Chapter 8.2 Elemental Plane Flow Solutions

Recall that for 2D we can define a stream function such that:

$$u = \psi_y$$

$$v = -\psi_x$$

$$\omega_z = v_x - u_y = \frac{\partial}{\partial x}(-\psi_x) - \frac{\partial}{\partial y}(\psi_y) = -\nabla^2\psi = 0$$

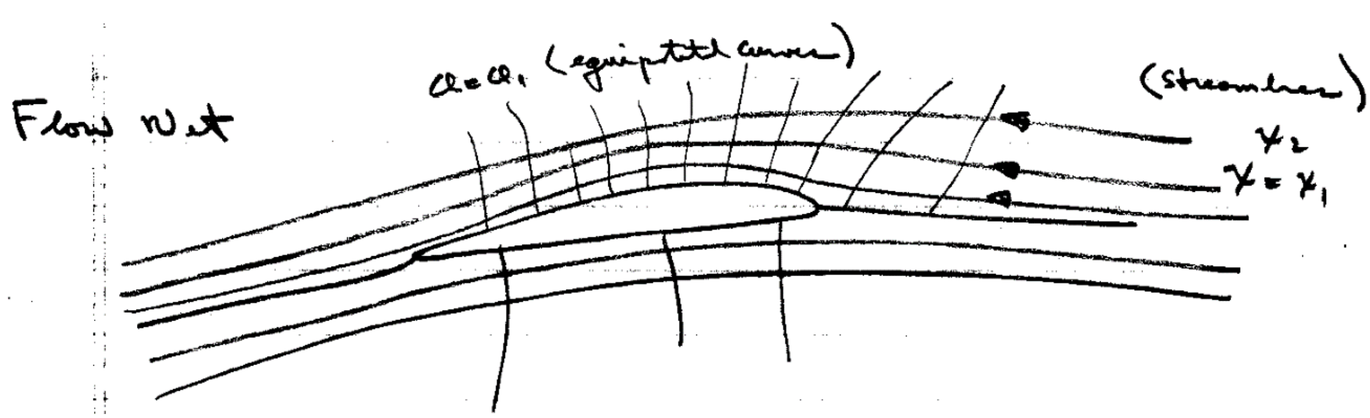
i.e., $\nabla^2\psi = 0$. Also recall that φ and ψ are orthogonal:

$$u = \psi_y = \varphi_x \quad \text{and} \quad v = -\psi_x = \varphi_y$$

$$d\varphi = \varphi_x dx + \varphi_y dy = udx + vdy$$

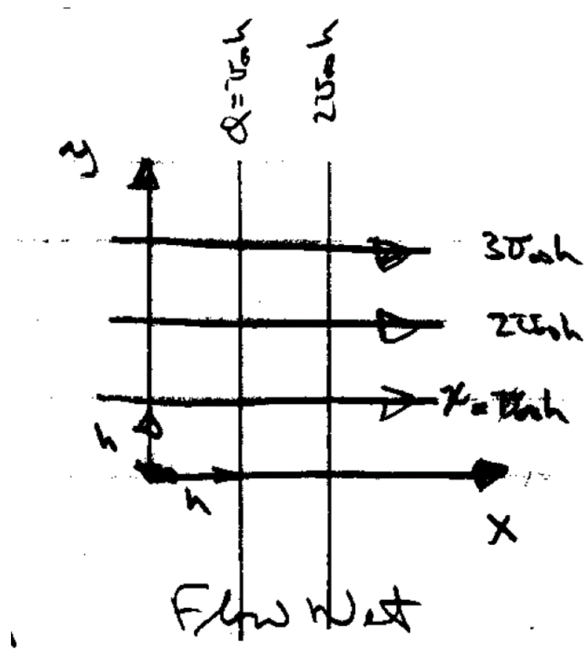
$$d\psi = \psi_x dx + \psi_y dy = -vdx + udy$$

i.e.: $$\left.\frac{dy}{dx}\right|_{\varphi=constant} = -\frac{u}{v} = \frac{-1}{\left.\frac{dy}{dx}\right|_{\psi=constant}}$$ [1]



[1] If the slopes of the tangents to two curves are related by $m_1 m_2 = -1$, then the two curves are perpendicular.

