Centrifugal Instability: Taylor Vortices

Experiments indicate initially instability appears as axisymmetric disturbances such that $\frac{\partial}{\partial \varphi} = 0$.

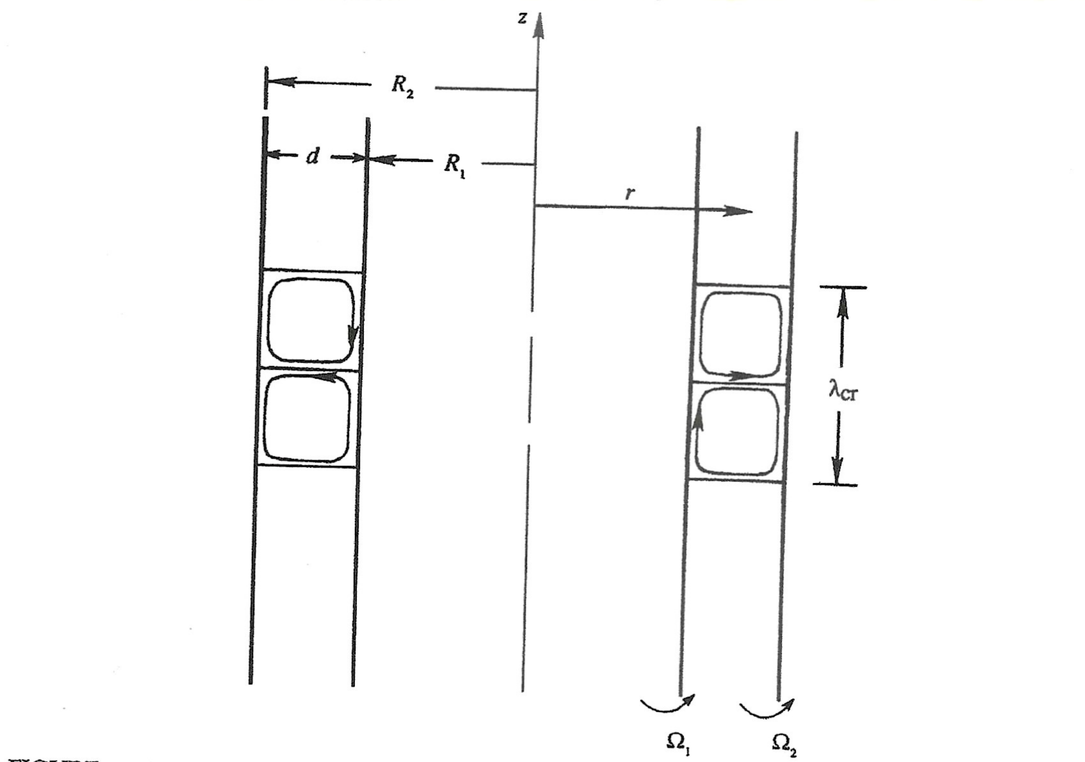



**FIGURE 11.16** Geometry of the flow and the instability in rotating Couette flow. The fluid resides between rotating cylinders with radii $R_1$ and $R_2$. As for Bénard convection, the resulting instability forms as counter-rotating rolls with a wavelength that is approximately twice the gap between the cylinders.

Assuming $\frac{\partial}{\partial \varphi} = 0$: ~ implied

Equation of continuity: $\cancel{\frac{\partial \rho}{\partial t}} + \frac{1}{R}\frac{\partial}{\partial R}(R\rho u_R) + \cancel{\frac{1}{R}\frac{\partial}{\partial \varphi}(\rho u_\varphi)} + \frac{\partial}{\partial z}(\rho u_z) = 0$

Navier-Stokes equations with constant $\rho$, constant $\nu$, and no body force:

$$\frac{\partial u_R}{\partial t} + (\mathbf{u}\cdot\nabla)u_R - \frac{u_\varphi^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial R} + \nu\left(\nabla^2 u_R - \frac{u_R}{R^2} - \cancel{\frac{2}{R^2}\frac{\partial u_\varphi}{\partial \varphi}}\right),$$

$$\frac{\partial u_\varphi}{\partial t} + (\mathbf{u}\cdot\nabla)u_\varphi + \frac{u_R u_\varphi}{R} = \cancel{-\frac{1}{\rho R}\frac{\partial p}{\partial \varphi}} + \nu\left(\nabla^2 u_\varphi + \cancel{\frac{2}{R^2}\frac{\partial u_R}{\partial \varphi}} - \frac{u_\varphi}{R^2}\right), \qquad (1)$$

$$\frac{\partial u_z}{\partial t} + (\mathbf{u}\cdot\nabla)u_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 u_z$$

$R^{-1}\frac{\partial}{\partial R} + \frac{\partial^2}{\partial R^2}$

where: $\mathbf{u}\cdot\nabla = u_R\frac{\partial}{\partial R} + \cancel{\frac{u_\varphi}{R}\frac{\partial}{\partial \varphi}} + u_z\frac{\partial}{\partial z}$ and $\nabla^2 = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\right) + \cancel{\frac{1}{R^2}\frac{\partial^2}{\partial \varphi^2}} + \frac{\partial^2}{\partial z^2}$.

