

7.4 Momentum Integral Methods

Historically similarity and other AFD methods used for idealized flows and momentum integral methods for practical applications, including pressure gradients, but failure 3D methods motivated 3D BL theory which quickly progressed to modern day CFD.

Momentum integral equation, which is valid for both laminar and turbulent flow:

$$\int_{y=0}^{\infty} (\text{steady flow BL equation} + (u - U)\text{continuity}) dy$$

$$\frac{\tau_w}{\rho U^2} = \frac{1}{2} C_f = \frac{d\theta}{dx} + (2 + H) \frac{\theta}{U} \frac{dU}{dx}$$

For flat plate equation $\rightarrow \frac{dU}{dx} = 0$

$$\theta = \int_0^{\delta} \frac{u}{U} \left(1 - \frac{u}{U}\right) dy$$

$$\delta^* = \int_0^{\delta} \left(1 - \frac{u}{U}\right) dy$$

$$H = \frac{\delta^*}{\theta}$$

Momentum: $uu_x + vu_y = -\frac{\partial}{\partial x} \left(\frac{p}{\rho}\right) + \frac{1}{\rho} \frac{\partial \tau}{\partial y}$ where $\tau = \mu \frac{\partial u}{\partial y}$ (for laminar flow) The pressure gradient evaluated from the outer potential flow using Bernoulli equation.

$$p + \frac{1}{2} \rho U^2 = \text{constant}$$

$$p_x + \frac{1}{2} \rho 2U U_x = 0$$

$$-p_x / \rho = U U_x$$

$$(u - U) \underbrace{(u_x + v_y)}_{\text{Continuity}} = uu_x + uv_y - Uu_x - Uv_y$$

$$\underbrace{uu_x + uv_y - UU_x - \frac{1}{\rho}\tau_y}_0 + \underbrace{uu_x + uv_y - Uu_x - Uv_y}_0 = 0$$

$$\begin{aligned} -\frac{1}{\rho}\tau_y &= -2uu_x - uv_y + UU_x - uv_y + Uu_x + Uv_y \\ &= \frac{\partial}{\partial x}(uU - u^2) + (U - u)U_x + \frac{\partial}{\partial y}(vU - vu) \end{aligned}$$

$$\int_0^\infty -\frac{1}{\rho}\tau_y dy = -(\cancel{\tau_\infty}^0 - \tau_w) / \rho = \frac{\partial}{\partial x} \int_0^\infty u(U - u) dy + U_x \int_0^\infty (U - u) dy + \cancel{(vU - vu)}^0 \Big|_0^\infty$$

$$\frac{\tau_w}{\rho} = \frac{\partial}{\partial x} \left[U^2 \int_0^\infty \frac{u}{U} \left(1 - \frac{u}{U} \right) dy \right] + U_x U \int_0^\infty \left(1 - \frac{u}{U} \right) dy$$

$$= U^2 \theta_x + 2UU_x \theta + UU_x \delta^*$$

$$\frac{\tau_w}{\rho U^2} = \frac{1}{2} C_f = \theta_x + (2\theta + \delta^*) \frac{1}{U} \frac{dU}{dx}$$

$$\frac{C_f}{2} = \frac{d\theta}{dx} + (2 + H) \frac{\theta}{U} U_x$$

Method Solution Momentum Integral Equation

Historically two approaches:

- (1) One parameter velocity profiles
- (2) Empirical correlations: Thwaites method

(1) Kármán-Pohlhausen Method

① $u(x, y) = U(x) f[\eta/\delta, \lambda(x)]$ guess form velocity profile

$$u/u_\infty = 2\eta - 2\eta^3 + \eta^4 + \frac{\lambda}{6} [\eta(1-\eta)^2] \quad \eta = y/\delta$$

- ② use 5 BC for u/u_∞
- ③ compute: $\theta, \delta^*, H, \tau_w$
- ④ Substitute momentum integral equation for 1st order ODE $\delta(x)$
- ⑤ with $\delta(x)$ known all variables also known

$$\lambda = \frac{\delta^2}{\nu} U_x$$

= Pohlhausen parameter

Accuracy not as good Thwaites method

Issues:

Recall using quadratic guessed profile
flat plate velocity profile only 10% accuracy.

Accuracy depends type of guessed profile.

