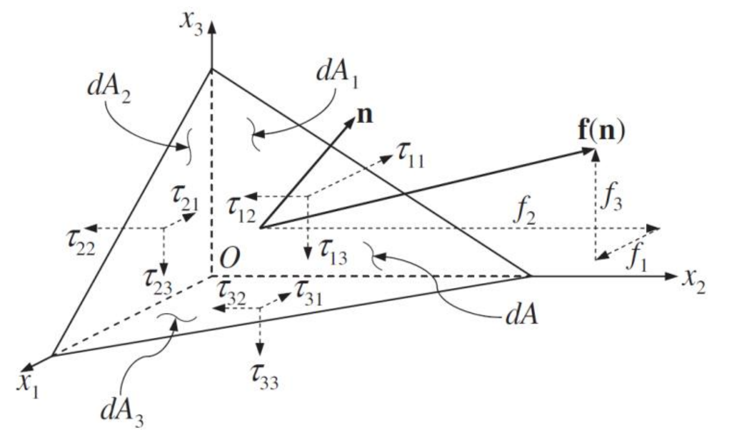

FIGURE 2.5 Force **f** per unit area on a surface element whose outward normal is **n**. The areas of the tetrahedron's faces that are perpendicular to the *i*th coordinate axis are dA_i. The area of the largest tetrahedron face is dA. As in Figure 2.4, the directions of positive normal and shear stresses are shown.

Arbitrarily oriented dA with normal $\underline{n} = n_i$.

Surface force $\underline{f}(\underline{n}, \underline{x}, t) = f_i = n_j\, T_{ij}$ per unit area.

$$f_1 = n_1 T_{11} + n_2 T_{21} + n_3 T_{31} \quad T_{ij} = stress\ tensor$$

$$f_2 = n_1 T_{12} + n_2 T_{22} + n_3 T_{32}$$

$$f_3 = n_1 T_{13} + n_2 T_{23} + n_3 T_{33}$$

Normal component = $\underline{n} \cdot \underline{f} = n_i f_i$

Tangential component vector = $\underline{f} - \left(\underline{n} \cdot \underline{f}\right) \underline{n}$

$$= f_k - (n_i f_i) n_k$$

$$\underline{n} \cdot \underline{f} = n_1 f_1 + n_2 f_2 + n_3 f_3$$

Cubical element

x-face: $\underline{n} = (1,0,0)$

$\underline{f_x} = T_{11}\hat{\imath} + T_{12}\hat{\jmath} + T_{13}\hat{k}$

$f_1 = T_{11}$

$f_2 = T_{12}$

$f_3 = T_{13}$

$\underline{n} \cdot \underline{f_x} = T_{11}$ Tangential component $\underline{f_t} = \underline{f_x} - T_{11}\hat{\imath}$

$= T_{12}\hat{\jmath} + T_{13}\hat{k}$

y-face: $\underline{n} = (0,1,0)$

$\underline{f_y} = T_{21}\hat{\imath} + T_{22}\hat{\jmath} + T_{23}\hat{k}$

$f_1 = T_{21}$

$f_2 = T_{22}$

$f_3 = T_{23}$

z-face: $\underline{n} = (0,0,1)$

$\underline{f_z} = T_{31}\hat{\imath} + T_{32}\hat{\jmath} + T_{33}\hat{k}$

$f_1 = T_{31}$

$f_2 = T_{32}$

$f_3 = T_{33}$

Writing momentum equation for MV: $\underline{u_r} = 0$, V* = V, A* = A

$$\frac{d}{dt}\int_V \rho \underline{u}\, dV = \int_V \frac{\partial}{\partial t}(\rho \underline{u})\, dV + \int_A \rho \underline{u}\, \underline{u} \cdot \underline{n}\, dA$$

$$= \int_V \rho \underline{g}\, dV + \int_A \underline{f}\, dA$$

$\int_A \rho \underline{u}\, \underline{u} \cdot \underline{n}\, dA = \int_V \nabla \cdot (\rho \underline{u}\, \underline{u})\, dV \; = \; \int_V \frac{\partial}{\partial x_j}(\rho u_i u_j)\, dV$

$\int_A \underline{f}\, dA = \int_A n_i T_{ij}\, dA = \int_V \frac{\partial T_{ij}}{\partial x_j}\, dV$

$$\int_V \left[\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) - \rho g_i - \frac{\partial}{\partial x_j}(T_{ij})\right] dV = 0$$

$$\lim_{dV \to 0}: \quad \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \rho g_i + \frac{\partial}{\partial x_j}(T_{ij})$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \frac{\partial \rho}{\partial t} u_i + \rho \frac{\partial u_i}{\partial t} + u_i \frac{\partial}{\partial x_j}(\rho u_j) + \rho u_j \frac{\partial u_i}{\partial x_j}$$

$$= \rho \frac{\partial u_i}{\partial t} + u_i \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)\right] + \rho u_j \frac{\partial u_i}{\partial x_j}$$