Uniform stream



$$u = U_\infty = \psi_y = \varphi_x$$

$$v = 0 = -\psi_x = \varphi_y$$

Where U_∞ = constant

i.e.: $\varphi = U_\infty x$

$$\psi = U_\infty y$$

Note: $\nabla^2\varphi = \nabla^2\psi = 0$ is satisfied.

$$\underline{V} = \nabla\phi = U_\infty \hat{\imath}$$

Say a uniform stream is at an angle α to the x-axis:

$$u = U_\infty \cos\alpha = \frac{\partial\psi}{\partial y} = \frac{\partial\phi}{\partial x}$$

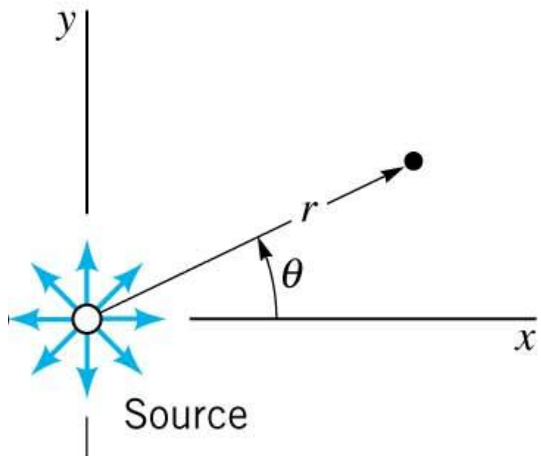
$$v = U_\infty \sin\alpha = -\frac{\partial\psi}{\partial x} = \frac{\partial\phi}{\partial y}$$

After integration, we obtain the following expressions for the stream function and velocity potential:

$$\psi = U_\infty(y\cos\alpha - x\sin\alpha)$$

$$\phi = U_\infty(x\cos\alpha + y\sin\alpha)$$

2D Source or Sink



$$x = r\cos\theta \qquad y = r\sin\theta$$

Imagine that fluid comes out radially at the origin with uniform rate in all directions. (singularity at origin where velocity is infinite).

Consider a circle of radius r enclosing this source. Let v_r be the radial component of velocity associated with this source (or sink). Then, from conservation of mass, for a cylinder of radius r, and width b perpendicular to the paper,

$$Q = \int_A \underline{V} \cdot \underline{dA} = \text{constant}$$

Where $\underline{V} = v_r \widehat{e_r}$ $\quad \underline{n} = \widehat{e_r}$ $\quad \underline{dA} = \underline{n}\text{dA}$ $\quad dA = rd\theta b$

$$Q = v_r(2\pi r)(b)$$

$$v_r = \frac{Q}{2\pi br}$$

$$\Rightarrow v_r = \frac{m}{r} \quad v_\theta = 0$$

Where $m = \frac{Q}{2\pi b}$ is the source strength with units m^2/s velocity × length (m>0 for source and m<0 for sink).

Note that $\underline{V}$ is singular at (0,0) since $v_r \rightarrow \infty$.

In a polar coordinate:

$$\underline{V} = \nabla\phi = \frac{\partial\phi}{\partial r}\widehat{e_r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\widehat{e_\theta}$$

$$\nabla = \frac{\partial}{\partial r}\widehat{e_r} + \frac{1}{r}\frac{\partial}{\partial\theta}\widehat{e_\theta}$$

And:

$$\nabla \cdot \underline{V} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(v_\theta) = 0$$

i.e.

$$v_r = \text{Radial velocity} = \frac{\partial\varphi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

$$v_\theta = \text{Tangential velocity} = \frac{1}{r}\frac{\partial\varphi}{\partial\theta} = -\frac{\partial\psi}{\partial r}$$

Such that $\nabla \cdot \underline{V} = 0$ by definition.