Decompose motion into background state plus perturbation:

$$\tilde{\underline{u}} = \underline{U} + \underline{u} \qquad \tilde{p} = P + p \qquad (2)$$

where $\underline{U} = (0, U_\varphi, 0)$ $\quad U_\varphi = AR + B/R$

$$\rho^{-1} P_R = U_\varphi^2 / R$$

$$A = (\Omega_2 R_2^2 - \Omega_1 R_1^2)/(R_2^2 - R_1^2)$$
$$B = (\Omega_2 - \Omega_1) R_1^2 R_2^2 / (R_2^2 - R_1^2)$$

Substituting (2) into (1), neglecting non-linear terms, and subtract background state:

$$\frac{\partial u_R}{\partial t} - \frac{2U_\varphi u_\varphi}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial R} + \nu\left(\nabla^2 u_R - \frac{u_R}{R^2}\right), \quad \frac{\partial u_\varphi}{\partial t} + \left(\frac{dU_\varphi}{dR} + \frac{U_\varphi}{R}\right)u_R = \nu\left(\nabla^2 u_\varphi - \frac{u_\varphi}{R^2}\right),$$
$$\frac{\partial u_z}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 u_z, \quad \text{and} \quad \frac{1}{R}\frac{\partial}{\partial R}(Ru_R) + \frac{\partial u_z}{\partial z} = 0. \qquad (11.50)$$

As the coefficients in these equations depend only on $R$, the equations admit solutions that depend on $z$ and $t$ exponentially. We therefore consider normal mode solutions of the form:

$$(u_R, u_\varphi, u_z, p) = (\hat{u}_R(R), \hat{u}_\varphi(R), \hat{u}_z(R), \hat{p}(R))\exp\{ikz + \sigma t\}.$$

The requirement that the solutions remain bounded as $z \to \pm\infty$ implies that the axial wave number $k$ must be real. After substituting the normal modes into (11.50) and eliminating $\hat{u}_z$ and $\hat{p}$, we get a coupled system of equations in $\hat{u}_R$ and $\hat{u}_\phi$. Under the *narrow-gap approximation*, for which $d = R_2 - R_1$ is much smaller than $(R_1 + R_2)/2$, these equations finally become (see Chandrasekhar, 1961 for details):

$$(d^2/dR^2 - k^2 - \sigma)(d^2/dR^2 - k^2)\hat{u}_R = (1 + \alpha x)\hat{u}_\varphi, \quad \text{and} \quad (d^2/dR^2 - k^2 - \sigma)\hat{u}_\varphi = -\mathrm{Ta}k^2\hat{u}_R, \qquad (11.51)$$

where:

$$\alpha \equiv (\Omega_2/\Omega_1) - 1, \quad x \equiv (R - R_1)/d, \quad d \equiv R_2 - R_1,$$

and Ta is the Taylor number:

$$Ta \equiv 4\left(\frac{\Omega_1 R_1^2 - \Omega_2 R_2^2}{R_2^2 - R_1^2}\right)\frac{\Omega_1 d^4}{\nu^2}. \qquad (11.52)$$

It is the ratio of the centrifugal force to viscous force, and equals $2(\Omega_1 R_1 d/\nu)^2(d/R_1)$ when only the inner cylinder is rotating and the gap is narrow.

**Exercise 11.8.** Consider the centrifugal instability problem of Section 11.6. Making the narrow-gap approximation, work out the algebra of going from (11.50) to (11.51).

**Solution 11.8.** The perturbation equations (11.50) are:

$$\frac{\partial u_R}{\partial t}-\frac{2U_\varphi u_\varphi}{R}=-\frac{1}{\rho}\frac{\partial p}{\partial R}+\nu\left(\nabla^2 u_R-\frac{u_R}{R^2}\right),\quad \frac{\partial u_\varphi}{\partial t}+\left(\frac{dU_\varphi}{dR}+\frac{U_\varphi}{R}\right)u_R=\nu\left(\nabla^2 u_\varphi-\frac{u_\varphi}{R^2}\right),$$

$$\frac{\partial u_z}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 u_z,\ \text{and}\ \frac{\partial}{\partial R}(Ru_R)+\frac{\partial u_z}{\partial z}=0. \tag{1}$$

Substituting in

$u_R=u(R)e^{\sigma t}\cos kz$, $u_\varphi=v(R)e^{\sigma t}\cos kz$, $u_z=w(R)e^{\sigma t}\sin kz$, and $p/\rho=\hat{p}(R)e^{\sigma t}\cos kz$,

the equation set (1) becomes

$$\nu\left(\frac{d}{dR}\left(\frac{d}{dR}+\frac{1}{R}\right)-k^2-\frac{\sigma}{\nu}\right)u+2\frac{U_\varphi}{R}v=\frac{d\hat{p}}{dR},$$

$$\nu\left(\frac{d}{dR}\left(\frac{d}{dR}+\frac{1}{R}\right)-k^2-\frac{\sigma}{\nu}\right)v-\left(\frac{dU_\varphi}{dR}+\frac{U_\varphi}{R}\right)u=0,$$

$$\nu\left(\left(\frac{d}{dR}+\frac{1}{R}\right)\frac{d}{dR}-k^2-\frac{\sigma}{\nu}\right)w=-k\hat{p},\ \text{and}$$

$$\left(\frac{d}{dR}+\frac{1}{R}\right)u=-kw. \tag{2}$$

Eliminating $w$ between the third and fourth equations of set (2) produces:

$$\frac{\nu}{k^2}\left(\left(\frac{d}{dR}+\frac{1}{R}\right)\frac{d}{dR}-k^2-\frac{\sigma}{\nu}\right)\left(\frac{d}{dR}+\frac{1}{R}\right)u=\hat{p}.$$