= 0 continuity

$$= \rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho \frac{D u_i}{D t}$$

$\rho \frac{D u_i}{D t} = \rho g_i + \frac{\partial}{\partial x_j}(T_{ij})$ Cauchy equation of motion

Unknowns: ρ, u_i, T_{ij} = 1 + 3 + 9 = 13 need stress–strain relationship

Equations: 1 + 3 + 2 = 6

↑

Thermodynamic equations (ρ, p)

Inner Product

row x column

1×3

V = 1st order tensor (column matrix)

$$\underline{u} \cdot \underline{v} \quad u_i v_i \quad U^T V \quad U^T = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

m n n p m p

$$A = 1 \times 3 \quad B = 3 \times 1 \quad C = 1 \times 1$$

$u^T v = 1 \times 1 = 0$ order tensor

$$c_{ij} = \sum_{k=1}^{n=3} a_{ik} b_{kj} \quad i = 1, m = 1 \quad j = 1, p = 1 = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$

$$= u_1 v_1 + u_2 v_2 + u_3 v_3$$

Vector Product

$$\underline{u} \cdot \underline{v} \quad u_i v_j \quad U\, V^T$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$$

$3 \times 1 \quad 1 \times 3 \quad = \quad 3 \times 3$

m n n p m p

$$c_{ij} = \sum_{k=1}^{n=1} a_{i1} b_{1j} \quad i = 1,3 \quad j = 1,3$$

$$\begin{matrix} a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} \\ a_{21}b_{11} & a_{21}b_{12} & a_{21}b_{13} \\ a_{31}b_{11} & a_{31}b_{12} & a_{31}b_{13} \end{matrix} \qquad \begin{matrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{matrix}$$

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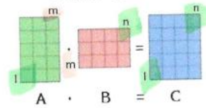
Matrix multiplication

In mathematics, particularly in linear algebra, **matrix multiplication** is a binary operation that produces a matrix from two matrices. For matrix multiplication, the number of columns in the first matrix must be equal to the number of rows in the second matrix. The resulting matrix, known as the **matrix product**, has the number of rows of the first and the number of columns of the second matrix. The product of matrices $\mathbf{A}$ and $\mathbf{B}$ is denoted as $\mathbf{AB}$.[1]

columns A = rows B



For matrix multiplication, the number of columns in the first matrix must be equal to the number of rows in the second matrix. The result matrix has the number of rows of the first and the number of columns of the second matrix.

C = rows A
& columns B

Matrix multiplication was first described by the French mathematician Jacques Philippe Marie Binet in 1812,[2] to represent the composition of linear maps that are represented by matrices. Matrix multiplication is thus a basic tool of linear algebra, and as such has numerous applications in many areas of mathematics, as well as in applied mathematics, statistics, physics, economics, and engineering.[3][4] Computing matrix products is a central operation in all computational applications of linear algebra.

Notation

This article will use the following notational conventions: matrices are represented by capital letters in bold, e.g. $\mathbf{A}$; vectors in lowercase bold, e.g. $\mathbf{a}$; and entries of vectors and matrices are italic (they are numbers from a field), e.g. A and a. Index notation is often the clearest way to express definitions, and is used as standard in the literature. The entry in row i, column j of matrix $\mathbf{A}$ is indicated by $(\mathbf{A})_{ij}$, A_{ij} or a_{ij}. In contrast, a single subscript, e.g. $\mathbf{A}_1$, $\mathbf{A}_2$, is used to select a matrix (not a matrix entry) from a collection of matrices.

Definition

If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times p$ matrix,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}$$

the *matrix product* $\mathbf{C} = \mathbf{AB}$ (denoted without multiplication signs or dots) is defined to be the $m \times p$ matrix[5][6][7][8]

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{pmatrix}$$

such that

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj},$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, p$.

That is, the entry c_{ij} of the product is obtained by multiplying term-by-term the entries of the ith row of $\mathbf{A}$ and the jth column of $\mathbf{B}$, and summing these n products. In other words, c_{ij} is the dot product of the ith row of $\mathbf{A}$ and the jth column of $\mathbf{B}$.

