Chapter 7: Properties of Turbulent Free Shear Flow (Chap. 11 Bernard)

Part 3: Turbulent Jet

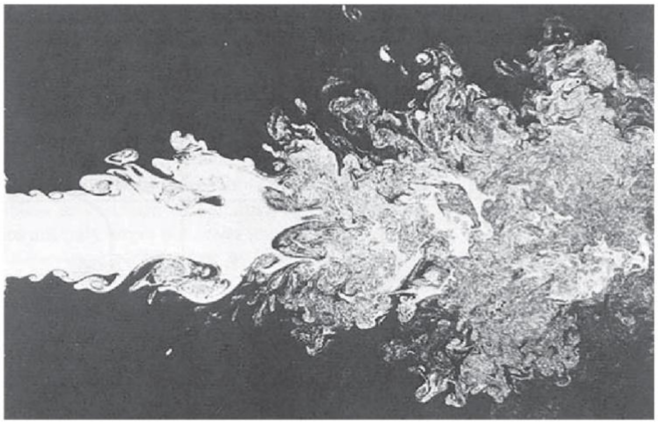


Figure 1.4 Transition to turbulence in a jet. Courtesy of J.-L. Balint and L. Ong.

Jet in crossflow

Round jet $\overline{w} = 0$, i.e., without swirl.

Herein, plane jets are considered.

Near nozzle exit mixing layers due to ΔU as potential core shrinks, and flow becomes fully developed, transitions to turbulence, and becomes self-similar at $x/d \approx 50$ such that:

$$\frac{\overline{U}}{\Delta U} = f(\eta) \quad (1)$$

Where $\eta = y/l(x)$ and $\Delta U = \overline{U}_{max}(x)$, and both η and ΔU are $f(x)$. $\overline{U}$ reaches self-similarity before $\overline{u_i u_j}$.

Introducing a stream function $\overline{\psi}(x, y)$ defined as

$$\overline{\psi} = l \Delta U F(\eta)$$

Where:

$$F'(\eta) = f(\eta) \quad (2)$$

and the coefficient $l\Delta U$ is chosen for dimensional consistency, i.e., $\overline{\psi}$ has dimensions m^2/s.

By the definition of $\overline{\psi}$:

$$\overline{U} = \overline{\psi}_y \quad (3)$$
$$\overline{V} = -\overline{\psi}_x \quad (4)$$

From Eq. (3):

$$\overline{U} = l\Delta U \underbrace{\frac{dF}{d\eta}}_{\boxed{F'}} \frac{d\eta}{dy}$$

$$\boxed{\frac{d\eta}{dy} = \frac{d\left(\frac{y}{l}\right)}{dy} = \frac{1}{l}}$$

$$\overline{U} = \Delta U F' \quad (5)$$

From Eq. (4):

$$\boxed{\begin{aligned}\frac{d\eta}{dx} &= \frac{d\left(\frac{y}{l}\right)}{dx} \\ &= -\frac{y}{l^2}\frac{dl}{dx} = -\frac{\eta}{l}\frac{dl}{dx}\end{aligned}}$$

$$\overline{V} = -\frac{d(l\Delta U)}{dx}F - l\Delta U \frac{dF}{d\eta}\frac{d\eta}{dx}$$

$$\overline{V} = -\frac{d(l\Delta U)}{dx}F + \eta\Delta U \frac{dl}{dx}F' \quad (6)$$

Recall BL streamwise mean momentum equation:

$$\overline{U}\frac{\partial \overline{U}}{\partial x} + \overline{V}\frac{\partial \overline{U}}{\partial y} + \frac{\partial \overline{uv}}{\partial y} = 0$$

And

$$-\overline{uv} = (\Delta U)^2 g(\eta)$$

Which differs from wake scaling where $\Delta U = U_e - \overline{U}_{min}(x)$, whereas for jet flow $\Delta U = \overline{U}_{max}(x)$.

