# Stability Nearly Parallel Viscous Flows

Assume 2D flow for simplicity: $U(x,y), V(x,y)$ and $V \ll U$ and $U_x \ll U_y$ with perturbation $\underline{u} = (u, v, w)$ and $p$

$$\underline{v} = (U+u, v, w) \qquad p = P + p$$

Both steady mean flow $U, V, P$ and $\underline{u}, p$ satisfy NS. All variables are nondimensional using $U_\infty, L, \rho$.

x-momentum: $u_t + (U+u)(U+u)_x + (V+v)(U+u)_y = -(P+p)_x + Re^{-1}\nabla^2(U+u)$ total

$v_t + (U+u)v_x + v v_y + w v_z = -(P+p)_y + Re^{-1}\nabla^2 v$

$w_t + (U+u)w_x + v w_y + w w_z = -(P+p)_z + Re^{-1}\nabla^2 w$

$UU_x + VU_y = -P_x + Re^{-1}\nabla^2 U$ mean

$-P_y = 0 \quad -P_z = 0$

Expand total, linearize by neglecting products perturbation, and subtract mean:

$$\begin{aligned} u_t + U u_x + v U_y &= -p_x + Re^{-1}\nabla^2 u \\ v_t + U v_x &= -p_y + Re^{-1}\nabla^2 v \qquad (1) \\ w_t + U w_x &= -p_z + Re^{-1}\nabla^2 w \end{aligned}$$

Continuity: $\nabla^2 \underline{u}$ and $\nabla^2 \underline{v} = 0 \quad \underline{u} = (u, v, w)$

(2)

(1) + (2) linear system for $\underline{v}$ and $p$ as f $(U(y))$

$U(x)$ suppressed by "nearly parallel" assumption.

For either channel flows or unbounded $\underline{u}(\pm h)$ or $\underline{u}(\pm\infty) = 0$ and BC $p$ are not required.

Assume normal mode solutions:

$$\exp[i(\alpha x + \beta z - \alpha c t)]$$

$$v_i = \hat{v}_i(y)\exp[i(\alpha x + \beta z - \alpha c_R t)]\exp(\alpha c_I t)$$
$$p = \hat{p}(y)\exp[i(\alpha x + \beta z - \alpha c_R t)]\exp(\alpha c_I t) \quad (3)$$

For wave numbers $(\alpha, \beta)$ real and specified Re, the substitution (3) into (1) + (2) is an eigenvalue problem for $\hat{v}_i(y)$ and $\hat{p}(y)$ eigenfunctions for specific values $c$ = complex for temporal instability = eigenvalue. $c = c_R + i c_I$

$$i\alpha(U-c)\hat{u} + \hat{v}U_y = -i\alpha\hat{p} + Re^{-1}[\hat{u}_{yy} - (\alpha^2+\beta^2)\hat{u}]$$
$$i\alpha(U-c)\hat{v} = -\hat{p}_y + Re^{-1}[\hat{v}_{yy} - (\alpha^2+\beta^2)\hat{v}] \quad (4)$$
$$i\alpha(U-c)\hat{w} = -i\beta\hat{p} + Re^{-1}[\hat{w}_{yy} - (\alpha^2+\beta^2)\hat{w}]$$

$$i\alpha\hat{u} + i\beta\hat{w} + \hat{v}_y = 0 \quad (5)$$

(4) + (5) governs the viscous stability of a normal mode for any "nearly parallel" flow.