TABLE 4-1

Boundary-layer predictions from five piecewise analytic profiles with their errors relative to the classic Blasius values

$\frac{u}{U} = F(\xi)$	$\eta^* = \frac{\delta^*}{\delta}$	$\theta^* = \frac{\theta}{\delta}$	$H = \frac{\delta^*}{\theta}$	$\frac{\delta}{x} \sqrt{Re_x}$	$C_f \sqrt{Re_x} = \frac{\delta^*}{x} \sqrt{Re_x}$	$\frac{\delta^*}{x} \sqrt{Re_x}$	L_2 error
1 $2\xi - \xi^2$	0.333 3.1%	0.133 0.25%	2.500 3.5%	5.477 9.5%	0.730 10%	1.826 6.1%	0.020
2 $\frac{3}{5}\xi - \frac{1}{5}\xi^3$	0.375 9.0%	0.139 4.7%	2.692 4.0%	4.641 7.2%	0.646 2.6%	1.740 1.1%	0.034
3 $2\xi - 2\xi^3 + \xi^4$	0.300 13%	0.118 12%	2.554 1.4%	5.836 17%	0.685 3.2%	1.751 1.8%	0.054
4 $\sin(\frac{1}{2} \pi \xi)$	0.363 5.6%	0.137 2.7%	2.660 2.7%	4.795 4.1%	0.655 1.3%	1.743 1.3%	0.021
5 $\frac{3}{5}\xi - \xi^3 + \frac{1}{5}\xi^4$	0.350 1.7%	0.134 0.52%	2.618 1.1%	4.993 0.13%	0.668 0.53%	1.748 1.6%	0.008
Blasius (1908)	0.344	0.133	2.59	5	0.664	1.72	n/a

$$L_2 = \left[\int_0^1 (F - F_{\text{Blasius}})^2 d\eta \right]^{1/2}$$

- 1, 2, 3 Pohlhausen (1921)
4 Schlichting (1979)
5 Majdalani & Xuon (2020)

Pohlhausen paradox: missing order, i.e., $\theta \ll \delta$ profile
can satisfy does not improve
accuracy

- BC: $u(x, 0) = 0$ 1 no slip
 $u(x, \delta) = U$ 2 match
 $u_y(x, \delta) = 0$ 3 smooth merge U
 $\mu u_{yy}(x, 0) = p_x$ 4 correct balance momentum $y=0$
 $\mu u_{yy}(x, \delta) = 0$ 5 $\mu \frac{\partial^2 u}{\partial y^2} = 0$ zero shear stress at δ

However, the Blasius profile does not
in fact satisfy all these conditions!

Note: 1-3 satisfy 1-3 of 2-5 satisfy 1-4
but only 3 satisfy 1-5
& largest L_2 error

BC is true at $y \rightarrow \infty$ but not true $y=5$
where $\eta_{y=5} = -0.7085$
Blasius

Also higher initial
slope $F'(0)$

Other differences
such as $F(1)$ & $F'(1)$

TABLE 4-2

Endpoint properties of the piecewise analytic velocity profiles and their corresponding Blasius values

$\frac{u}{U} = F(\xi)$	$F(0)$	$F'(0)$	$F''(0)$	$F(1)$	$F'(1)$	$F''(1)$
$2\xi - \xi^2$	0	2.000	-2	1.000	0	-2.000
$\frac{3}{2}\xi - \frac{1}{2}\xi^2$	0	1.500	0	1.000	0	-3.000
$2\xi - 2\xi^2 + \xi^4$	0	2.000	0	1.000	0	0.000
$\sin(\frac{1}{2}\pi\xi^2)$	0	1.571	0	1.000	0	-2.467
$\frac{3}{2}\xi - \xi^2 + \frac{1}{3}\xi^4$	0	1.667	0	1.000	0	-2.000
Blasius (1908)	0	1.630	0	0.990	0.0904	-0.709

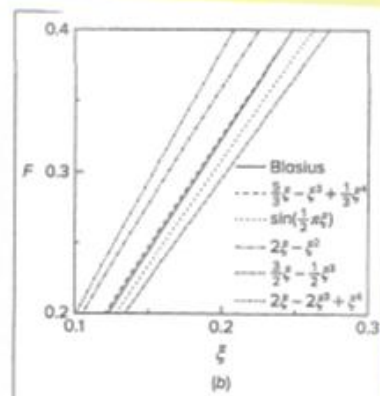
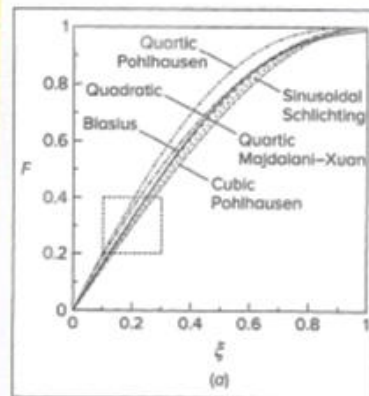