Substitution of Eqs. (5), (6), and $\overline{uv}$ into mean momentum equation gives:

$$\Delta U F'^2 \frac{d(\Delta U)}{dx} + \Delta U^2 F' \underbrace{\frac{dF'}{d\eta}}_{\boxed{F''}} \frac{d\eta}{dx} + \left(-\frac{d(l\Delta U)}{dx} F + \eta \Delta U \frac{dl}{dx} F'\right) \Delta U \underbrace{\frac{dF'}{d\eta}}_{\boxed{F''}} \frac{d\eta}{dy} - \Delta U^2 \frac{dg(\eta)}{d\eta}\frac{d\eta}{dy} = 0$$

$$\Delta U F'^2 \frac{d(\Delta U)}{dx} - \cancel{\eta \Delta U^2 \frac{dl}{dx} \frac{F'\Delta U F''}{l}} - \frac{d(l\Delta U)}{dx}\frac{F\Delta U F''}{l} + \cancel{\eta\Delta U^2 \frac{dl}{dx}\frac{F'\Delta U F''}{l}} - \frac{\Delta U^2}{l} g' = 0$$

$$\Delta U F'^2 \frac{d(\Delta U)}{dx} - \frac{d\Delta U}{dx} F\Delta U F'' - \frac{dl}{dx}\frac{F\Delta U^2 F''}{l} - \frac{\Delta U^2}{l} g' = 0$$

Multiply by $l/\Delta U^2$:

$$\frac{lF'^2}{\Delta U}\frac{d(\Delta U)}{dx} - \frac{l}{\Delta U}\frac{d\Delta U}{dx} FF'' - \frac{dl}{dx} FF'' = g'$$

$$\underbrace{\frac{l}{\Delta U}\frac{d(\Delta U)}{dx}}_{\boxed{\beta}}\left(F'^2 - FF''\right) - \underbrace{\frac{dl}{dx}}_{\boxed{\alpha}} FF'' = g' \quad (7)$$

Where:

$$\frac{dl}{dx} = \alpha \quad (8)$$

$$\frac{l}{\Delta U}\frac{d(\Delta U)}{dx} = \beta \quad (9)$$

Self-similarity can only be achieved if α and β are not $f(x)$, i.e., either constant or $f(\eta)$. One way to achieve similarity is to assume they are constant.

Integration of Eqs. (8) and (9) gives:

$$l(x) = \alpha(x - x_0) \quad (10)$$

$$\Delta U(x) = C(x - x_0)^m \quad (11)$$

$m \equiv \beta/\alpha$ is a constant which needs to be determined, x_0 represents the virtual origin and C is a constant.

Integration of

$$\frac{\partial}{\partial x}\left[\overline{U}\left(\overline{U} - U_e\right)\right] + \frac{\partial}{\partial y}\left[\overline{V}\left(\overline{U} - U_e\right)\right] + \frac{\partial}{\partial y}\overline{uv} = 0$$

showed that:

$$\frac{d}{dx}\int_{-\infty}^{\infty}\overline{U}\left(\overline{U} - U_e\right)dy = 0 \quad (12)$$

Changing the integration variable to η, using Eq. (5) and the fact that $U_e = 0$ for a jet with no co-flow:

$$\frac{d}{dx}\int_{-\infty}^{\infty}\overline{U}^2 l d\eta = \frac{d}{dx}\left(l\Delta U^2\int_{-\infty}^{\infty}F'^2 d\eta\right) = 0$$

And substituting Eqs. (10) and (11) for l and ΔU^2:

$$\frac{d}{dx}\left(C\alpha(x - x_0)^{1+2m}\int_{-\infty}^{\infty}F'^2 d\eta\right) = 0$$

Which shows that $1 + 2m = 0$ (i.e., $m = -1/2$) for $l\Delta U^2 \neq f(x)$.

Substituting this value for m into Eq. (11) gives:

$$\Delta U = C(x - x_0)^{-1/2}$$

i.e., l grows linearly and ΔU decreases as $x^{-1/2}$.

The Reynolds number:

$$Re = \frac{l\Delta U}{\nu} = \frac{\alpha(x - x_0) \times C(x - x_0)^{-1/2}}{\nu} = \frac{\alpha C\sqrt{x - x_0}}{\nu}$$

increases with distance by $\sqrt{x - x_0}$ such that the thin layer assumptions are increasingly well justified.