Therefore, $\mathbf{AB}$ can also be written as

$$\mathbf{C} = \begin{pmatrix} a_{11}b_{11} + \cdots + a_{1n}b_{n1} & a_{11}b_{12} + \cdots + a_{1n}b_{n2} & \cdots & a_{11}b_{1p} + \cdots + a_{1n}b_{np} \\ a_{21}b_{11} + \cdots + a_{2n}b_{n1} & a_{21}b_{12} + \cdots + a_{2n}b_{n2} & \cdots & a_{21}b_{1p} + \cdots + a_{2n}b_{np} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{11} + \cdots + a_{mn}b_{n1} & a_{m1}b_{12} + \cdots + a_{mn}b_{n2} & \cdots & a_{m1}b_{1p} + \cdots + a_{mn}b_{np} \end{pmatrix}$$

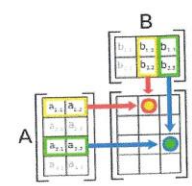
Thus the product $\mathbf{AB}$ is defined if and only if the number of columns in $\mathbf{A}$ equals the number of rows in $\mathbf{B}$,[1] in this case n.

In most scenarios, the entries are numbers, but they may be any kind of mathematical objects for which an addition and a multiplication are defined, that are associative, and such that the addition is commutative, and the multiplication is distributive with respect to the addition. In particular, the entries may be matrices themselves (see block matrix).

Illustration

The figure to the right illustrates diagrammatically the product of two matrices $\mathbf{A}$ and $\mathbf{B}$, showing how each intersection in the product matrix corresponds to a row of $\mathbf{A}$ and a column of $\mathbf{B}$.

$$\overset{4\times 2 \text{ matrix}}{\begin{bmatrix} a_{11} & a_{12} \\ \cdot & \cdot \\ a_{31} & a_{32} \\ \cdot & \cdot \end{bmatrix}} \overset{2\times 3 \text{ matrix}}{\begin{bmatrix} \cdot & b_{12} & b_{13} \\ \cdot & b_{22} & b_{23} \end{bmatrix}} = \overset{4\times 3 \text{ matrix}}{\begin{bmatrix} \cdot & c_{12} & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & c_{33} \\ \cdot & \cdot & \cdot \end{bmatrix}}$$



The values at the intersections, marked with circles in figure to the right, are:

$c_{12} = a_{11} b_{12} + a_{12} b_{22}$

$c_{33} = a_{31} b_{13} + a_{32} b_{23}$

13

Consider the equation for Newtonian fluid

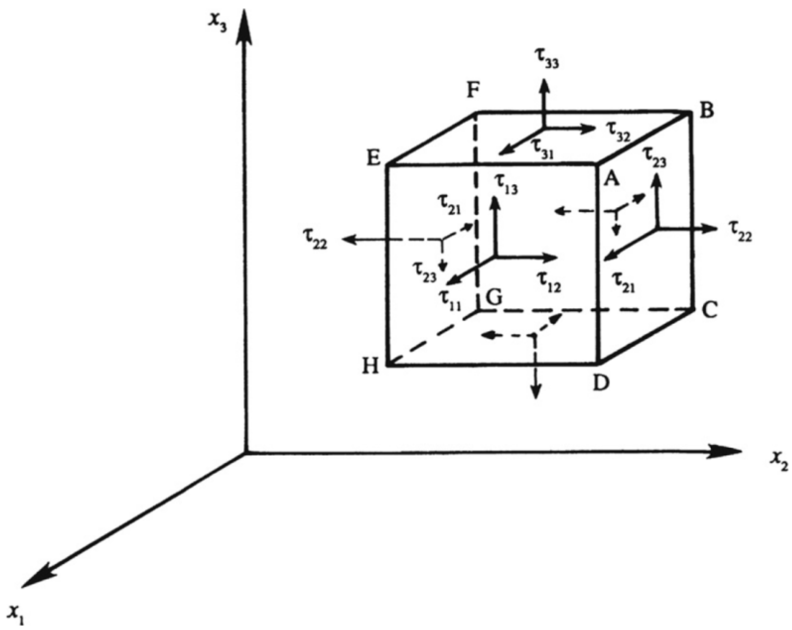


FIGURE 2.4 Illustration of the stress field at a point via stress components on a cubic volume element. Here each surface may experience one normal and two shear components of stress. The directions of positive normal and shear stresses are shown. For clarity, the stresses on faces FBCG and CDHG are not labeled.

Stress at a point fully described by T_{ij}: 9 components

However,

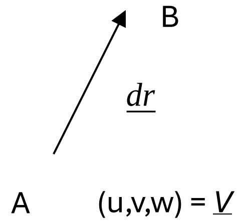
$$T_{ij} = T_{ji}$$

is symmetric such that only six independent components; since, the stresses themselves cause no rotation, which is shown by considering the differential equation of angular momentum for limit $dV = dx_1\, dx_2\, dx_3 \rightarrow 0$ assuming no external body force moments $\propto \rho$ such as electric field or polarized fluid molecules.

$$T_{ij} = f(u_{ij}) = \text{constitutive equation}$$

$$= \propto u_{ij} \text{ Newtonian fluid}$$

Next, we need to relate the stresses σ_{ij} to the fluid motion, i.e., the velocity field. To this end, we examine the relative motion between two neighboring fluid particles.