Inserting this equation for $\hat{p}$ into the first equation of set (2), and working on the algebra eventually leads to:

$$\frac{\nu}{k^2}\left(\frac{d}{dR}\left(\frac{d}{dR}+\frac{1}{R}\right)-k^2-\frac{\sigma}{\nu}\right)\left(\frac{d}{dR}\left(\frac{d}{dR}+\frac{1}{R}\right)-k^2\right)u=2\frac{U_\varphi}{R}v. \tag{3}$$

The second equation of set (2) and equation (3) are a pair of equations relating $u$ and $v$.

Using the radius of the outer cylinder $R_2$, define new dimensionless variables and parameters:

$$r=R/R_2,\quad k^2=K^2/R_2^2,\quad \text{and}\quad \omega=\sigma R_2^2/\nu,$$

so that the relevant equation pair becomes:

$$\left(\frac{d}{dr}\left(\frac{d}{dr}+\frac{1}{r}\right)-K^2-\omega\right)\left(\frac{d}{dr}\left(\frac{d}{dr}+\frac{1}{r}\right)-K^2\right)u=2\frac{BK^2}{\nu}\left(\frac{1}{r^2}+\frac{AR_2^2}{B}\right)v,\ \text{and}$$

$$\left(\frac{d}{dr}\left(\frac{d}{dr}+\frac{1}{r}\right)-K^2-\omega\right)v=2\frac{A}{\nu}R_2^2 u,$$

where $U_\varphi = Ar + B/r$. It is convenient to make the transformation

$$2\frac{AR_2^2}{\nu}u\to u,$$

so that the equations take the more convenient forms:

$$\left(\frac{d}{dr}\left(\frac{d}{dr}+\frac{1}{r}\right)-K^2-\omega\right)\left(\frac{d}{dr}\left(\frac{d}{dr}+\frac{1}{r}\right)-K^2\right)u=-TaK^2\left(\frac{1}{r^2}-\kappa\right)v,\ \text{and}$$



$$\left(\frac{d}{dr}\left(\frac{d}{dr}+\frac{1}{r}\right)-K^2-\omega\right)v=u,$$

where $Ta=-4\frac{AB}{\nu^2}R_2^2=4\frac{\Omega_1^2R_1^4}{\nu^2}\frac{(1-\mu)(1-\mu/\eta^2)}{(1-\eta^2)^2}$, $\kappa=-\frac{A}{B}R_2^2=\frac{1-\mu/\eta^2}{1-\mu}$, $\mu=\frac{\Omega_2}{\Omega_1}$, $\eta=\frac{R_1}{R_2}$,

$A=-\Omega_1\eta^2\frac{1-\mu/\eta^2}{1-\eta^2}$, and $B=\Omega_1R_1^2\frac{1-\mu}{1-\eta^2}$.

The no-slip and boundary conditions at the walls require:

$$u=v=0\ \text{and}\ (d/dr)u=0\ \text{at}\ r=\eta\ \text{and}\ 1,$$

where the last of the three conditions is equivalent to $w = 0$ (see the final equation of set (1)).

Now consider the narrow gap approximation that is valid when

$$R_2-R_1<<\tfrac{1}{2}(R_2+R_1).$$

When this is true, $d/dr >> 1/r$ so

$$\frac{d}{dr}+\frac{1}{r}\cong\frac{d}{dr},\ \text{and}\ A+\frac{B}{r^2}\cong\Omega_1\left[1-(1-\mu)\frac{r-R_1}{R_2-R_1}\right].$$

Now convert the independent radial coordinate ($R$) to one ($x$) that starts on the inner cylinder using the gap dimension, $d=R_2-R_1$, as the length scale, and let $k = K/d$, and $w = \sigma d^2/\nu$ to find:

$$\left(\frac{d^2}{dx^2}-K^2-\omega\right)\left(\frac{d^2}{dx^2}-K^2\right)u=\frac{2\Omega_1d^4}{\nu}K^2(1-(1-\mu)x)v,\ \text{and}$$

$$\left(\frac{d^2}{dx^2}-K^2-\omega\right)v=\frac{2Ad^4}{\nu}u,$$

as the relevant equation set. By the further transformation $u\to\frac{2\Omega_1d^2K^4}{\nu}u$, these equations become:

$$\left(\frac{d^2}{dx^2}-K^2-\omega\right)\left(\frac{d^2}{dx^2}-K^2\right)u=(1+\alpha x)v,\ \text{and}$$

$$\left(\frac{d^2}{dx^2}-K^2-\omega\right)v=-TaK^2u,$$

where

$$Ta=-\frac{4A\Omega_1}{\nu^2}d^4,\ \text{and}\ \alpha=-(1-\mu).$$

These are the same equations as (11.51).