Therefore,

$$v_r = \frac{m}{r} = \frac{\partial \varphi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

$$v_\theta = 0 = \frac{1}{r}\frac{\partial \varphi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

φ and ψ are obtained by integration:

$$\varphi = m \ln r = m \ln \sqrt{x^2 + y^2}$$

$$\psi = m\theta = m \tan^{-1}\frac{y}{x}$$

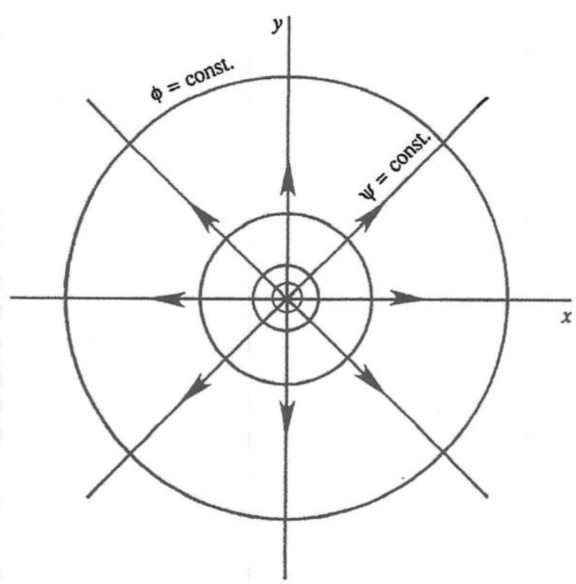
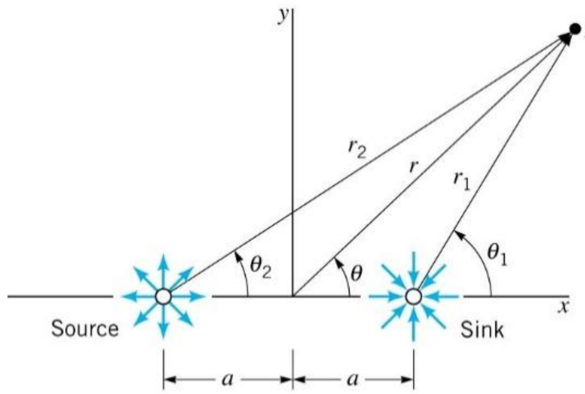


FIGURE 7.5 The flow field of an ideal source located at the origin of coordinates in two dimensions. The streamlines are radials and the potential lines are circles.

Doublets



The doublet is derived from a source and sink equally spaced apart about the origin along the x axis:

$$\Psi = -\frac{m}{2\pi}\left(\underbrace{\theta_1}_{sink} - \underbrace{\theta_2}_{source}\right) \rightarrow \theta_1 - \theta_2 = -\frac{2\pi\Psi}{m}$$

$$tan\left(-\frac{2\pi\Psi}{m}\right) = tan(\theta_1 - \theta_2) = \frac{tan\,\theta_1 - tan\,\theta_2}{1 + tan\,\theta_1\,tan\,\theta_2}$$

$$tan\,\theta_1 = \frac{r\,sin\,\theta}{r\,cos\,\theta - a} \quad tan\,\theta_2 = \frac{r\,sin\,\theta}{r\,cos\,\theta + a}$$

$$tan\left(-\frac{2\pi\Psi}{m}\right) = \frac{2ar\sin\theta}{r^2 - a^2}$$

Therefore

$$\Psi = -\frac{m}{2\pi}\tan^{-1}\left(\frac{2ar\sin\theta}{r^2 - a^2}\right)$$

For small distance

$$\Psi = -\frac{m}{\cancel{2}\pi}\frac{\cancel{2}ar\sin\theta}{r^2 - a^2} = -\frac{mar\sin\theta}{\pi(r^2 - a^2)}$$

The doublet is formed by letting $a \to 0$ while increasing the strength m ($m \to \infty$) so that doublet strength $K = \frac{ma}{\pi}$ remains constant. The flow direction is from the sink towards the source.

$$\Psi = -\frac{K\sin\theta}{r}$$

Corresponding potential

$$\phi = \frac{K\cos\theta}{r}$$

By rearranging

$$\Psi = -\frac{K\,r\sin\theta}{r^2} = -\frac{Ky}{x^2 + y^2} \to x^2 + \left(y + \frac{K}{2\Psi}\right)^2 = \left(\frac{K}{2\Psi}\right)^2 = R^2$$

Plots of lines constant Ψ reveal that streamlines for the doublet are circles through the origin tangent to the x axis as shown in Figure below (equation circle radius R center (h,k) is $(x-h)^2+(y-k)^2=R^2$). Circles show various Ψ = constant above/below x axis

