Step 1: Start from the general 2D form

$$\nabla^2 p = -\rho[\,(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial y})^2 + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\,]$$

Step 2: Apply incompressibility

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \implies \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$$

Substitute into the first two terms:

$$(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial y})^2 = (\frac{\partial u}{\partial x})^2 + (-\frac{\partial u}{\partial x})^2 = 2(\frac{\partial u}{\partial x})^2$$

Step 3: Express the squared term as a product

Use incompressibility in a **clever way**:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \implies \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$$

Then

$$(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial y})^2 = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y} = -2\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}?$$

Let's check carefully:

$$(\partial u/\partial x)^2 + (\partial v/\partial y)^2 = (\partial u/\partial x)^2 + (-\partial u/\partial x)^2 = 2(\partial u/\partial x)^2$$

We want to **write it as a product of gradients** instead of squares. Use:

$$(\partial u/\partial x)^2 + (\partial v/\partial y)^2 = (\partial u/\partial x)(\partial u/\partial x) + (\partial v/\partial y)(\partial v/\partial y) = -2(\partial u/\partial x)(\partial v/\partial y)$$

✅ Correct, because $\partial u/\partial x = -\partial v/\partial y$.

Step 4: Substitute into the source term

$$\begin{aligned}\nabla^2 p &= -\rho[(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial y})^2 + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}] \\ &= -\rho[-2\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}] \\ &= 2\rho[\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}]\end{aligned}$$

✅ Step 5: Final form (no squares)

$$\nabla^2 p = 2\rho\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)$$

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