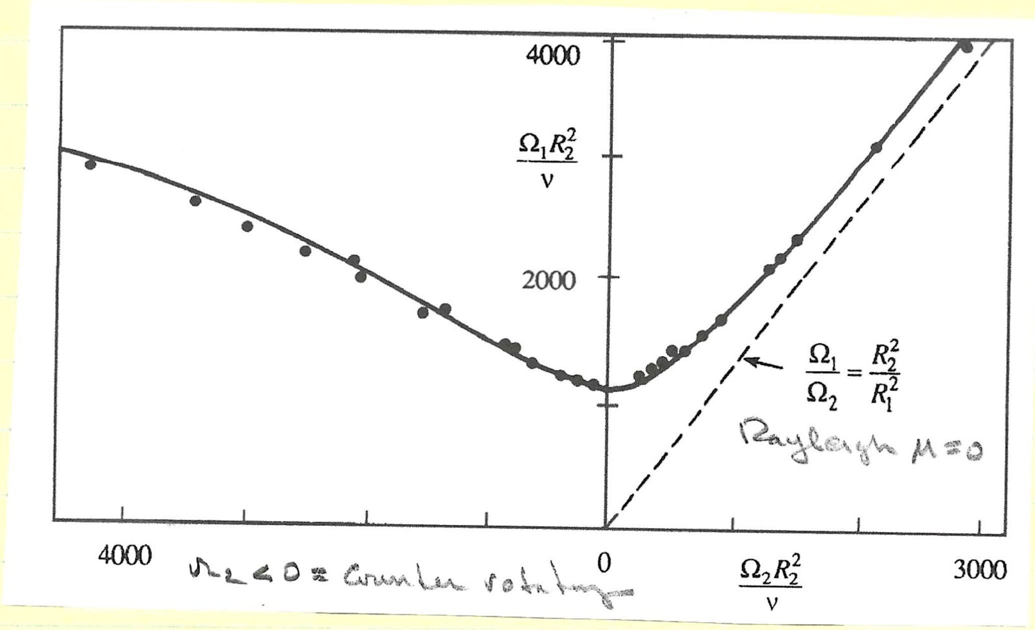
The BC are: $\hat{u}_R = d\hat{u}_R/dR = \hat{u}_\varphi = 0$ at $x = 0$ and $x = 1$. (11.53)

$\sigma = 0$ gives eigenvalues marginal state under assumption principle exchange of stabilities. ie via solution (11.51) subject (11.53)

Solution small gap eigenvalue problem shows excellent agreement EFD

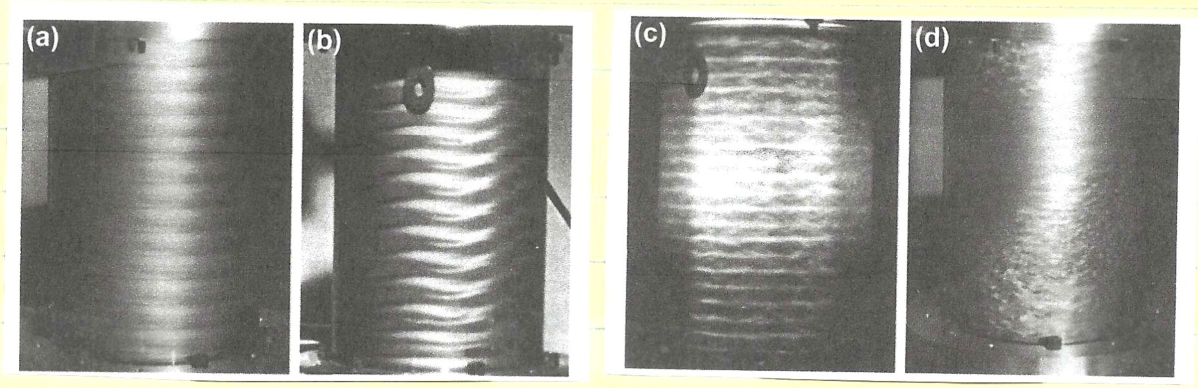
$\mu$ stabilizes the flow

$$Ta_{critical} = \frac{1708}{(1/2)(1+\Omega_2/\Omega_1)}$$




$k_{cr} = 3.12 \Rightarrow \lambda_{cr} = 2\pi d / k_{cr} = 2d$ ie height on cell $= d$
& vortices are square

At the time 1923 Taylor solution & EFD was considered confirmation NS & $\sigma_{ij} = -p\delta_{ij} + 2\mu \varepsilon_{ij}$