@ B: $\underline{V} + \underline{dV} = \underline{V} + \nabla\underline{V} \cdot \underline{dr}$ 1st order Taylor Series

$$u_b = u_a + u_x\,dx + u_y\,dy + u_z\,dz + u_{xx}\frac{dx^2}{2} + \cdots$$

$$v_b = v_a + v_x\,dx + v_y\,dy + v_z\,dz + v_{xx}\frac{dx^2}{2} + \cdots$$

$$w_b = w_a + w_x\,dx + w_y\,dy + w_z\,dz + w_{xx}\frac{dx^2}{2} + \cdots$$

$\underline{dV} = (u_B - u_A,\ v_B - v_A,\ w_B - w_A)$

$$\underline{dV} = \nabla\underline{V} \cdot \underline{dr} = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix}\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = e_{ij}\,dx_j$$

relative motion

$\underline{dV} = dV_i = (dV_1, dV_2, dV_3)$

deformation rate tensor = e_{ij}

$$e_{ij} = \frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\substack{symmetric\ part \\ \varepsilon_{ij}=\varepsilon_{ji}}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\substack{anti-symmetric\ part \\ \omega_{ij}=-\omega_{ji}}} = \varepsilon_{ij} + \omega_{ij}$$

$$\omega_{ij} = \begin{bmatrix} 0 & \frac{1}{2}(u_y - v_x) & \overbrace{\frac{1}{2}(u_z - w_x)}^{\eta} \\ \underbrace{\frac{1}{2}(v_x - u_y)}_{\zeta} & 0 & \frac{1}{2}(v_z - w_y) \\ \frac{1}{2}(w_x - u_z) & \underbrace{\frac{1}{2}(w_y - v_z)}_{\xi} & 0 \end{bmatrix} = \textit{rigid body rotation of fluid element}$$

where ξ = *rotation about x axis*
η = *rotation about y axis*
ς = *rotation about z axis*

Note that the components of ω_{ij} are related to the <u>vorticity vector defined by</u>:

$$\underline{\omega} = \nabla \times \underline{V} = \underbrace{(w_y - v_z)}_{2\xi}\hat{i} + \underbrace{(u_z - w_x)}_{2\eta}\hat{j} + \underbrace{(v_x - u_y)}_{2\zeta}\hat{k} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}$$

= 2 × angular velocity of fluid element

$$\varepsilon_{ij} = \textit{rate of strain tensor}$$

$$= \begin{bmatrix} u_x & \frac{1}{2}(u_y + v_x) & \frac{1}{2}(u_z + w_x) \\ \frac{1}{2}(v_x + u_y) & v_y & \frac{1}{2}(v_z + w_y) \\ \frac{1}{2}(w_x + u_z) & \frac{1}{2}(w_y + v_z) & w_z \end{bmatrix}$$

$$u_x + v_y + w_z = \nabla \cdot \underline{V} = \textit{elongation (or volumetric dilatation) of fluid element} = \frac{1}{\forall}\frac{D\forall}{Dt}$$

$\frac{1}{2}(u_y + v_x)$ = distortion wrt (x,y) plane

$\frac{1}{2}(u_z + w_x)$ = distortion wrt (x,z) plane

$\frac{1}{2}(v_z + w_y)$ = distortion wrt (y,z) plane

Thus, general motion consists of:

1) pure translation described by $\underline{V}$
2) rigid-body rotation described by $\underline{\omega}$
3) volumetric dilatation described by $\nabla \cdot \underline{V}$
4) distortion in shape described by ε_{ij} $\quad i \neq j$

It is now necessary to make certain postulates concerning the relationship between the fluid stress tensor (σ_{ij}) and rate-of-deformation tensor (e_{ij}). These postulates are based on physical reasoning and experimental observations and have been verified experimentally even for extreme conditions. For a Newtonian fluid:

1) When the fluid is at rest the stress is hydrostatic, and the pressure is the thermodynamic pressure

2) Since there is no shearing action in rigid body rotation, it causes no shear stress.

3) τ_{ij} is linearly related to ε_{ij} and only depends on ε_{ij}.

4) There is no preferred direction in the fluid, so that the fluid properties are point functions (condition of isotropy).

Using statements 1-3

$$\sigma_{ij} = -p\delta_{ij} + k_{ijmn}\varepsilon_{mn} \qquad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

k_{ijmn} = 4th order tensor with 81 components (3x3x3x3) such that each stress is linearly related to all nine components of ε_{mn}.

