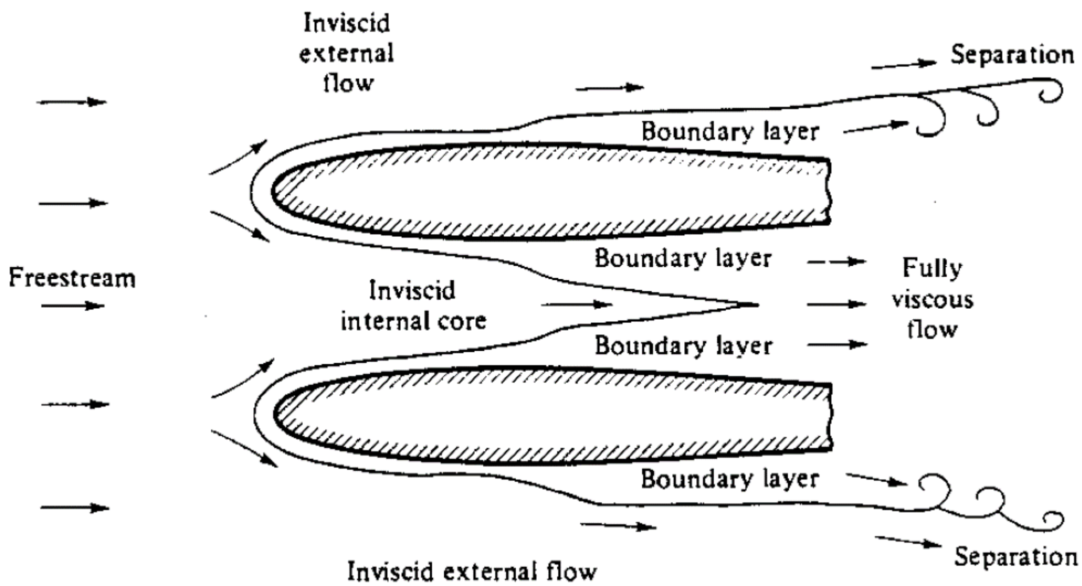
# Chapter 8.1: Introduction, Theory, and Solution Techniques

## Introduction




For high Re external flow about streamlined bodies viscous effects are confined to the boundary layer and wake region. For regions where the BL is thin i.e. favorable or weak adverse pressure gradient regions, viscous/inviscid interaction is weak and traditional BL theory can be used. For regions where BL is thick and/or the flow is separated i.e. strong adverse pressure gradient regions more advanced boundary layer theory must be used including viscous/inviscid interactions, whereas the current state-of-the-art is, of course, CFD methods.

For internal flows at high Re viscous effects are always important except near the entrance. Recall that vorticity is generated in regions with large shear. Therefore, outside the BL and wake and if there is no upstream vorticity, as per Kelvins Circulation Theorem, then $\underline{\omega} = 0$ is a good approximation.

Note that for compressible flow this is not the case in regions of large entropy gradients. Also, we are neglecting non-inertial effects and other mechanisms of vorticity generation.

Navier-Stokes equations for constant property flow (ρ and μ constant:

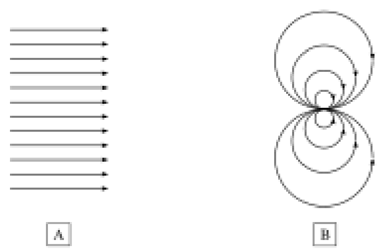
$$\rho\underline{a} = -\nabla(p) - \rho g\hat{k} + \mu\nabla^2\underline{V} = -\nabla(p+\gamma z) + \mu\nabla^2\underline{V}$$

$$\rho\left[\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right]$$

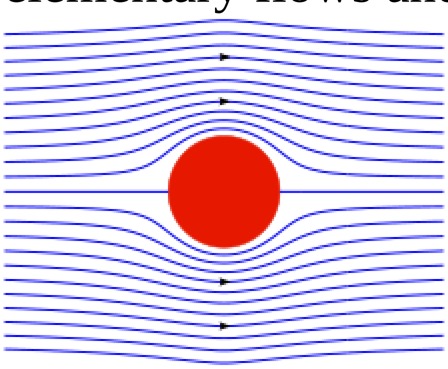
$$= -\nabla(p+\gamma z) + \mu\left[\nabla(\nabla\cdot\underline{V}) - \nabla\times(\nabla\times\underline{V})\right]$$

Viscous term=0 for ρ=constant and $\underline{\omega}=0$, i.e., potential flow solutions are also solutions of the NS under such conditions. But cannot satisfy the no slip condition and suffers from D'Alembert's paradox that drag = 0.

In fluid dynamics, d'Alembert's paradox (or the hydrodynamic paradox) is a contradiction reached in 1752 by French mathematician Jean le Rond d'Alembert. D'Alembert proved that – for incompressible and inviscid potential flow – the drag force is zero on a body moving with constant velocity relative to the fluid. Zero drag is in direct contradiction to the observation of substantial drag on bodies moving relative to fluids, such as air and water, especially at high velocities corresponding with high Reynolds numbers. It is a particular example of the reversibility paradox.




A potential flow is constructed by adding simple elementary flows and observing the result.