Squires Theorem: for any unstable 3D disturbance there is a corresponding 2D disturbance ($\hat{w}=0$) that is more unstable.

$$\alpha^* = (\alpha^2+\beta^2)^{1/2} \qquad c^* = c$$

$$\alpha^* u^* = \alpha\hat{u} + \beta\hat{w} \qquad v^* = \hat{v} \qquad (5)$$

$$p^*/\alpha^* = \hat{p}/\alpha \qquad \alpha^* Re^* = \alpha Re \qquad Re^{-1} = \frac{\alpha}{\alpha^*} Re^{*-1}$$

Transform (3) + (4) using (5). 1st & 3rd (4) are added & 2nd & 4th simply transformed

$$i\alpha^*(U-c^*)u^* + v^* U_y = -i\alpha^* p^* + Re^{*-1}(u^*_{yy} - \alpha^{*2}u^*)$$
$$i\alpha^*(U-c^*)v^* = -p^*_y + Re^{*-1}(v^*_{yy} - \alpha^* v^*) \qquad (6)$$

$$i\alpha^* u^* + v^*_y = 0 \qquad (7)$$

$$i\alpha\hat{u} = i\alpha^* u^* - i\beta\hat{w} \qquad \hat{v}_y = v^*_y \qquad (5) \rightarrow (7) \quad ✓$$
$$i\alpha^* u^* + v^*_y = 0$$

$$i\alpha(U-c^*)\hat{v} = -\frac{\alpha}{\alpha^*}p^*_y + \frac{\alpha}{\alpha^*}Re^{*-1}\left[v^*_{yy} - \alpha^{*2}v^*\right]$$

$$i\alpha^*(U-c^*)v^* = -p^*_y + Re^{*-1}\left[v^*_{yy} - \alpha^{*2}v^*\right] \quad ✓$$

$$i\alpha(U-c^*)\hat{u} + v^* U_y = -i\frac{\alpha^2}{\alpha^*}p^* + \frac{\alpha}{\alpha^*}Re^{*-1}\left[\hat{u}_{yy} - \alpha^{*2}\hat{u}\right]$$

$$i\alpha(U-c^*)\hat{w} = -i\beta\frac{\alpha}{\alpha^*}p^* + \frac{\alpha}{\alpha^*}Re^{*-1}\left[\hat{w}_{yy} - \alpha^{*2}\hat{w}\right]$$

$$i\alpha(U-c^*)\alpha\hat{u} + \alpha v^* U_y = -i\frac{\alpha^3}{\alpha^*}p^* + \frac{\alpha}{\alpha^*}Re^{*-1}\left[\alpha\hat{u}_{yy} - \alpha^{*2}\hat{u}\alpha\right]$$

$$i\alpha(U-c^*)\beta\hat{w} = -i\beta^2\frac{\alpha}{\alpha^*}p^* + \frac{\alpha}{\alpha^*}Re^{*-1}\left[\beta\hat{w}_{yy} - \alpha^{*2}\hat{w}\beta\right]$$

$$i\alpha(U-c^*)(\alpha\hat{u}+\beta\hat{w}) + \alpha v^* U_y = -i\frac{\alpha}{\alpha^*}(\alpha^2+\beta^2)p^* + \frac{\alpha}{\alpha^*}Re^{*-1}\left[\alpha\hat{u}_{yy} + \beta\hat{w}_{yy} - \alpha^{*2}(\hat{u}\alpha+\hat{w}\beta)\right]$$

$$i\cancel{\alpha}(U-c^*)\alpha^* u^* + \cancel{\alpha} v^* U_y = -\cancel{i\alpha}\alpha^* p^* + \frac{\cancel{\alpha}}{\cancel{\alpha^*}}R^{-1}\left[\cancel{\alpha^*}u^*_{yy} - \alpha^{*2}\cancel{\alpha^*}u^*\right]$$

$$i(U-c^*)\alpha^* u^* + v^* U_y = -i\alpha^* p^* + Re^{*-1}\left[u^*_{yy} - \alpha^{*2}u^*\right] \checkmark$$

(6) + (7) = (4) + (5) for $\beta = 0$ & $\hat{w} = 0$

Therefore, any solution 2D disturbance $u^*, v^*$ with wave number $\alpha^*$ & speed $c^*$ = equivalent 3D with wave number $\alpha, \beta$ & speed $c$.