FIGURE 4-5

Comparison of five analytic approximations to the Blasius solution (solid line) including Pohlhausen's quadratic, cubic, and quartic polynomials (chained lines) as well as Schlichting's sinusoidal (dotted) and Majdalani-Xuan's quartic (dashed) profiles across (a) the boundary layer and (b) a designated quadrant where individual deviations from the Blasius curve are magnified.

Presumably these differences compounded
for $\eta_x \neq 0$.

Thwaites Method (1949)

Pressure gradient parameter $\lambda = \frac{\theta^2}{\nu} \frac{dU}{dx} = \left(\frac{\theta}{\delta}\right)^2 \Lambda$ where $\Lambda = \frac{\delta^2}{\nu} \frac{dU}{dx} = -p_x \frac{\delta^2}{\mu U}$ is the Pohlhausen parameter.

Multiply momentum integral equation by $\frac{U\theta}{\nu}$

$$\frac{\tau_w \theta}{\mu U} = \frac{U\theta}{\nu} \frac{d\theta}{dx} + \frac{\theta^2}{\nu} \frac{dU}{dx} (2 + H)$$

The equation is dimensionless and, LHS and H can be correlated with λ as shear and shape-factor correlations:

$$\frac{\tau_w \theta}{\mu U} = S(\lambda) = (\lambda + 0.09)^{0.62}$$

$$H = \delta^*/\theta = H(\lambda) = \sum_{i=0}^5 a_i (0.25 - \lambda)^i$$

$$a_i = (2, 4.14, -83.5, 854, -3337, 4576)$$

Note

$$\frac{U\theta}{\nu} \frac{d\theta}{dx} = \frac{1}{2} U \frac{d}{dx} \left(\frac{\theta^2}{\nu} \right)$$

Substitute above into momentum integral equation.

$$S(\lambda) = \frac{1}{2} U \frac{d}{dx} \left(\frac{\theta^2}{\nu} \right) + \lambda (2 + H)$$

$$U \frac{d(\lambda/U_x)}{dx} = 2[S - \lambda(2 + H)] = F(\lambda)$$

$F(\lambda) = 0.45 - 6\lambda$ based on AFD and EFD

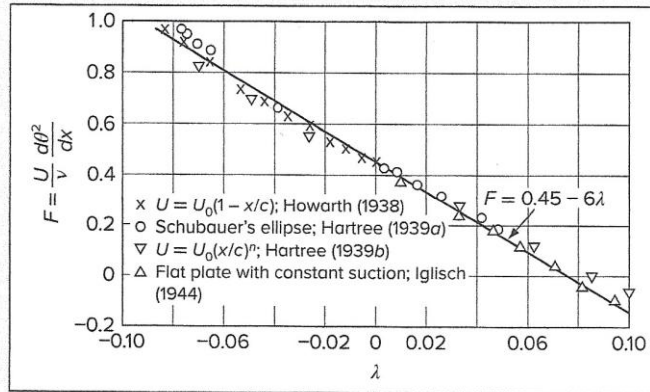


FIGURE 4-27
Empirical correlation of the boundary-layer function in Eq. (4-156). [After Thwaites (1949).]

Define $z = \frac{\theta^2}{\nu}$ so that $\lambda = z \frac{dU}{dx}$

$$U \frac{dz}{dx} = 0.45 - 6\lambda = 0.45 - 6z \frac{dU}{dx}$$

$$U \frac{dz}{dx} + 6z \frac{dU}{dx} = 0.45$$

$$\frac{1}{U^5} \frac{d}{dx} (zU^6) = U \frac{dz}{dx} + 6z \frac{dU}{dx} = 0.45$$

$$d(zU^6) = 0.45U^5 dx$$

$$zU^6 = 0.45 \int_0^x U^5 dx + C$$

$$\rightarrow \theta^2 = \theta_0^2 + \frac{0.45\nu}{U^6} \int_0^x U^5 dx$$

$\theta_0(x = 0) = 0$ and $U(x)$ known from potential flow solution.