However, statement (4) requires that the fluid has no directional preference, i.e., σ_{ij} is independent of rotation of the coordinate system, which means k_{ijmn} is an isotropic tensor = 4th order tensor made up of products of δ_{ij}.

$$k_{ijmn} = \lambda\delta_{ij}\delta_{mn} + \mu\delta_{im}\delta_{jn} + \gamma\delta_{in}\delta_{jm}$$

$$(\lambda, \mu, \gamma) = scalars$$

Lastly, the symmetry condition $\sigma_{ij} = \sigma_{ji}$ requires:

$$k_{ijmn} = k_{jimn}$$

$$k_{ijmn} = \lambda\delta_{ij}\delta_{mn} + \mu\delta_{im}\delta_{jn} + \gamma\delta_{in}\delta_{jm}$$
$$k_{jimn} = \lambda\delta_{ji}\delta_{mn} + \mu\delta_{jm}\delta_{in} + \gamma\delta_{jn}\delta_{im}$$

Equating the two:

$$\lambda\delta_{ij}\delta_{mn} + \mu\delta_{im}\delta_{jn} + \gamma\delta_{in}\delta_{jm} = \lambda\delta_{ji}\delta_{mn} + \mu\delta_{jm}\delta_{in} + \gamma\delta_{jn}\delta_{im}$$

$$\lambda\delta_{mn}(\delta_{ij} - \delta_{ji}) + \mu(\delta_{im}\delta_{jn} - \delta_{jm}\delta_{in}) \\ + \gamma(\delta_{in}\delta_{jm} - \delta_{jn}\delta_{im}) = 0$$

The first term is zero since $\delta_{ij} = \delta_{ji}$, therefore:

$$\mu(\delta_{im}\delta_{jn} - \delta_{jm}\delta_{in}) + \gamma(\delta_{in}\delta_{jm} - \delta_{jn}\delta_{im}) = 0$$

If $i = m$ and $j = n$:

$$\mu(1 - \delta_{nm}\delta_{mn}) - \gamma(1 - \delta_{mn}\delta_{nm}) = 0$$

If $j = m$ and $i = n$:

$$\mu(\delta_{nm}\delta_{mn} - 1) - \gamma(\delta_{mn}\delta_{nm} - 1) = 0$$

i.e.,

$$\mu = \gamma = \text{viscosity}$$

The stress tensor can be written as:

$$\sigma_{ij} = -p\delta_{ij} + \mu\delta_{im}\delta_{jn}\varepsilon_{mn} + \mu\delta_{in}\delta_{jm}\varepsilon_{mn} + \lambda\delta_{ij}\delta_{mn}\varepsilon_{mn}$$

Take $\mu\delta_{im}\delta_{jn}\varepsilon_{mn} \rightarrow \delta_{im} \neq 0$ if $m = i$ and $\delta_{jn} \neq 0$ if $n = j \rightarrow$ equivalent to $\mu\varepsilon_{ij}$. Similar reasoning for other terms:

$$\sigma_{ij} = -p\delta_{ij} + 2\mu\varepsilon_{ij} + \lambda \underbrace{\varepsilon_{mm}}_{\nabla\cdot\underline{V}} \delta_{ij}$$

λ and μ can be further related if one considers mean normal stress vs. thermodynamic p.

$$\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \sigma_{ii} = -3p + (2\mu + 3\lambda)\nabla \cdot \underline{V}$$

$$p = \underbrace{-\frac{1}{3}\sigma_{ii}}_{\bar{p} = \text{mean normal stress}} + \left(\frac{2}{3}\mu + \lambda\right)\nabla \cdot \underline{V}$$

$$p - \bar{p} = \left(\frac{2}{3}\mu + \lambda\right)\nabla \cdot \underline{V}$$

Incompressible flow: $p = \bar{p}$ and absolute pressure is indeterminant since there is no equation of state for p. Equations of motion determine ∇p.

Compressible flow: $p \neq \bar{p}$ and λ = bulk viscosity which must be determined; however, it is a very difficult measurement requiring large $\nabla \cdot \underline{V} = -\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{1}{\forall}\frac{D\forall}{Dt}$, e.g., within shock waves.