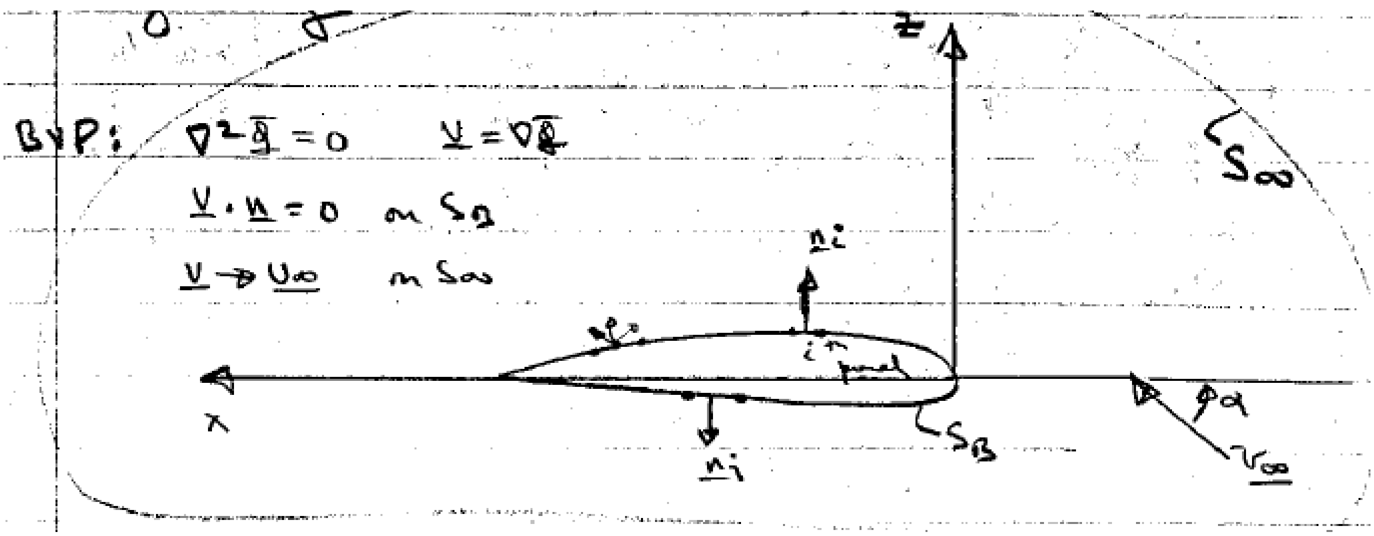


Streamlines for the incompressible potential flow around a circular cylinder in a uniform onflow.

# Potential Flow Theory

Primarily for external flow applications we now consider inviscid ($\mu = 0$), incompressible $\rho$ = constant, and irrotational flow $\underline{\omega}$=0, i.e., ideal flow theory.

1) Determine $\varphi$ from solution to Laplace equation (different from NS BVP, p not needed and determined post facto from Bernoulli equation)




$$\frac{DF}{Dt} = 0 \to \frac{\partial F}{\partial t} + \underline{V}.\nabla F = 0 \to \underline{V}.\underline{n} = -\frac{1}{|\nabla F|}\frac{\partial F}{\partial t} \quad \text{on } S_B$$

F = surface function = z - $S_B$

for steady flow $\underline{V}.\underline{n} = 0$

2) Determine $\underline{V}$ from $\underline{V} = \nabla\phi$ and p(x) from Bernoulli equation

# Euler equation for Incompressible Flow:

$$\nabla \cdot \underline{V} = 0$$

$$\rho \frac{D\underline{V}}{Dt} = -\nabla p + \rho \underline{g}$$

$$\rho \frac{\partial V}{\partial t} + \rho \underline{V} \cdot \nabla \underline{V} = -\nabla(p + \gamma z)$$

$$\underline{V} \cdot \nabla \underline{V} = \nabla \frac{V^2}{2} - \underline{V} \times \underline{\omega}$$

Where $\underline{\omega} = \nabla \times \underline{V}$ = vorticity = 2 x fluid angular velocity

$$\Rightarrow \rho \frac{\partial \underline{V}}{\partial t} + \nabla(p + \frac{1}{2}\rho V^2 + \gamma z) = \rho \underline{V} \times \underline{\omega}$$

For $\underline{\omega} = 0$, i.e., $\nabla \times \underline{V} = 0$, then $\underline{V} = \nabla\phi$ and the unsteady potential flow Bernoulli equation follows:

$$\rho \frac{\partial \phi}{\partial t} + p + \frac{1}{2}\rho \nabla\phi \cdot \nabla\phi + \gamma z = B(t)$$

The continuity equation shows that the GDE for $\varphi$ is the Laplace equation which is a 2nd order linear PDE i.e. superposition principle is valid. (Linear combination of solution is also a solution)

$$\nabla \cdot \underline{V} = \nabla \cdot \nabla \phi = \nabla^2 \phi = 0$$

$$\phi = \phi_1 + \phi_2$$

$$\nabla^2 \phi = 0 \Rightarrow \nabla^2(\phi_1 + \phi_2) = 0 \Rightarrow \nabla^2 \phi_1 + \nabla^2 \phi_2 = 0$$

$$\Rightarrow \begin{cases} \nabla^2 \phi_1 = 0 \\ \nabla^2 \phi_2 = 0 \end{cases}$$

Techniques for solving Laplace equation:

1) Superposition of elementary solution (simple geometries)
2) Surface singularity method (integral equation)
3) FD or FE
4) Electrical or mechanical analogs
5) Conformal mapping (for 2D flow)
6) Analytical for simple geometries (separation of variable etc.)