Complete solution:

$$\lambda = \lambda(\theta) = \frac{\theta^2}{\nu} \frac{dU}{dx}$$

$$\frac{\tau_w \theta}{\mu U} = S(\lambda)$$

$$\delta^* = \theta H(\lambda)$$

Accuracy: mild $p_x \pm 5\%$ and strong adverse p_x (τ_w near 0) $\pm 15\%$

TABLE 4-8

Shear and shape functions correlated by Thwaites (1949)

λ	$H(\lambda)$	$S(\lambda)$	λ	$H(\lambda)$	$S(\lambda)$
+0.25	2.00	0.500	-0.056	2.94	0.122
0.20	2.07	0.463	-0.060	2.99	0.113
0.14	2.18	0.404	-0.064	3.04	0.104
0.12	2.23	0.382	-0.068	3.09	0.095
0.10	2.28	0.359	-0.072	3.15	0.085
+0.080	2.34	0.333	-0.076	3.22	0.072
0.064	2.39	0.313	-0.080	3.30	0.056
0.048	2.44	0.291	-0.084	3.39	0.038
0.032	2.49	0.268	-0.086	3.44	0.027
0.016	2.55	0.244	-0.088	3.49	0.015
0.0	2.61	0.220	-0.090	3.55	0.000
				(Separation)	
-0.016	2.67	0.195			
-0.032	2.75	0.168			
-0.040	2.81	0.153			
-0.048	2.87	0.138			
-0.052	2.90	0.130			

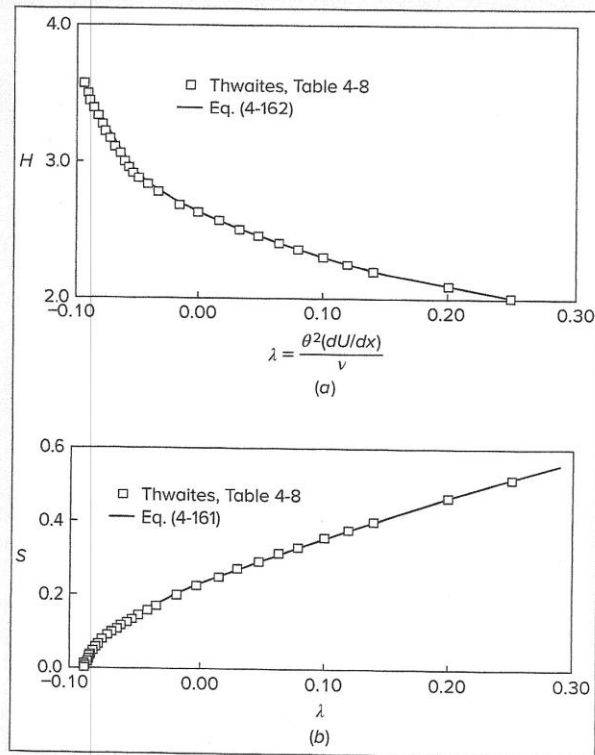


FIGURE 4-28
The laminar boundary-layer correlated functions by Thwaites (1949): (a) shape factor; (b) shear stress with curve fits.

Separation predicted within 4%; however, large scale separation causes viscous/inviscid interaction and alters imposed external $U(x)$ and $p_x(x)$

TABLE 4-9

Laminar-separation-point prediction by Thwaites' method

$U(x)$	x_{sep} (exact)	Thwaites	
		x_{sep}	Error [%]
Howarth (1938)			
$1 - x$	0.120	0.123	+2.5
Tani (1949)			
$1 - x^2$	0.271	0.268	-1.1
$1 - x^4$	0.462	0.449	-2.8
$1 - x^8$	0.640	0.621	-3.0
Terrill (1960)			
$\sin(x)$	1.823	1.800	-1.3
Curle (1958)			
$x - x^3$	0.655	0.648	-1.1
Görtler (1957)			
$\cos(x)$	0.389	0.384	-1.3
$(1 - x)^{1/2}$	0.218	0.221	+1.3
$(1 - x)^2$	0.0637	0.0652	+2.4
$(1 + x)^{-1}$	0.151	0.158	+4.6
$(1 + x)^{-2}$	0.0713	0.0739	+3.6