To obtain the similarity form of the mean velocity field, a model is needed for g' to be related to F. Recall the gradient law and combine with $\overline{uv} = -(\Delta U)^2 g(\eta)$:

$$\overline{uv} = -\nu_t \frac{\partial \overline{U}}{\partial y} = -\nu_t \frac{\Delta U}{l} F'' = -(\Delta U)^2 g(\eta)$$

$$g(\eta) = R_t^{-1} F'' \quad (13)$$

$$R_t = \frac{l\Delta U}{\nu_t}$$

Differentiating Eq. (13) gives:

$$g'(\eta) = R_t^{-1} F''' \quad (14)$$

Recall

$$m \equiv \frac{\beta}{\alpha} = -\frac{1}{2}$$

$$2\beta = -\alpha \rightarrow \beta = -\alpha/2$$

Substituting this relation and Eq. (14) into (7) yields:

$$-\frac{\alpha}{2}\left(F'^2 - FF''\right) - \alpha FF'' = R_t^{-1}F'''$$

$$\frac{\alpha}{2}\left(F'^2 + FF''\right) + R_t^{-1}F''' = 0 \quad (15)$$

For Eq. (15) to have a similarity solution, it must be that R_t is constant, which implies that $\nu_t \propto \sqrt{x - x_0}$.

Boundary conditions for Eq. (15) are given by:

$F(0) = 0 \rightarrow y = 0$ symmetry line is a streamline, i.e.,

$\overline{\Psi}(0) = l\Delta UF(\eta = 0) = \alpha C\sqrt{x - x_0}\, F(0) =$ constant = 0

$$F'(0) = \frac{\overline{U}(x,0)}{\Delta U(x,0)} = \frac{\overline{U}_{max}(x,0)}{\overline{U}_{max}(x,0)} = 1$$

$\lim_{\eta\to\infty} F'(\eta) = 0$ since $\overline{U}(x,\eta) \rightarrow 0$ as $\eta \rightarrow \infty$

$\lim_{\eta\to\infty} F''(\eta) = 0$ since $\overline{U}_y(x,\eta) \rightarrow 0$ as $\eta \rightarrow \infty$

Integrating Eq. (15) twice and applying BCs gives

$$F^2 + \frac{4}{\alpha R_t}(F' - 1) = 0$$

Appendix A.1

Which represents an example of a Riccati equation.

The solution is given by:

$$F(\eta) = \frac{2}{\sqrt{\alpha R_t}}\tanh\left(\frac{\sqrt{\alpha R_t}}{2}\eta\right) \quad (16)$$

Taking a derivative of Eq. (16) and using Eq. (5) gives

$$F'(\eta) = \frac{\overline{U}}{\Delta U} = \left[1 - \tanh^2\left(\frac{\sqrt{\alpha R_t}}{2}\eta\right)\right]$$

$$\overline{U} = \Delta U\left[1 - \tanh^2\left(\frac{\sqrt{\alpha R_t}}{2}\eta\right)\right] \quad (17)$$

For simplicity, assume $\alpha = 4/R_t$, Eq. (17) becomes:

$$\overline{U} = \Delta U(1 - \tanh^2 \eta) \quad (18)$$

Where:

$$\eta = \frac{yR_t}{4(x - x_0)} = \frac{y}{l}$$

Eq. (18), in combination with $\Delta U = C(x - x_0)^{-1/2}$ shows that:

$$\overline{U} = C(x - x_0)^{-1/2}(1 - \tanh^2 \eta) = f(R_t, C)$$

And C can be expressed in terms of the momentum flux, M:

$$M = \rho \int_{-\infty}^{\infty} \overline{U}^2 \, l d\eta \rightarrow C = \sqrt{\frac{3MR_t}{16\rho}}$$

Appendix A.2

From EFD with $R_t = \frac{l\Delta U}{\nu_t} = 25.7$ and associated M and C values plots for $\overline{U}$, ΔU and l can be generated. It can be observed from Eq. (18) that when $\eta = 1$, i.e., $y = l = \alpha(x - x_0)$, $\overline{U} = \Delta U(1 - \tanh^2 1) = 0.420\Delta U$.