Taylor vortices for increasing Ta

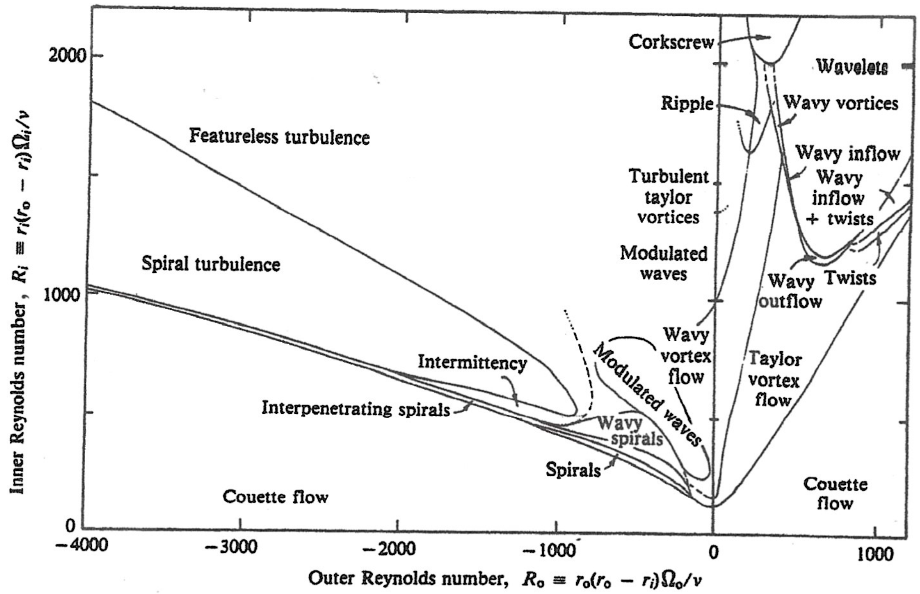



**Figure 25.15** Stability chart for Taylor vortex behavior. Reprinted with permission from Andereck et al. (1986).

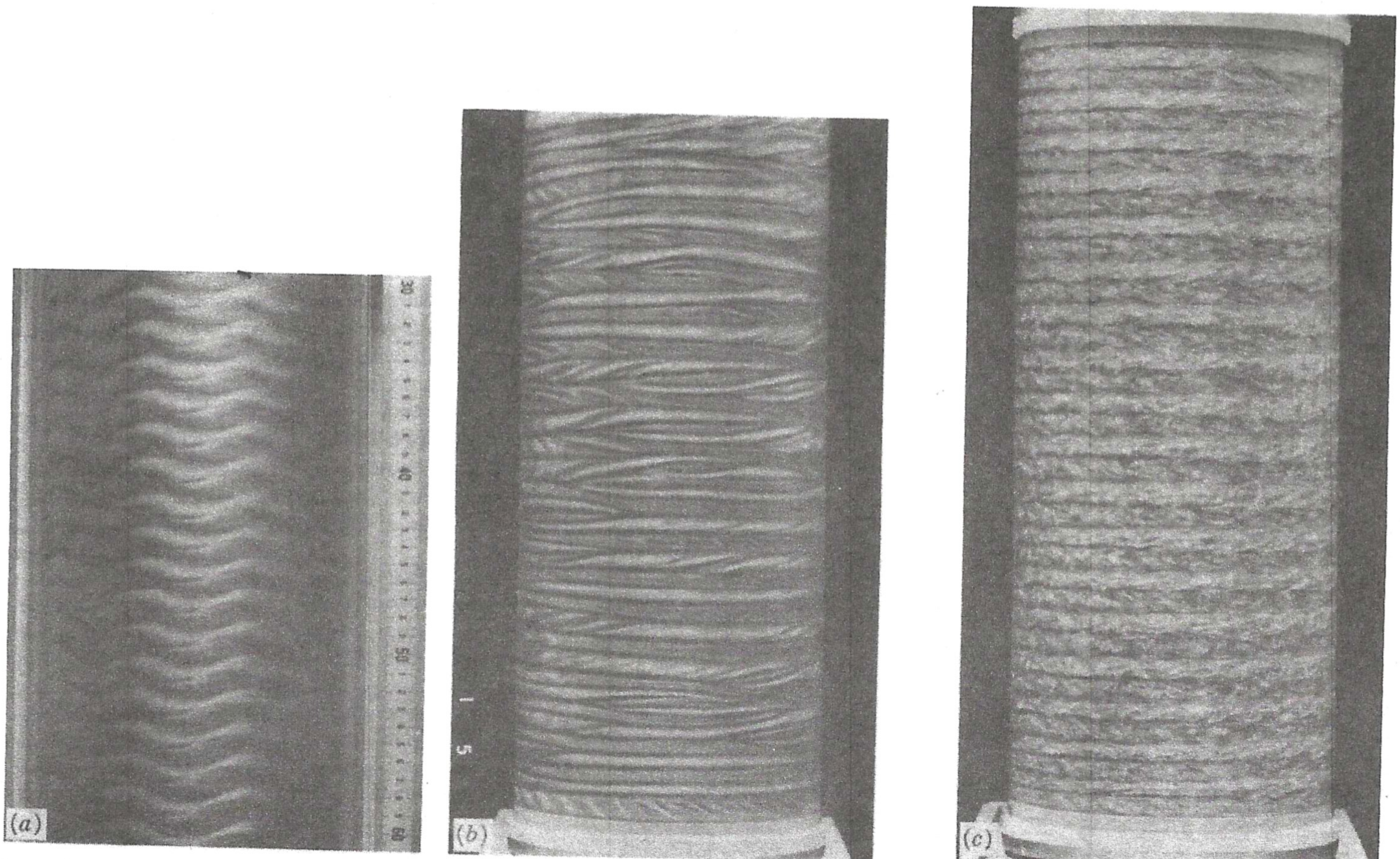



**Figure 25.16** (*a*) Wavy Taylor vortices. Reprinted with permission from Koschmieder (1979). (*b*) Braided Taylor vortices. From Andereck et al. (1983). (*c*) Turbulent Taylor vortices. Courtesy of Zhang and Swinney (1985), University of Texas. Reprinted with permission.