Note temporal growth rate = $\exp(\alpha c_I t)$. Thus, for 3D disturbance equivalent 2D has larger x-direction wave number since $\alpha^* = (\alpha^2+\beta^2)^{1/2}$ & is more unstable $\alpha^* c_I^* > \alpha c_I$ (note $c_I^* = c_I$). Moreover $Re^* = Re\,\alpha/\alpha^*$ i.e. $Re^* < Re$.

Interested most unstable marginal stability curve so Squire theorem shows that 2D disturbance provides this condition. Therefore use (4)+(5) with $\beta = 0$ & $\hat{w} = 0$.

Orr-Sommerfeld Equation

$u = \chi_y \qquad v = -\chi_x \qquad \chi = a(y)\exp[i(\alpha x + \beta z - ct)]$

normal mode complex amplitude $a$

$\hat{u} = a_y \qquad \hat{v} = -i\alpha a$

continuity: $i\alpha a_y + (-i\alpha a_y) = 0$

$$i\alpha(U-c)a_y - (i\alpha a)U_y = -i\alpha p + Re^{-1}(a_{yyy} - \alpha^2 a_y)$$

$$\cancel{i\alpha U_y a_y} + i\alpha(U-c)a_{yy} - \cancel{i\alpha a_y U_y} - i\alpha a U_{yy} = -i\alpha p_y + Re^{-1}(a^{IV} - \alpha^2 a_{yy})$$

$$\frac{\phantom{x}}{i\alpha}$$

$$(U-c)a_{yy} - aU_{yy} = -p_y + \frac{1}{i\alpha Re}(a^{IV} - \alpha^2 a_{yy})$$

$$i\alpha(U-c)(-i\alpha a) = -p_y + Re^{-1}(-i\alpha a_{yy} - \alpha^2(-i\alpha a))$$

$$\alpha^2(U-c)a = -p_y - \frac{i\alpha}{Re}(a_{yy} - \alpha^2 a)$$

$$\alpha^2(U-c)a = (U-c)a_{yy} - aU_{yy} - \frac{1}{i\alpha Re}(a^{IV} - \alpha^2 a_{yy}) - \frac{i\alpha}{Re}(a_{yy} - \alpha^2 a)$$

$$(U-c)[a_{yy} - \alpha^2 a] - aU_{yy} = \frac{1}{i\alpha Re}(a^{IV} - \alpha^2 a_{yy}) + \frac{i\alpha}{Re}(a_{yy} - \alpha^2 a)$$

$$\left(\frac{d^2}{dy^2} - \alpha^2\right)^2 = \frac{d^4}{dy^4} - 2\alpha^2\frac{d^2}{dy^2} + \alpha^4$$

$$= \frac{1}{i\alpha Re}(a^{IV} - \alpha^2 a_{yy}) - \frac{1}{i\alpha Re}(\alpha^2 a_{yy} - \alpha^4 a)$$

$$= \frac{1}{i\alpha Re}\left(\frac{d^2}{dy^2} - \alpha^2\right)^2 a$$

OS with no-slip $\hat{v} = \hat{v}_y = 0$ at $\pm h$ for specified $\bar{U}(y)$, Re, & $\alpha$ is eigenvalue problem for for eigenfunction $\hat{v}(y)$ & eigenvalue $c = c_R + ic_I$. If BL flow it is unbounded also no slip at say $y = 0$ & $\hat{v}$ & $\hat{v}_y \to 0$ at $|y| = \infty$

For example, for given flow:

$$c_R = c_R(\alpha, Re)$$
$$c_I = c_I(\alpha, Re)$$

unstable $c_I > 0$ & neutral stab for $c_I = 0$, ie $c_I = c_I(\alpha, Re) = 0$ provides curve of neutral stability. If $c_I$ changes sign across $c_I = 0$ then also curve marginal stability, which separates stable & unstable regions in $\alpha$, Re space. Similarly for spatial stability with $\alpha = \alpha_R + i\alpha_I$ & $\alpha_I = 0$ etc.