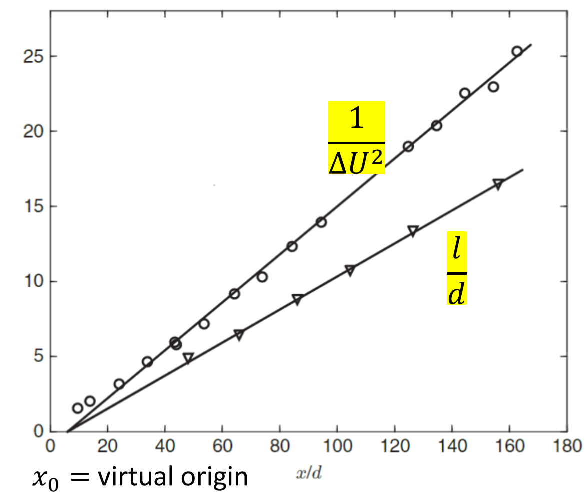


Figure 11.4 Centerline mean velocity and jet width development of a turbulent plane jet at $Re_d = 3.4 \times 10^4$. Data from [13]. o, $1/(\Delta U)^2$; ▿, ℓ/d.

$$Re_d = \frac{U_j d}{\nu}$$

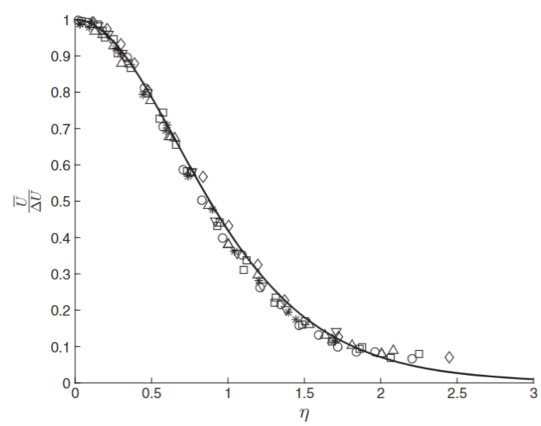
U_j = jet exit velocity

d = jet width at nozzle exit

$$\Delta U = C(x - x_0)^{-1/2} \rightarrow \Delta U^{-2} = \frac{x - x_0}{C^2}$$

$$l/d = \alpha(x - x_0)/d$$

ΔU^{-2} linear for $x/d \geq 45$ and l linear for $x/d \geq 65$. Linear growth confirms similarity analysis.



$$\eta = \frac{y}{(x - x_0)}$$

$$\alpha = \frac{4}{R_t} = 1$$

Figure 11.5 Mean streamwise velocity profiles of turbulent plane jet at $Re_d = 3.4 \times 10^4$ for ◇, $x/d = 47$; o, 65; □, 85; ▿, 103; △, 125; ∗, 155; and, —, Eq. (11.68). Data from [13].

Good agreement except near jet edge due to intermittency of turbulence and $\nu_t \neq$ constant.

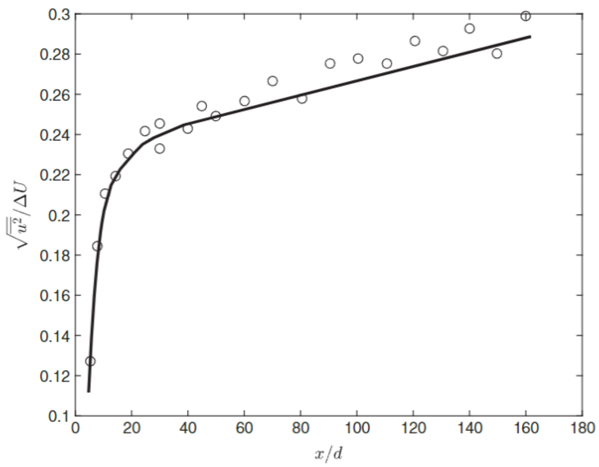


Figure 11.6 Growth of $\sqrt{\overline{u^2}}$ along the centerline of a turbulent plane jet at $Re_d = 3.4 \times 10^4$. o, data from [13]; —, fit to the data.

$$u_{rms} = \sqrt{\langle u^2 \rangle}\Big|_{y=0} = \text{linear for } x/d \geq 45$$