**Exercise 11.9**. Consider the centrifugal instability problem of Section 11.6. From (11.51) and (11.53), the eigenvalue problem for determining the marginal state ($\sigma = 0$) is

$$\left(d^2/dR^2 - k^2\right)^2 \hat{u}_R = (1+\alpha x)\hat{u}_\varphi\,,\ \left(d^2/dR^2 - k^2\right)^2 \hat{u}_\varphi = -\mathrm{Ta}k^2\hat{u}_R\,, \qquad (11.92,93)$$

with $\hat{u}_R = d\hat{u}_R/dR = \hat{u}_\varphi = 0$ at $x = 0$ and 1. Conditions on $\hat{u}_\varphi$ are satisfied by assuming solutions of the form

$$\hat{u}_\varphi = \sum_{m=1}^{\infty} C_m \sin(m\pi x)\ . \qquad (11.94)$$

Inserting this into (11.92), obtain an equation for $\hat{u}_R$, and arrange so that the solution satisfies the four remaining conditions on $\hat{u}_R$. With $\hat{u}_R$ determined in this manner and $\hat{u}_\varphi$ given by (11.94), (11.93) leads to an eigenvalue problem for Ta($k$). Following Chandrasekhar (1961, p. 300), show that the minimum Taylor number is given by (11.54) and is reached at $k_{cr} = 3.12$.

**Solution 11.9**. Continuing the effort from Exercise 11.8, insert (11.94) into (11.92) to find:

$$\left(d^2/dx^2 - K^2\right)^2 u = (1+\alpha x)v = (1+\alpha x)\sum_{m=1}^{\infty} C_m \sin(m\pi x)\,, \qquad (5)$$

where a switch has been made to the dimensionless variables of Exercise 11.8. Now arrange that the solution satisfies the four remaining boundary conditions on $u$. With u determined in this fashion and $v$ given by (11.94), (11.93) will lead to an equation for Ta.

The solution of (5) is straightforward. The general solution can be written in the form:

$$u = \sum_{m=1}^{\infty}\frac{C_m}{(m^2\pi^2+K^2)^2}\left\{\begin{array}{l} A_1^{(m)}\cosh Kx + B_1^{(m)}\sinh Kx + A_2^{(m)}x\cosh Kx + B_2^{(m)}x\sinh Kx \\ \qquad + (1+\alpha x)\sin(m\pi x) + \dfrac{4\alpha m\pi}{m^2\pi^2+K^2}\cos(m\pi x)\end{array}\right\}, \qquad (6)$$

where the constants are determined by the boundary conditions $u = (d/dx)u = 0$ at $x = 0$ and 1. These conditions lead to the four equations:

$$A_1^{(m)} = -\frac{4\alpha m\pi}{m^2\pi^2+K^2},\quad KB_1^{(m)} + A_2^{(m)} = -m\pi\,,$$

$$A_1^{(m)}\cosh K + B_1^{(m)}\sinh K + A_2^{(m)}\cosh K + B_2^{(m)}\sinh K = (-1)^{m+1}\frac{4\alpha m\pi}{m^2\pi^2+K^2}\,, \qquad (7)$$

$$A_1^{(m)}K\sinh K + B_1^{(m)}K\cosh K + A_2^{(m)}(\cosh K + K\sinh K) + B_2^{(m)}(\sinh K + K\cosh K) = (-1)^{m+1}(1+\alpha)m\pi$$

The solution of these equations is:

$$A_1^{(m)} = -\frac{4\alpha m\pi}{m^2\pi^2+K^2}$$

$$B_1^{(m)} = \frac{m\pi}{\Delta}\left\{K + \beta_m(\sinh K + K\cosh K) - \gamma_m \sinh K\right\}$$

$$A_2^{(m)} = -\frac{m\pi}{\Delta}\left\{\sinh^2 K + \beta_m K(\sinh K + K\cosh K) - \gamma_m K\sinh K\right\}$$

$$B_2^{(m)} = \frac{m\pi}{\Delta}\left\{(\sinh K\cosh K - K) + \beta_m K^2\sinh K - \gamma_m(K\cosh K - \sinh K)\right\} \qquad (8)$$

where:

$$\Delta = \sinh^2 K - K^2,\quad \beta_m = \frac{4\alpha}{m^2\pi^2+K^2}\left\{(-1)^{m+1} + \cosh K\right\},\quad \text{and}$$

$$\gamma_m = (-1)^{m+1}(1-\alpha) + \frac{4\alpha}{m^2\pi^2+K^2}K\sinh K\,.$$



Substituting $u$ from (6) and $v$ from (11.94) into (11.93) produces:

$$\sum_{n=1}^{\infty} C_n\left(n^2\pi^2 + K^2\right)\sin n\pi x$$

$$= TaK^2\sum_{m=1}^{\infty}\frac{C_m}{(m^2\pi^2+K^2)^2}\left\{\begin{array}{l} A_1^{(m)}\cosh Kx + B_1^{(m)}\sinh Kx + A_2^{(m)}x\cosh Kx + B_2^{(m)}x\sinh Kx \\ \qquad + (1+\alpha x)\sin(m\pi x) + \dfrac{4\alpha m\pi}{m^2\pi^2+K^2}\cos(m\pi x)\end{array}\right\}. \qquad (9)$$

