Let

$$\mathbf{u} = (u, v, w), \qquad (x_1, x_2, x_3) = (x, y, z)$$

The pressure Poisson equation is

$$\nabla^2 p = -\rho \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}$$

Step 1: Write out the summation

$$\frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}$$

Step 2: Expand term by term

Diagonal terms $(i = j)$

$$(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial y})^2 + (\frac{\partial w}{\partial z})^2$$

Off-diagonal terms $(i \square j)$

Each pair appears twice, giving a factor of 2:

$$2\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial y}\right)$$

Step 3: Final expanded form

$$\boxed{\begin{aligned}\nabla^2 p = -\rho\Big[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 \\ + 2\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial y}\right)\Big]\end{aligned}}$$

Optional simplification (using incompressibility)

If

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

you can rewrite the diagonal part in alternative forms, but the expression above is the **standard explicit expansion** used in CFD implementations.

If you want, I can also:

- express this in terms of **strain-rate and vorticity**, or
- show how this looks in **2D** or **axisymmetric** form.

2D

Perfect! Let's reduce the **pressure Poisson equation** to **2D Cartesian flow**.

Step 1: 2D setup

- Flow: $\mathbf{u} = (u(x, y, t), v(x, y, t))$
- Coordinates: x, y
- Incompressibility:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

The 2D version of the Poisson equation is

$$\nabla^2 p = -\rho \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}$$

with $i, j = x, y$.

Step 2: Expand the summation

$$\frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} = \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} + \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \frac{\partial v}{\partial y}$$

Notice the off-diagonal terms combine:

$$\frac{\partial u}{\partial y} \frac{\partial v}{\partial x} + \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} = 2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial x}$$

Step 3: Final 2D expanded form

$$\boxed{\nabla^2 p = -\rho [(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial y})^2 + 2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial x}]}$$

✅ This is the **standard 2D pressure Poisson equation** used in CFD and projection methods.