Pohlhausen Velocity Profile:

$$\frac{u}{U} = f(\eta) = a\eta + b\eta^2 + c\eta^3 + d\eta^4 \text{ with } \eta = \frac{y}{\delta}$$

a, b, c, d determined from boundary conditions:

- 1) $y = 0 \rightarrow u = 0, u_{yy} = -\frac{U}{\nu} U_x$
- 2) $y = \delta \rightarrow u = U, u_y = 0, u_{yy} = 0$

$$\rightarrow \frac{u}{U} = F(\eta) + \Lambda G(\eta), -12 \leq \Lambda \leq 12 \quad \Lambda = \frac{\delta^2}{\nu} \frac{dU}{dx} = -p_x \frac{\delta^2}{\mu U}$$

separation

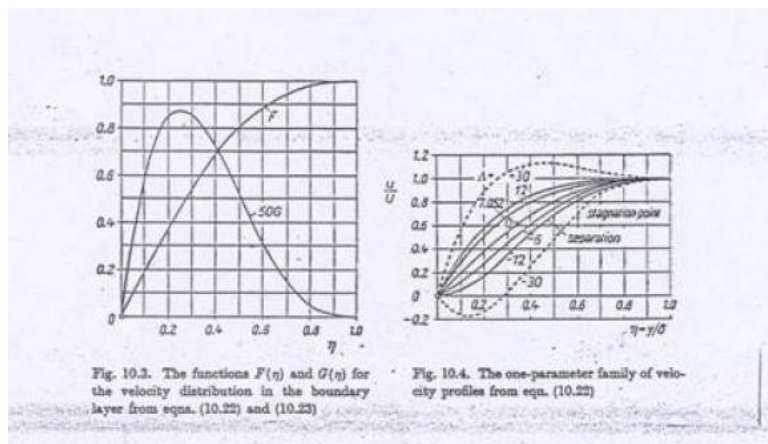
(experiment: $\Lambda_{\text{separation}} = -5$)

$$F(\eta) = 2\eta - 2\eta^3 + \eta^4$$

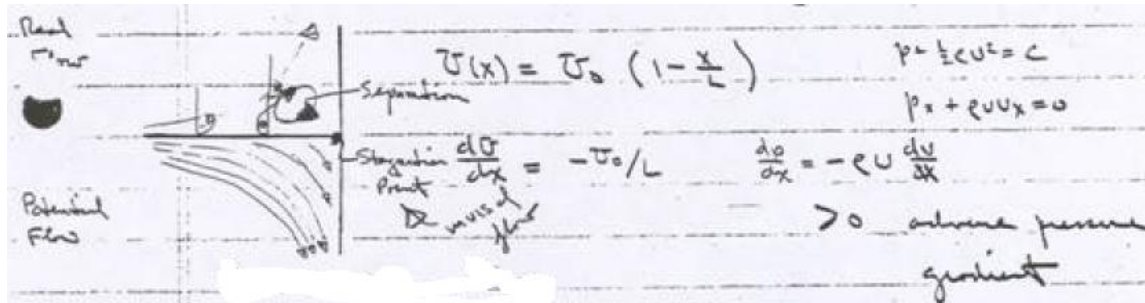
$$G(\eta) = \frac{\eta}{6} (1 - \eta)^3$$

$$\lambda = \lambda(\Lambda) = \left(\frac{37}{315} - \frac{\Lambda}{945} + \frac{\Lambda^2}{9072} \right) \Lambda$$

Profiles are realistic, except near separation. In guessed profile methods u/U directly used to solve momentum integral equation numerically, but accuracy not as good as empirical correlation methods; therefore, use Thwaites method to get λ , etc., and then use λ to get Λ and plot u/U .



Howarth linearly decelerating flow (example of solution of steady state 2D boundary layer using Thwaites method)



Howarth proposed a linearly decelerating external velocity distribution $U(x) = U_0 \left(1 - \frac{x}{L}\right)$ as a theoretical model for laminar boundary layer study. Use Thwaites's method to compute:

- X_{sep}
- $C_f \left(\frac{x}{L} = 0.1\right)$

Note $U_x = -U_0/L$

Solution

$$\theta^2 = \frac{0.45\nu}{U_0^6 \left(1 - \frac{x}{L}\right)^6} \int_0^x U_0^5 \left(1 - \frac{x}{L}\right)^5 dx = 0.075 \frac{\nu L}{U_0} \left[\left(1 - \frac{x}{L}\right)^{-6} - 1 \right]$$

can be evaluated for given L , $Re_L = U_0 L / \nu$

$$\lambda = \frac{\theta^2}{\nu} \frac{dU}{dx} = -0.075 \left[\left(1 - \frac{x}{L}\right)^{-6} - 1 \right]$$

$$\lambda_{sep} = -0.09 \Rightarrow \frac{X_{sep}}{L} = 0.123$$

3% higher than exact solution = 0.1199

$C_f \left(\frac{x}{L} = 0.1 \right) \rightarrow$ i.e. just before separation

$$\lambda = -0.0661$$

$$S(\lambda) = 0.099 = \frac{1}{2} C_f Re_\theta$$

$$C_f = \frac{2(0.099)}{Re_\theta}$$

Compute Re_θ in terms of Re_L

$$\theta^2 = 0.075 \frac{\nu L}{U_0} \left[(1 - 0.1)^{-6} - 1 \right] = 0.0661 \frac{\nu L}{U_0}$$


$$\frac{\theta^2}{L^2} = 0.0661 \frac{\nu}{LU_0} = \frac{0.0661}{Re_L}$$

$$\frac{\theta}{L} = \frac{0.257}{Re_L^{1/2}}$$

$$Re_\theta = \frac{\theta}{L} Re_L = 0.257 Re_L^{1/2}$$

$$C_f = \frac{2(0.099)}{0.257} Re_L^{-1/2} = 0.77 Re_L^{-1/2}$$

To complete
solution must
specify Re_L



Consider the complex potential

$$F(z) = \frac{a}{2} z^2 = \frac{a}{2} r^2 e^{2i\theta}$$

$$\phi = \text{Re}[F(z)] = \frac{a}{2} r^2 \cos 2\theta$$

$$\psi = \text{Im}[F(z)] = \frac{a}{2} r^2 \sin 2\theta$$

Orthogonal rectangular hyperbolas

ϕ : asymptotes $y = \pm x$

ψ : asymptotes $x=0, y=0$

$$\underline{V} = \nabla \phi = \phi_r \hat{e}_r + \frac{1}{r} \phi_\theta \hat{e}_\theta$$

$$\left. \begin{aligned} v_r &= ar \cos 2\theta \\ v_\theta &= -ar \sin 2\theta \end{aligned} \right\} \quad \frac{\pi}{2} \leq \theta \leq 0 \text{ (flow direction as shown)}$$

$$\underline{V} = v_r (\cos \theta \hat{i} + \sin \theta \hat{j}) + v_\theta (-\sin \theta \hat{i} + \cos \theta \hat{j}) = (v_r \cos \theta - v_\theta \sin \theta) \hat{i} + (v_r \sin \theta + v_\theta \cos \theta) \hat{j}$$

Potential flow slips along surface: (consider $\theta = 90^\circ$)

1) determine a such that $v_r = U_0$ at $r=L, \theta = 90^\circ$

$$v_r = aL \cos(2 \times 90) = U_0 \Rightarrow aL = -U_0, \text{ i.e. } a = -\frac{U_0}{L}$$

2) let $U(x) = v_r$ at $x=L-r$:

$$\Rightarrow v_r = a(L-x) \cos(2 \times 90) = U(x)$$

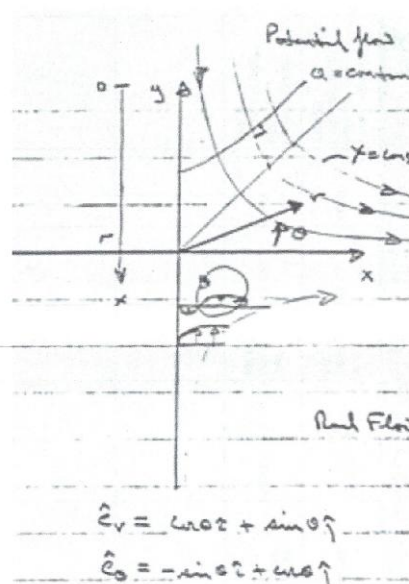
$$\text{Or: } U(x) = -a(L-x) = \frac{U_0}{L}(L-x) = U_0(1 - \frac{x}{L})$$

$$U_x = -\frac{U_0}{L}$$

$$p + \frac{1}{2} \rho U^2 = C$$

$$p_x + \rho U U_x = 0$$

$$p_x = -\rho U U_x = -\rho U_0(1 - \frac{x}{L})(-\frac{U_0}{L}) = \rho \frac{U_0^2}{L}(1 - \frac{x}{L})$$