Multiply this equation by $\sin(n\pi x)$ and integrate from $x = 0$ to $x = 1$, to obtain a system of linear homogeneous equations for the constants $C_m$. The requirement that these constants are not all zero leads to the equation:

$$\frac{n\pi}{n^2\pi^2+K^2}\left\{\begin{array}{l}\left[1+(-1)^{n+1}\cosh K\right]A_1^{(m)} + \left[(-1)^{n+1}\sinh K\right]B_1^{(m)} + (-1)^{n+1}\left[\cosh K - \dfrac{2K}{n^2\pi^2+K^2}\sinh K\right]A_2^{(m)} \\ \qquad + \left[(-1)^{n+1}\sinh K - \dfrac{2K}{n^2\pi^2+K^2}\left\{1+(-1)^{n+1}\cosh K\right\}\right]B_2^{(m)}\end{array}\right\}$$

$$+\alpha x_{nm} + \frac{1}{2}\delta_{nm} - \frac{1}{2}\left(n^2\pi^2 + K^2\right)^3\frac{\delta_{nm}}{K^2Ta} = 0 \qquad (10)$$

where

$$x_{nm} = \left\{\begin{array}{ll} 0 & \text{if } m+n \text{ is even and } m \ne n \\ 1/4 & \text{if } m = n \\ \dfrac{4nm}{n^2-m^2}\left\{\dfrac{2}{m^2\pi^2+K^2} - \dfrac{1}{\pi^2(n^2-m^2)}\right\} & \text{if } m+n \text{ is odd}\end{array}\right\}.$$

On using the first two equations of (7), equation (10) simplifies to:

$$\frac{n\pi}{n^2\pi^2+K^2}\left\{\frac{4m\pi\alpha}{m^2\pi^2+K^2}\left[(-1)^{m+n} - 1\right] - \frac{2K}{n^2\pi^2+K^2}\left[(-1)^{n+1}A_2^{(m)}\sinh K + (-1)^{n+1}B_2^{(m)}\cosh K + B_2^{(m)}\right]\right\}$$

$$+\alpha x_{nm} + \frac{1}{2}\delta_{nm} - \frac{1}{2}\left(n^2\pi^2+K^2\right)^3\frac{\delta_{nm}}{K^2Ta} = 0 \qquad (11)$$

After substituting for the constants $A_2^{(m)}$ and $B_2^{(m)}$ given by (8), (11) becomes:

$$\frac{4mn\pi^2\alpha}{\left(n^2\pi^2+K^2\right)\left(m^2\pi^2+K^2\right)}\left[(-1)^{m+n} - 1\right]$$

$$-\frac{2Kmn\pi^2}{\left(n^2\pi^2+K^2\right)\left(\sinh^2 K - K^2\right)}\left\{\begin{array}{l}(\sinh K\cosh K - K)\left[1+(1+\alpha)(-1)^{m+n}\right] \\ +(\sinh K - K\cosh K)\left[(-1)^{n+1} + (1+\alpha)(-1)^{m+1}\right] \\ -\dfrac{4k\alpha\sinh K}{m^2\pi^2+K^2}\left[\sinh K + K(-1)^{m+1}\right]\left[(-1)^{m+n} - 1\right]\end{array}\right\}$$

$$+\alpha x_{nm} + \frac{1}{2}\delta_{nm} - \frac{1}{2}\left(n^2\pi^2+K^2\right)^3\frac{\delta_{nm}}{K^2Ta} = 0 \qquad (12)$$

A first approximation to the solution of (12) is obtained by setting the (1,1)-element of the determinant zero. This implies:

$$\frac{1}{2}\left(\pi^2+K^2\right)^3\frac{\delta_{nm}}{K^2 Ta}=\frac{1}{4}\alpha+\frac{1}{2}-\frac{2K\pi^2(2+\alpha)}{\left(\pi^2+K^2\right)\left(\sinh^2 K-K^2\right)}\left[(\sinh K\cosh K-K)+(\sinh K-K\cosh K)\right]$$

and this simplifies to:

$$Ta=\frac{2}{2+\alpha}\frac{\left(\pi^2+K^2\right)^3}{K^2\left\{1-16K\pi^2\cosh^2(K/2)\Big/\left[\left(\pi^2+K^2\right)^2(\sinh K+K)\right]\right\}}$$

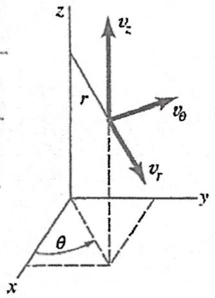
A plot of $Ta(2+\alpha)$ as a function of K from this solution shows that the minimum value of the Taylor number is:

$Ta_{cr}=3430/(2+\alpha)$, and is reached at $K_{cr}=3.12$.