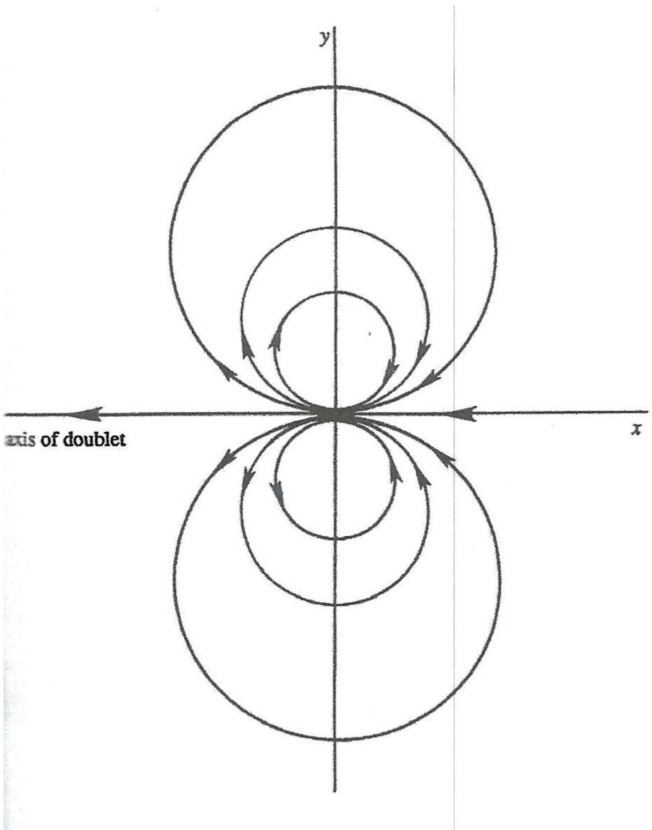


FIGURE 7.6 The flow field of an ideal two-dimensional doublet that points along the negative x-axis. The net source strength is zero so all streamlines begin and end at the origin. In this flow, the streamlines are circles tangent to the x-axis at the origin.

Axis of the dipole is from the sink to the source, i.e., the flow is into the sink and out of the source.

$\tan\theta_1 = \dfrac{r_1 \sin\theta_1}{r_1 \cos\theta_1}$ $\quad \tan\theta = \dfrac{r\sin\theta}{r\cos\theta - a}$

$r\sin\theta = r_1 \sin\theta_1 \quad r_1\cos\theta_1 = s \quad r\cos\theta = a + s$

$r\cos\theta - a = r_1\cos\theta_1$

$\tan\theta_1 = \dfrac{r\cos\theta}{r\cos\theta - a}$

$\tan\theta_2 = \dfrac{r_2\sin\theta_2}{r_2\cos\theta_2} \quad r\sin\theta = r_2\sin\theta_2$

$r_2\cos\theta_2 = 2a + s = r\cos\theta + a = 2a + s$

$= \dfrac{r\sin\theta}{r\cos\theta + a}$

$\tan\theta_1 - \tan\theta_2 = \dfrac{r\sin\theta(r\cos\theta + a) - r\sin\theta(r\cos\theta - a)}{(r\cos\theta - a)(r\cos\theta + a)}$

$= (2ra\sin\theta)/(r^2\cos^2\theta - a^2)$

$1 + \dfrac{r^2\sin^2\theta}{(r^2\cos^2\theta - a^2)} = \dfrac{r^2\cos^2\theta - a^2 + r^2\sin^2\theta}{r^2\cos^2\theta - a^2} = \dfrac{r^2 - a^2}{r^2\cos^2\theta - a^2}$

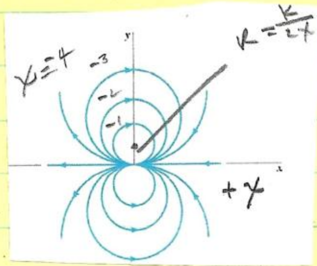
$\tan(\theta_1 - \theta_2) = \dfrac{2ra\sin\theta}{r^2 - a^2} = \tan\left(-\dfrac{2\pi\psi}{m}\right)$