$$\chi = Bxy \quad B = U/L$$

$$u = Bx = \chi_y$$

$$v = -By = -\chi_x$$

$$p + \frac{1}{2} \rho (u^2 + v^2) = C$$

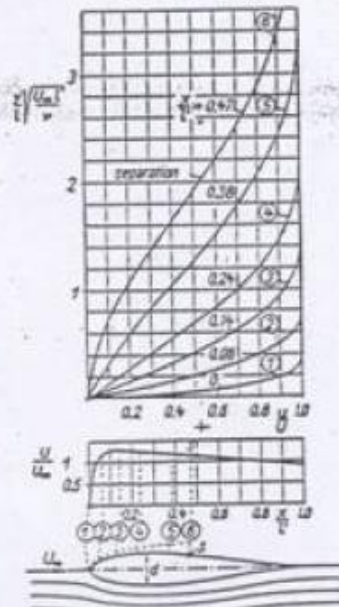
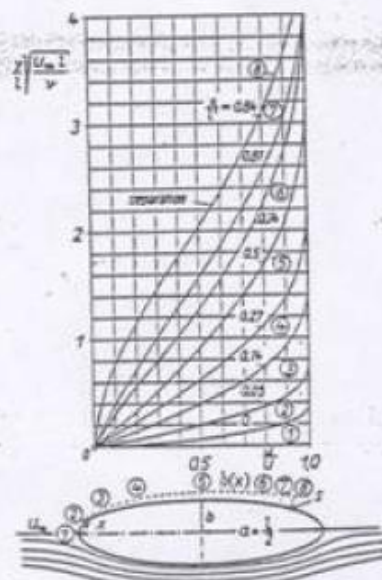
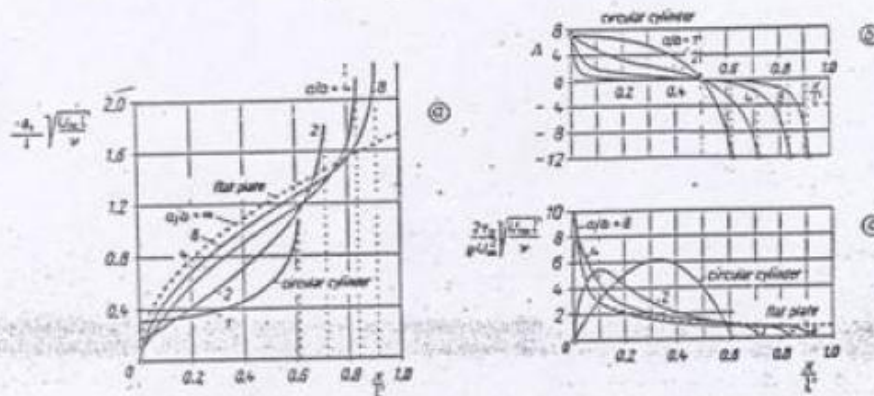
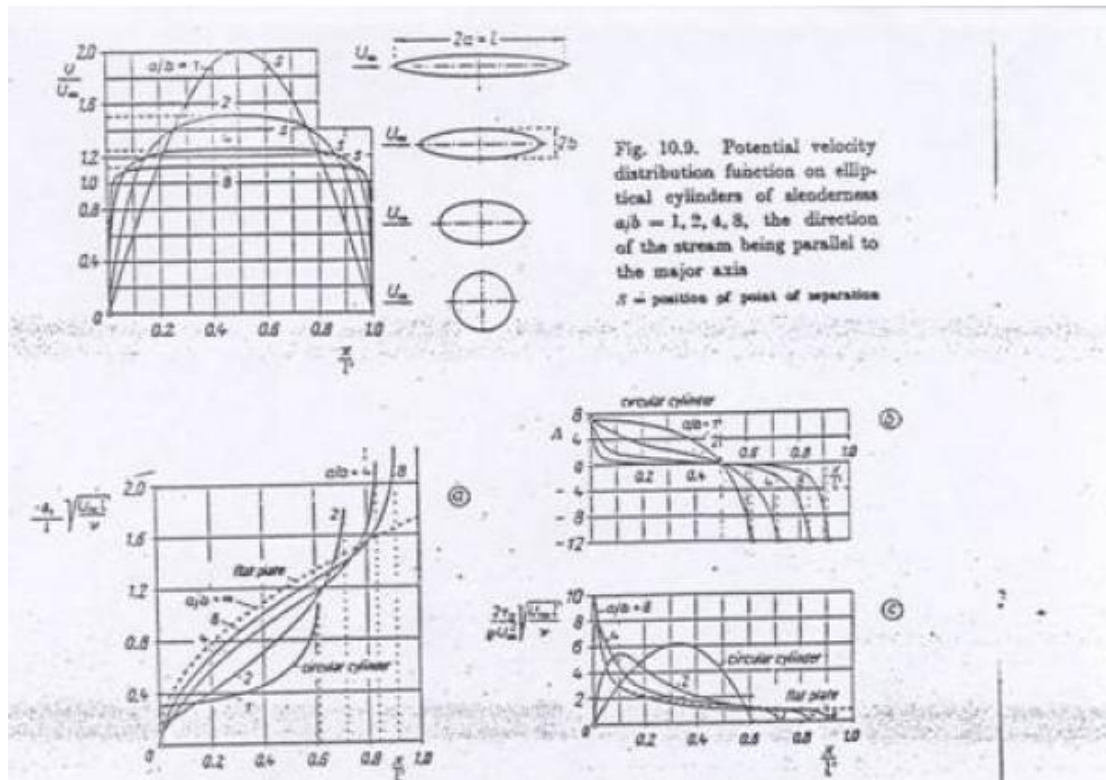
$$p + \frac{1}{2} \rho B^2 (x^2 + y^2) = C$$