Stokes Hypothesis also supported kinetic theory monotonic gas.

$$\lambda = -\tfrac{2}{3}\mu$$

$$p = \bar{p}$$

$$\sigma_{ij} = -\left(p + \frac{2}{3}\mu\nabla \cdot \underline{V}\right)\delta_{ij} + 2\mu\varepsilon_{ij} = -p\delta_{ij} + \tau_{ij}$$

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad i \neq j$$ *relates shear stress to strain rate*

Generalization $\tau = \mu\dfrac{du}{dy}$ for 3D flow.

$$\sigma_{ii} = -p - \frac{2}{3}\mu\nabla \cdot \underline{V} + 2\mu\left(\frac{\partial u_i}{\partial x_i}\right) = -p + \underbrace{2\mu\left[-\frac{1}{3}\nabla \cdot \underline{V} + \frac{\partial u_i}{\partial x_i}\right]}_{normal\ viscous\ stress}$$

Where the normal viscous stress is the difference between the extension rate in the x_i direction and average expansion at a point. Only differences from the average = $\frac{1}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$ generate normal viscous stresses. For incompressible fluids, average = 0 i.e., $\nabla \cdot \underline{V} = 0$.

EXAMPLE 4.8

Write out all the components of the stress tensor $\mathbf{T}$ in (x, y, z)-coordinates in terms of $\mathbf{u} = (u, v, w)$, and its derivatives.

Solution

Evaluate each component of (4.36) and abbreviate $S_{mm} = \partial u/\partial x + \partial v/\partial y + \partial w/\partial z = \nabla\cdot\mathbf{u}$ to find:

$$\mathbf{T} = \begin{bmatrix} -p + 2\mu\frac{\partial u}{\partial x} + \left(\mu_v - \frac{2}{3}\mu\right)\nabla\cdot\mathbf{u} & \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & -p + 2\mu\frac{\partial v}{\partial y} + \left(\mu_v - \frac{2}{3}\mu\right)\nabla\cdot\mathbf{u} & \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & -p + 2\mu\frac{\partial w}{\partial z} + \left(\mu_v - \frac{2}{3}\mu\right)\nabla\cdot\mathbf{u} \end{bmatrix}$$

Non-Newtonian fluids:

$\tau_{ij} \propto \varepsilon_{ij}$ for small strain rates $\dot{\theta}$, which works well for air, water, etc. Newtonian fluids

$$\tau_{ij} \propto \underbrace{\varepsilon_{ij}^{n}}_{non-linear} + \underbrace{\frac{\partial}{\partial t}\varepsilon_{ij}}_{history\ effect}$$

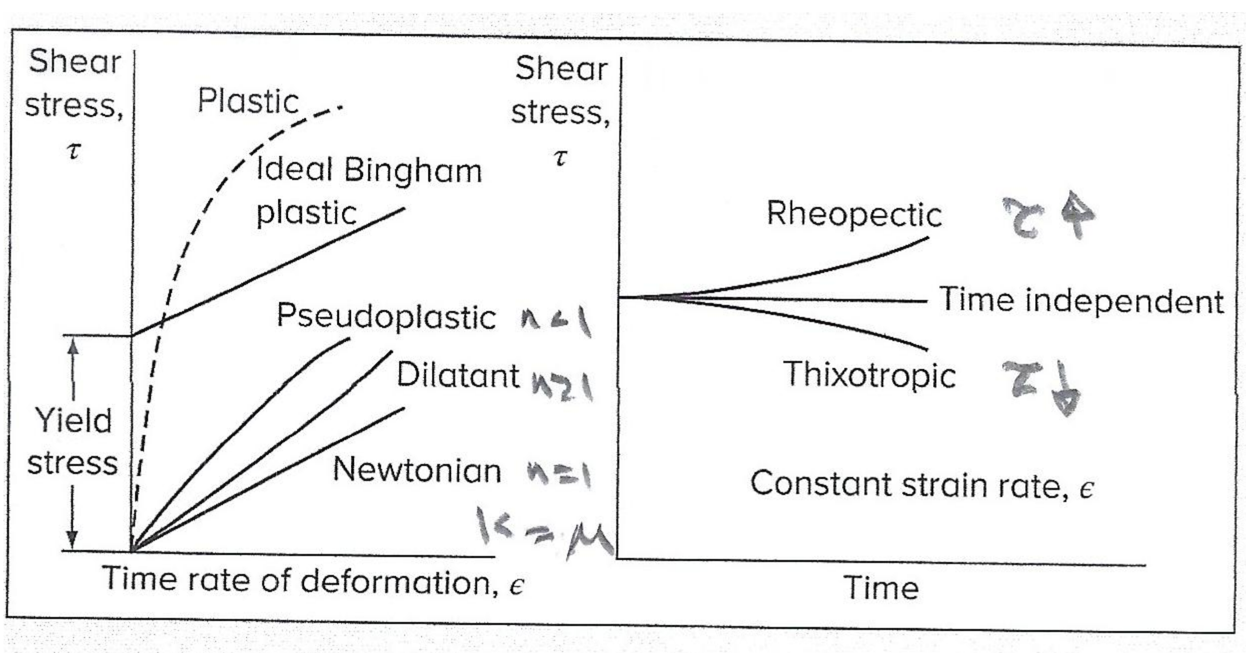
Non-Newtonian

Viscoelastic materials

Non-Newtonian fluids include:

(1) Polymer molecules with large molecular weights and form long chains coiled together in spongy ball shapes that deform under shear.