Taylor Solution Viscous Flow

Axisymmetric $(r, \theta, z)$ equations : $\frac{\partial}{\partial\theta} = 0$

$$\mathbf{V} = v_r\hat{\mathbf{e}}_r + v_\theta\hat{\mathbf{e}}_\theta + v_z\hat{\mathbf{e}}_z$$




($r$ direction)

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \cancel{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \cancel{\rho g_r} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) - \frac{v_r}{r^2} + \cancel{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}} - \cancel{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}} + \frac{\partial^2 v_r}{\partial z^2}\right]$$

($\theta$ direction)

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \cancel{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = \cancel{-\frac{1}{r}\frac{\partial p}{\partial \theta}} + \cancel{\rho g_\theta} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} + \cancel{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}} + \cancel{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}} + \frac{\partial^2 v_\theta}{\partial z^2}\right]$$

($z$ direction)

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \cancel{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \cancel{\rho g_z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}} + \frac{\partial^2 v_z}{\partial z^2}\right]$$

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \cancel{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}} + \frac{\partial v_z}{\partial z} = 0$$

Decompose motion into basic solution plus perturbation

$$\hat{\underline{u}} = \underline{v} + \underline{u} \qquad \hat{p} = P + p$$

Basic Solution:

$v_r = v_z = 0$

$v_\theta = V(r) = Ar + B/r$

$\frac{\partial P}{\partial r} = \rho V^2/r$

Basic solution as already given and discussed. Perturbation equations obtained by substituting motion decomposition into axisymmetric $(r, \theta, z)$ equations, neglecting nonlinear terms and subtracting the basic solution

$$u_{r_t} - \frac{2V}{r} u_\theta = -p_r/\rho + \nu\left(\nabla^2 u_r - u_r/r^2\right)$$

$$u_{\theta_t} + \left(V_r + \frac{V}{r}\right) u_r = \nu\left(\nabla^2 u_\theta - u_\theta^2/r\right)$$

$$u_{z_t} = -p_z/\rho + \nu \nabla^2 u_z$$

$$u_{r_r} + u_r/r + u_{z_z} = 0 \qquad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

Coefficients only $f(r)$ solutions depend on exp $z$ and $t$ so assume normal mode solutions

$$(\underline{u}, p) = (\hat{\underline{u}}, \hat{p})\, e^{\sigma t + ikz}$$

$z \to \pm\infty$ $(\hat{\underline{u}}, \hat{p})$ bounded $\Rightarrow$ $k$ = real

Substituting normal mode solutions into perturbation equations and eliminating $\hat{u}_z$ & $\hat{p}$ results in equations for $\hat{u}_r$ & $\hat{u}_\theta$, which are solved assuming narrow gap $d = r_2 - r_1 << (r_1 + r_2)/2$

$$(D^2 - k^2 - \sigma)(D^2 - k^2)\hat{u}_r = (1 + \alpha x)\,\hat{u}_\theta$$
$$(D^2 - k^2 - \sigma)\,\hat{u}_\theta = -Ta\, k^2\, \hat{u}_r$$

$\alpha = \omega_2/\omega_1 - 1$, $x = \frac{r - r_1}{d}$ $\quad D = \frac{d}{dr}$ $\quad Ta = 4\left(\frac{\omega_1 r_1^2 - \omega_2 r_2^2}{r_2^2 - r_1^2}\right)\frac{\omega_1 d^4}{\nu^2}$

BC: $\hat{u}_r = D\hat{u}_r = u_\theta = 0$ $\quad x = 0, 1$ $\qquad = 2\,(V_1 d/\nu)^2 (d/r_1)\Big|_{\omega_2 = 0}$

$V_1 = \omega_1 r_1$

$$\mathrm{Ta}_{cr} = \frac{1708}{(1/2)(1 + \Omega_2/\Omega_1)} \tag{11.54}$$

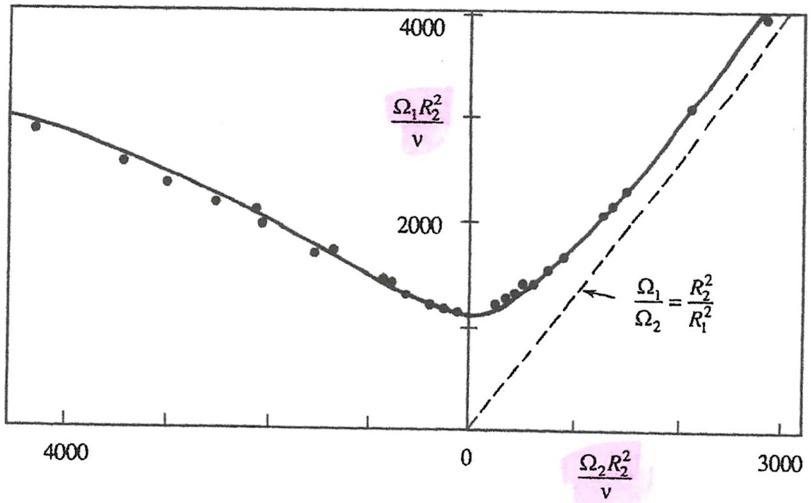



**FIGURE 11.17** Taylor's observation and narrow-gap calculation of marginal stability in rotating Couette flow of water. The ratio of radii is $R_2/R_1 = 1.14$. The region above the curve is unstable. The dashed line represents Rayleigh's inviscid criterion, with the region to the left of the line representing instability. The experimental and theoretical results agree well and suggest that viscosity acts to stabilize the flow.