Equation circle center $(h,k) = (x-h)^2 + (y-k)^2 = R^2$

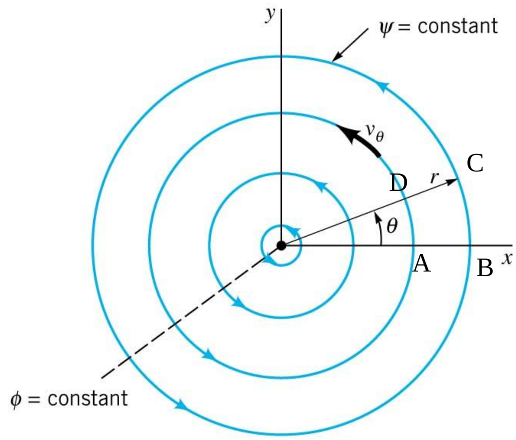
$\psi = -\dfrac{K r\sin\theta}{r} = \dfrac{-Ky}{x^2+y^2} \quad \psi(x^2+y^2) = -Ky$

$x^2 + y^2 = -Ky/\psi$

$x^2 + \underbrace{\left(y + \dfrac{K}{2\psi}\right)^2}_{y^2 + \frac{yK}{\psi} + \left(\frac{K}{2\psi}\right)^2} = \left(\dfrac{K}{2\psi}\right)^2 = R^2$



2D vortex



Suppose that value of the ψ and φ for the source are reversed.

$$v_r = 0$$

$$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r} = \frac{K}{r}$$

Purely circulatory flow with $v_\theta \rightarrow 0$ like 1/r and infinity as $r \rightarrow 0$.

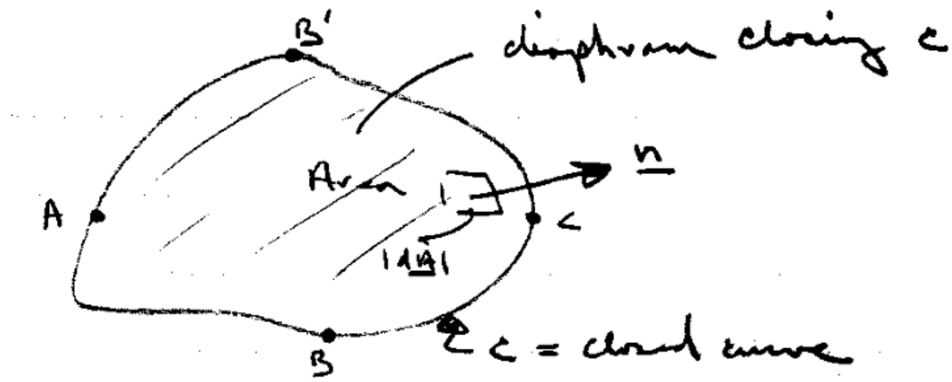
Integration results in:

$$\phi = K\theta$$

$$\psi = -K\, ln\, r \qquad \text{K=constant}$$

2D vortex is irrotational everywhere except at the origin where $\underline{V}$ and $\nabla \times \underline{V}$ are infinity.

Circulation



Circulation is defined by:

$$\Gamma = \oint_c \underline{V} \cdot \underline{ds}$$

or by using Stokes theorem: (if no singularity of the flow in A)

$$\Gamma = \oint_c \underline{V} \cdot \underline{ds} = \int \nabla \times \underline{V} \cdot \underline{dA} = \int \underline{\omega} \cdot \underline{n} dA = 0$$

Therefore, for potential flow $\Gamma = 0$ in general.

However, this is not true for the point vortex due to the singular point at vortex origin where $\underline{V}$ and $\nabla \times \underline{V}$ are infinity.

If singularity exists: Free vortex $v_\theta = \frac{K}{r}$

$$\Gamma = \int_0^{2\pi} \underbrace{v_\theta \hat{e}_\theta}_{\underline{V}} \cdot \underbrace{r d\theta \hat{e}_\theta}_{\underline{ds}} = \int_0^{2\pi} \frac{K}{r}(r d\theta)$$

$$= 2\pi K \text{ and } K = \frac{\Gamma}{2\pi}$$

Note for point vortex, flow is still irrotational everywhere except at origin itself where $\underline{V} \rightarrow \infty$.