(2) Emulsions and slurries containing suspended particles such as blood and water/clay.



Navier Stokes Equations:

$$\rho \underline{a} = \rho \frac{D\underline{V}}{Dt} = -\rho g \hat{k} + \nabla \cdot \sigma_{ij}$$

$$\rho \frac{D\underline{V}}{Dt} = -\rho g \hat{k} - \nabla p + \frac{\partial}{\partial x_j}\left[2\mu\varepsilon_{ij} - \frac{2}{3}\mu\nabla \cdot \underline{V}\delta_{ij}\right]$$

Recall μ = μ(T) and μ increases with T for gases, decreases with T for liquids, but if it is assumed that μ = constant:

$$\rho \frac{D\underline{V}}{Dt} = -\rho g \hat{k} - \nabla p + 2\mu \frac{\partial}{\partial x_j}\varepsilon_{ij} - \frac{2}{3}\mu \frac{\partial}{\partial x_j}\nabla \cdot \underline{V}$$

$$2\frac{\partial}{\partial x_j}\varepsilon_{ij} = \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{\partial^2 u_i}{\partial x_j \partial x_j} = \nabla^2 u_i = \nabla^2 \underline{V}$$

$$\rho \frac{D\underline{V}}{Dt} = -\rho g\ \hat{k} - \nabla p + \mu\left[\nabla^2 \underline{V} - \frac{2}{3}\frac{\partial}{\partial x_j}\nabla \cdot \underline{V}\right]$$

For incompressible flow $\nabla \cdot \underline{V} = 0$

$$\rho \frac{D\underline{V}}{Dt} = \underbrace{-\rho g \hat{k} - \nabla p}_{\substack{-\nabla \hat{p} \text{ where } \hat{p} = p + \gamma z \\ \text{piezometric pressure}}} + \mu \nabla^2 \underline{V}$$

For μ = 0

$$\rho \frac{D\underline{V}}{Dt} = -\rho g\ \hat{k} - \nabla p \qquad \text{Euler Equation}$$

NS equations for ρ, μ constant

$$\rho \frac{D\underline{V}}{Dt} = -\nabla\hat{p} + \mu\nabla^2\underline{V}$$

$$\rho\left[\frac{\partial\underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right] = -\nabla\hat{p} + \mu\nabla^2\underline{V}$$

$$\left[\frac{\partial\underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right] = -\frac{1}{\rho}\nabla\hat{p} + \nu\nabla^2\underline{V} \quad \nu = \frac{\mu}{\rho}$$ *kinematic viscosity/ diffusion coefficient*

Non-linear 2nd order PDE, as is the case for ρ, μ not constant.

Combine with $\nabla\cdot\underline{V}$ for 4 equations for 4 unknowns $\underline{V}$, p and can be, albeit difficult, solved subject to initial and boundary conditions for $\underline{V}$, p at t = t_0 and on all boundaries i.e. "well posed" IBVP.

Summary GDE for compressible non-constant property fluid flow

Continuity: $\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{V}) = 0$

Momentum: $\rho\frac{D\underline{V}}{Dt} = \rho\underline{g} - \nabla p + \nabla\cdot\sigma_{ij}$

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\nabla\cdot\underline{V}\delta_{ij}$$

$$\underline{g} = -g\hat{k}$$

Energy $\rho \frac{Dh}{Dt} = \frac{Dp}{Dt} + \nabla \cdot (k \nabla T) + \Phi$

Primary variables: p, $\underline{V}$, T

Auxiliary relations: ρ = ρ (p,T) μ = μ (p,T)

(equations of state) h = h (p,T) k = k (p,T)

Restrictive Assumptions:

1) Continuum
2) Newtonian fluids
3) Thermodynamic equilibrium
4) $\underline{g} = -g\hat{k}$
5) heat conduction follows Fourier's law.
6) no internal heat sources.

For incompressible constant property fluid flow

$d\hat{u} = c_v \, dT$ $c_v, \mu, k, \rho \sim constant$

$$\rho c_v \frac{DT}{Dt} = k \nabla^2 T + \Phi$$

For static fluid or $\underline{V}$ small

$\rho c_p \frac{\partial T}{\partial t} = k \nabla^2 T$ *heat conduction equation (also valid for solids)*

Summary GDE for incompressible constant property fluid flow ($c_v \sim c_p$)

$$\nabla \cdot \underline{V} = 0$$

$$\rho \frac{D\underline{V}}{Dt} = -\rho g \hat{k} - \nabla p + \mu \nabla^2 \underline{V} \qquad \text{“elliptic”}$$

$$\rho c_p \frac{DT}{Dt} = k \nabla^2 T + \Phi \qquad \text{where } \Phi = \tau_{ij} \frac{\partial u_i}{\partial x_j}$$

Continuity and momentum uncoupled from energy; therefore, solve separately and use solution post facto to get T.

