Chapter 9: Boundary Layer Flows (Chapter 7.3 Pope)

Part 2: Length scales and the mixing length

Three fundamental properties of the log law region:

1) $S = \langle U \rangle_y = u_\tau / ky$ or $\frac{du^+}{dy^+} = \frac{1}{ky^+}$
2) $P/\varepsilon \approx 1$
3) $-\langle uv \rangle / k \sim 0.3$

A fourth property that follows is:

4)
$$\frac{Sk}{\varepsilon} = \left|\frac{k}{\langle uv \rangle}\right| \frac{P}{\varepsilon} \approx 3$$

i.e., near constancy of turbulence to mean shear timescale ratio.

Turbulence length scale $L = k^{3/2}/\varepsilon$ varies as

$$L = ky \frac{|\langle uv \rangle|^{\frac{1}{2}}}{u_\tau} \left(\frac{P}{\varepsilon}\right) \left|\frac{\langle uv \rangle}{k}\right|^{-3/2}$$

For high Re, in overlap region $(50\delta_\nu < y < 0.1\delta)$, RS almost constant, such that

$$L = C_L y$$

With

$$C_L \approx k \left(\frac{P}{\varepsilon}\right) \left|\frac{\langle uv \rangle}{k}\right|^{-\frac{3}{2}} \approx 2.5$$

Notice that $S, P, \varepsilon \propto y^{-1}$, whereas L and $\tau = k/\varepsilon \propto y$.

Recall definition of turbulent viscosity:

$$-\langle uv \rangle = \nu_t \frac{d\langle U \rangle}{dy}$$

$$\nu_t = u^* l_m = f(y)$$

One between u^* and l_m can be specified at will, for example:

$$u^* = |\langle uv \rangle|^{1/2}$$

$$\rightarrow u^* = l_m \frac{d\langle U \rangle}{dy}$$

In the overlap region

$$-\langle uv \rangle \approx u_\tau^2$$

And

$$\frac{d\langle U \rangle}{dy} = \frac{u_\tau}{ky}$$

Consequently,

$$u^* = u_\tau \rightarrow l_m = ky$$

In summary, this represents Prandtl's mixing-length hypothesis:

$$\nu_t = u^* l_m = l_m^2 \left| \frac{d\langle U \rangle}{dy} \right|$$

Eddy viscosity and mixing length

Analogy stress/strain momentum exchange laminar and turbulent flow:

$$\frac{\tau_{lam}}{\rho} = \nu \frac{\partial U}{\partial y}$$

$$\nu = \text{fluid property}$$

$$= a\lambda$$

For gas due molecular motions for which kinetic theory gives $a =$ rms speed molecular motion, $\lambda =$ mean free path, i.e., average distance travelled between collisions.

Analogy:

$$\frac{\tau_{turb}}{\rho} = -\overline{uv} = \nu_t \frac{\partial U}{\partial y}$$

Where $\nu_t =$ eddy viscosity $= f(flow)$.

Gross approximation $l \propto$ large scale eddies.

Free shear flows: $l_m = c\delta$, with $c = f$(mixing layer, jet, wake).

BL: $l_m = ky$, eddy size $\propto y$.

Prandtl: $u^* = \sqrt{\tau_w/\rho}$

$$\nu_t = u' l_m$$

$u' =$ scale $u_{rms} =$ order U or u^*

$l_m =$ mixing length

$$\nu_t = k u^* y$$

$$\tau_{turb} = \rho {u^*}^2 = \rho k u^* y \frac{\partial U}{\partial y}$$

$y^+ > 5$ but still near wall

$$\frac{\partial U}{\partial y} = \frac{u^*}{ky} \rightarrow u^+ = \frac{U}{u^*} = \frac{1}{k} \log y + B$$

Mixing length model (Kundu et al.)

$$\mathcal{R}_{ij} = \overline{u_i u_j} = \frac{2}{3} k \delta_{ij} - \nu_t \left(\overline{U_{i,j}} + \overline{U_{j,i}} \right)$$

$$\nu_t = l_t u_t$$

Characteristic length and time scales for turbulence in analogy to molecular theory,

$$\nu = a\lambda$$

a = rms speed molecular motion, λ = mean free path.

l_t = mixing length, u_t = velocity fluctuations

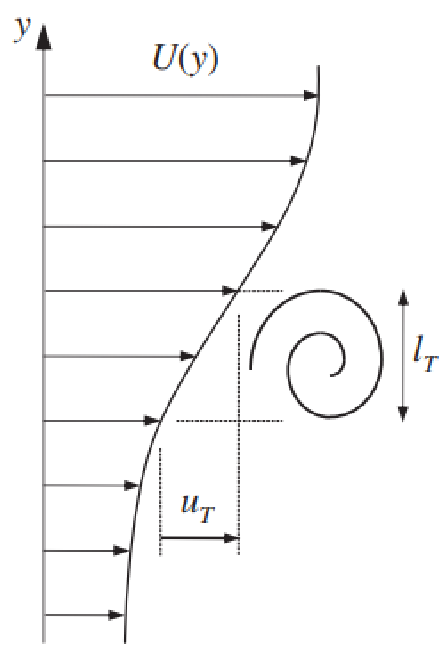

FIGURE 12.20 Schematic drawing of an eddy of size l_T in a shear flow with mean velocity profile $U(y)$. A velocity fluctuation, u or v, that might be produced by this eddy must be of order $l_T(dU/dy)$. Therefore, we expect that the Reynolds shear stress will scale like $\overline{uv} \sim l_T^2 (dU/dy)^2$.

$$\omega_z = -U_y$$

Turn over time = $|\omega_z^{-1}|$

$$u_t = \frac{l_t}{|\omega_z^{-1}|} = l_t U_y$$

Eddy size l_t driven by $U_y \rightarrow u_t = l_t U_y$.

$$-\overline{uv} = \nu_t U_y = l_t u_t U_y = l_t^2 U_y^2$$

For wall-bounded flow, assume $l_t \propto y \rightarrow l_t = ky$ such that streamwise momentum equation becomes:

$$0 = -\frac{1}{\rho}\frac{dP}{dx} + \frac{\partial}{\partial y}\left(\nu\frac{\partial U}{\partial y} - \overline{uv}\right)$$

$$0 = -\frac{1}{\rho}\frac{dP}{dx} + \frac{\partial}{\partial y}\left(\nu\frac{\partial U}{\partial y} + k^2 y^2 \left(\frac{dU}{dy}\right)^2\right)$$

Assuming negligible pressure gradient and integrating once:

$$\nu\frac{\partial U}{\partial y} + k^2 y^2 \left(\frac{dU}{dy}\right)^2 = \text{const.} = \frac{\tau_w}{\rho}$$

Where the last equality is obtained from evaluation of the expression on the left at $y = 0$.

For $y^+ > 50$ (outside viscous sublayer) $\nu\frac{\partial U}{\partial y} \ll k^2 y^2 \left(\frac{dU}{dy}\right)^2$ and solution of differential equation is:

$$\frac{dU}{dy} \approx \sqrt{\frac{\tau_w}{\rho}}\frac{1}{ky}$$

Or equivalently

$$\frac{U}{u^*} \approx \frac{1}{k}\log y + B$$