For a path not including (0,0) $\Gamma = 0$

$$\Gamma = \int_A^B v_\theta \widehat{e_\theta} \cdot \widehat{e_r} dr + \int_B^C v_\theta \widehat{e_\theta} \cdot r d\theta \widehat{e_\theta}$$

$$+ \int_C^D v_\theta \widehat{e_\theta} \cdot \widehat{e_r} dr + \int_D^A v_\theta \widehat{e_\theta} \cdot r d\theta \widehat{e_\theta} =$$

$$\Delta\theta K - \Delta\theta K = 0 \text{ since } v_\theta = \frac{K}{r}$$

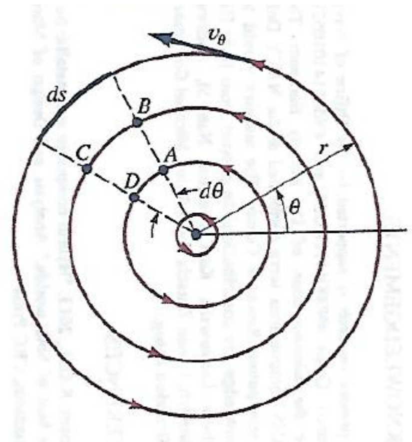


Figure 6.21 Circulation around various paths in a free vortex.

Stokes theorem can be used to show the existence of φ either using the fact that (1) $\oint_c \underline{V} \cdot \underline{ds}$ or (2) $\int \nabla \times \underline{V} \cdot \underline{dA}$ equal zero for conservative $\underline{V}$ or equivalently irrotational flow.

(1) With assumption for closed contour:

$$\int_{ABCB'A} \underline{V} \cdot \underline{ds} = 0$$

It follows that

$$\int_{ABC} \underline{V} \cdot \underline{ds} = \int_{AB'C} \underline{V} \cdot \underline{ds}$$

since $\underline{V}$ is a single value. Therefore, the integration is independent of path; and in general, for irrotational motion, a scalar function $\phi(\underline{s})$ at a point $\underline{s}$ can be defined as:

$$\phi(\underline{s}) = \int_{\underline{s}_0}^{\underline{s}} \underline{V} \cdot \underline{ds}\ , \ \ \underline{s}_0 = \text{a reference point}$$

$$d\phi = \underline{V} \cdot \underline{ds}, \quad d\phi = \underline{V} \cdot \hat{e}_s ds,\ (\underline{ds} = \hat{e}_s ds)$$

$$\text{Note that } \frac{d\phi}{ds} = \nabla\phi \cdot \hat{e}_s, d\phi = \nabla\phi \cdot \hat{e}_s ds$$

$$(\underline{V} - \nabla\phi) \cdot \hat{e}_s = 0$$

Since, $\hat{e}_s$is not zero:

$$\underline{V} = \nabla\phi$$

i.e., velocity vector is gradient of a scalar function φ if the ($\oint \underline{V} \cdot \underline{ds} = 0$).

(2)

To prove that a vector field **F** is the gradient of a scalar function ϕ, we need to show that **F** satisfi the condition for being irrotational, i.e., its curl is zero:

$$\mathbf{F} = \nabla\phi \implies \nabla \times \mathbf{F} = \mathbf{0}.$$

Steps to Prove:

1. **Definition of a Gradient Field**: If $\mathbf{F} = \nabla\phi$, then

$$\mathbf{F} = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right).$$

2. **Compute the Curl**: The curl of $\mathbf{F}$ is given by

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix},$$

where $F_x = \frac{\partial\phi}{\partial x}$, $F_y = \frac{\partial\phi}{\partial y}$, and $F_z = \frac{\partial\phi}{\partial z}$.

Expanding the determinant,

$$\nabla \times \mathbf{F} = \mathbf{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) - \mathbf{j}\left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right) + \mathbf{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right).$$

3. **Symmetry of Mixed Partial Derivatives**: If ϕ is a scalar field with continuous second-order partial derivatives (smoothness), then the mixed partial derivatives commute:

$$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}, \quad \text{and similarly for other pairs of variables.}$$

Substituting $F_x = \frac{\partial\phi}{\partial x}$, $F_y = \frac{\partial\phi}{\partial y}$, $F_z = \frac{\partial\phi}{\partial z}$ into the curl formula:

$$\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = \frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y} = 0,$$

and similarly for the other terms. Hence,

$$\nabla \times \mathbf{F} = \mathbf{0}.$$

Conclusion:

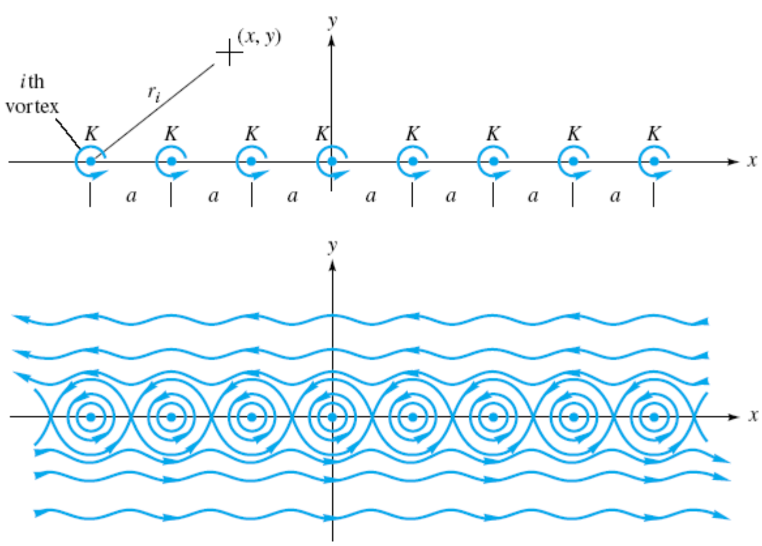
If the curl of a vector field $\mathbf{F}$ is zero ($\nabla \times \mathbf{F} = \mathbf{0}$), and the domain is simply connected (no holes), then $\mathbf{F}$ is the gradient of some scalar function ϕ.

The point vortex singularity is important in aerodynamics, since distribution of sources and sinks can be used to represent airfoils and wings as we shall discuss shortly. To see this, consider as an example an infinite row of vortices:

$$\psi = -K\sum_{i=1}^{\infty} ln\, r_i$$

$$= -\frac{1}{2}K\, ln\left[\frac{1}{2}\left(cosh\frac{2\pi y}{a} - cos\frac{2\pi x}{a}\right)\right]$$

Where r_i is radius from origin of ith vortex.



Superposition infinite row equally spaced vortices of equal strength

For $|y| \geq a$ the flow approaches uniform flow with

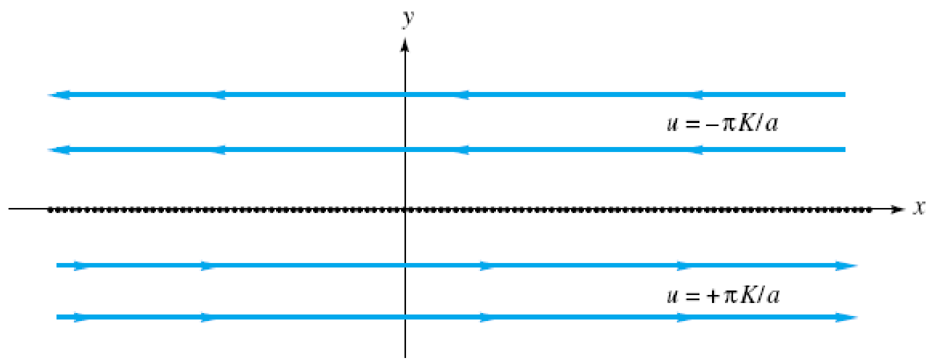
$$u = \frac{\partial \psi}{\partial y} = \pm \frac{\pi K}{a}$$

+ below x axis
- above x axis

Note that this flow is just due to infinite row of vortices and there isn't any pure uniform flow

Vortex sheet

From afar (i.e. $|y| \geq a$) looks like a thin sheet with velocity discontinuity.



Define $\gamma = \frac{2\pi K}{a}$ =strength of vortex sheet

$\underline{V} \cdot \underline{ds}$ (around closed contour)

$$d\Gamma = u_l dx - u_u dx = (u_l - u_u)dx = \frac{2\pi K}{a} dx$$

i.e., $\gamma = \frac{d\Gamma}{dx}$ = circulation per unit span

Note that there is no flow normal to the sheet so that vortex sheet can be used to simulate a body surface. This is the basis of airfoil theory where we let $\gamma = \gamma(x)$ to represent body geometry.

Vortex theorems of Helmholtz: (important role in the study of the flow about wings)

1) The circulation around a given vortex line is constant along its length

2) A vortex line cannot end in the fluid. It must form a closed path, end at a boundary or go to infinity.

3) No fluid particle can have rotation, if it did not originally rotate