For compressible flow, ρ solved from continuity equation, T from energy equation, and p = (ρ, T) from equation of state (e.g., ideal gas law). For incompressible flow, ρ = constant and T uncoupled from continuity and momentum equations, the latter of which contains ∇p such that reference p is arbitrary and specified post facto (i.e., for incompressible flow, there is no connection between p and ρ). The connection is between ∇p and $\nabla \cdot \underline{V} = 0$, i.e., a solution for p requires $\nabla \cdot \underline{V} = 0$.

NS:

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2}$$

$\nabla \cdot (NS)$:

$$\nabla \cdot \left[\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} = -\nabla\left(\frac{p}{\rho}\right) + \nu \nabla^2 \underline{V}\right]$$

$$\nabla \cdot \left(\frac{\partial \underline{V}}{\partial t} - \nu \nabla^2 \underline{V}\right) + \nabla \cdot \left(\underline{V} \cdot \nabla \underline{V}\right) = -\nabla^2\left(\frac{p}{\rho}\right)$$

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2\right) \nabla \cdot \underline{V} + \nabla \cdot \left(\underline{V} \cdot \nabla \underline{V}\right) = -\nabla^2\left(\frac{p}{\rho}\right)$$

$$\underline{V} \cdot \nabla \underline{V} = u_j \frac{\partial u_i}{\partial x_j}$$

$$\nabla \cdot \left(\underline{V} \cdot \nabla \underline{V}\right) = \frac{\partial}{\partial x_i}\left(u_j \frac{\partial u_i}{\partial x_j}\right) = \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} + u_j \cancel{\frac{\partial}{\partial x_i}\frac{\partial u_i}{\partial x_j}}$$

$$\nabla \cdot \left(\underline{V} \cdot \nabla \underline{V}\right) = \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}$$

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2\right) \nabla \cdot \underline{V} = -\frac{1}{\rho}\nabla^2 p - \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$$

For $\nabla \cdot \underline{V} = 0$: $\nabla^2 p = -\rho \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$

Poisson equation determines pressure up to additive constant.

Approximate Models:

1) Stokes Flow

For low $\mathrm{Re} = \dfrac{UL}{\nu} << 1, \quad \underline{V} \cdot \nabla \underline{V} \sim 0$

$$\nabla \cdot \underline{V} = 0$$
$$\frac{\partial \underline{V}}{\partial t} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \underline{V}$$

→ Linear, "elliptic"

Most exact solutions NS; and for steady flow superposition, elemental solutions, and separation of variables

$$\nabla \cdot (NS) \Rightarrow \nabla^2 p = 0$$

2) Boundary Layer Equations

For high Re >> 1 and attached boundary layers or fully developed free shear flows (wakes, jets, mixing layers), $v << U$, $\dfrac{\partial}{\partial x} << \dfrac{\partial}{\partial y}$, $p_y = 0$, and for free shear flow $p_x = 0$.

$$u_x + v_y = 0$$
$$u_t + u u_x + v u_y = -\hat{p}_x + \nu u_{yy}$$ *non-linear, "parabolic"*
$$p_y = 0$$
$$-\hat{p}_x = U_t + U U_x$$

Many exact solutions; similarity methods

3) Inviscid Flow

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \underline{V} \right) = 0$$

$$\rho \frac{D\underline{V}}{Dt} = \rho \underline{g} - \nabla p \qquad \text{Euler Equation, nonlinear, "hyperbolic"}$$

$$\rho \frac{Dh}{Dt} = \frac{Dp}{Dt} + \nabla \cdot (k \nabla T) \qquad p, \underline{V}, T \text{ unknowns} \quad \text{and } \rho, h, k = f(p, T)$$

4) Inviscid, Incompressible, Irrotational

$$\nabla \times \underline{V} = 0 \rightarrow \underline{V} = \nabla \varphi$$
$$\nabla \cdot \underline{V} = 0 \rightarrow \nabla^2 \varphi = 0 \quad \textit{linear} \text{ elliptic}$$

$\int$ Euler Equation → Bernoulli Equation:

$$p + \frac{\rho}{2} V^2 + \rho g z = const$$

Many elegant solutions: Laplace equation using superposition elementary solutions, separation of variables, complex variables for 2D, and Boundary Element methods.