$$p(0,0) = C = p_0$$

$$p = p_0 - \frac{1}{2} \rho \frac{U^2}{L^2} (x^2 + y^2)$$

$$p_x = -\rho \frac{U^2}{L^2} x$$

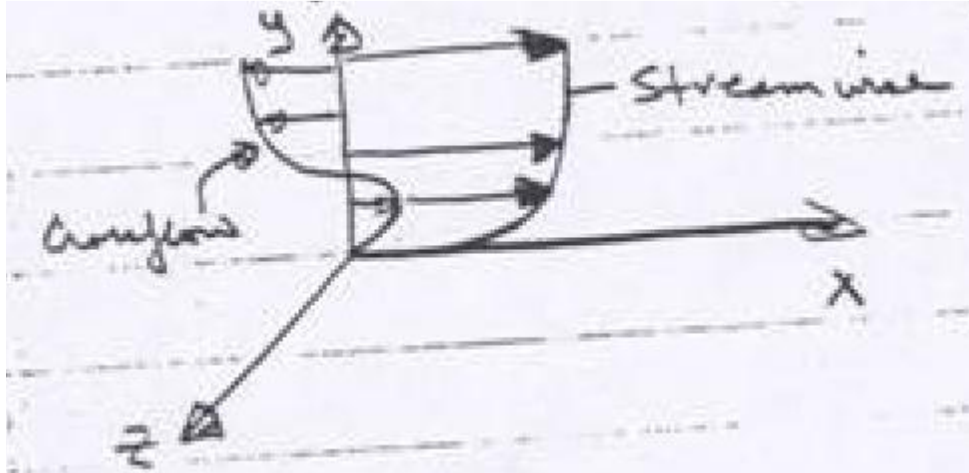
$$p_y = -\rho \frac{U^2}{L^2} y$$



Refer back to Table 4-6 (see 26) with regard to relative performance of integral methods

3-D Integral methods

Momentum integral methods perform well (i.e. compare well with experimental data) for a large class of both laminar and turbulent 2D flows. However, for **3D flows they do not**, primarily due to the inability of correlating the crossflow velocity components.



The cross flow is driven by $\frac{\partial p}{\partial z}$, which is imposed on BL from the outer potential flow $U(x,z)$.

3-D boundary layer equations

$$u_x + v_y + w_z = 0;$$

$$uu_x + vu_y + wu_z = -\frac{\partial}{\partial x}(p/\rho) + \nu u_{yy} - \frac{\partial}{\partial y}(\overline{u'v'})$$

$$uw_x + vw_y + ww_z = -\frac{\partial}{\partial z}(p/\rho) + \nu w_{yy} - \frac{\partial}{\partial y}(\overline{v'w'})$$

+ closure equations

Differential methods have been developed for this reason as well as for extensions to more complex and non-thin boundary layer flows.