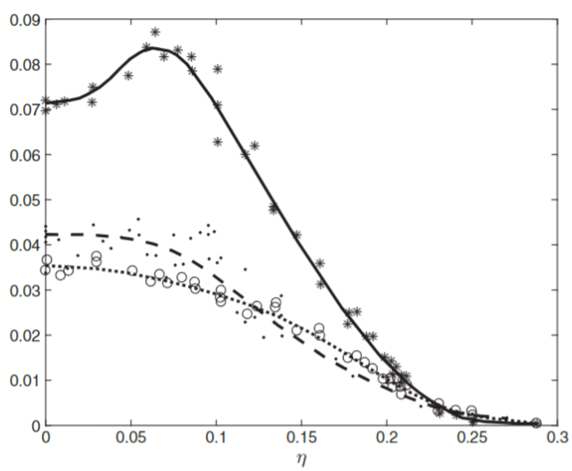


Figure 11.7 Velocity variances for turbulent plane jet at $Re_d = 3.4 \times 10^4$ and $x/d = 101$. Data from [13] with fitted curves: * and —, $\overline{u^2}/(\Delta U)^2$; o and · · ·, $\overline{v^2}/(\Delta U)^2$; • and − −, $\overline{w^2}/(\Delta U)^2$.

$x/d = 101$ required for $\overline{v^2}$ and $\overline{w^2}$

Peak of $\overline{u^2} > 2\overline{v^2}$ and $= 2\overline{w^2}$.

For $\eta \geq 0.3$, $\overline{u_i^2}/\Delta U^2 \approx 0$, i.e., RS become negligible compared to $\overline{U}_{max}$. This value of η can be expressed as a function of l:

$$\eta = \frac{y}{\Delta x} \rightarrow y = 0.3\Delta x = 0.3 \times 101d \sim 2.5l$$

Since $l/d \sim 12.5$ at $x/d = 101$ in Fig. 11.4.

In this region, jet flow is irrotational and outside turbulent core of the jet.

For $\eta \leq 0.15$ ($\sim 1.3l$ from $y = 0$) flow fully turbulent (only occasionally irrotational).

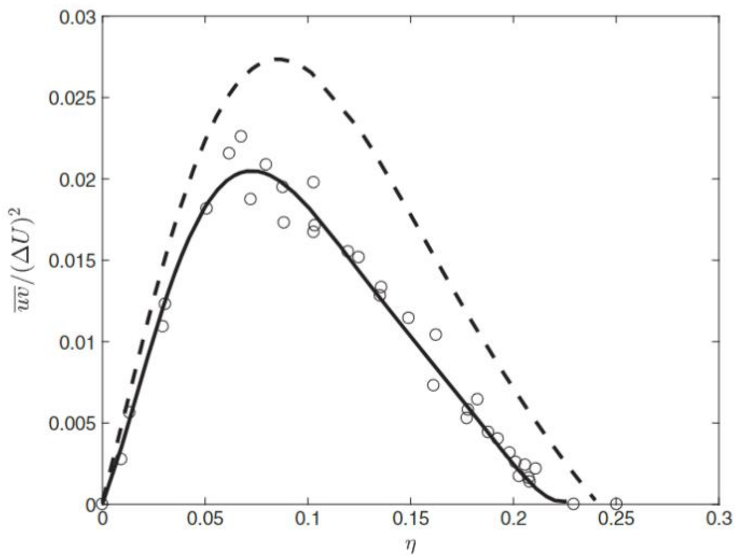


Figure 11.8 Reynolds shear stress distribution for turbulent plane jet at $Re_d = 3.4 \times 10^4$ and $x/d = 101$; ○, data from [13]; —, fit to data; – –, Eq. (11.70).

Good agreement g for $\eta \leq 0.08$, but poor agreement larger η probably due to EFD errors.

$\overline{uv}$ peaks at $\eta \sim 0.07$ ($0.6l$ from $y = 0$) and is 0 at $y = 0$ due to symmetry of mean flow.

$$g = -\frac{\sqrt{\alpha}}{R_t}\tanh\left(\frac{\sqrt{\alpha R_t}}{2}\eta\right)\left[1 - \tanh^2\left(\frac{\sqrt{\alpha R_t}}{2}\eta\right)\right]$$

Obtained from Eq. (14).

$$g'(\eta) = R_t^{-1}F''' \quad (14)$$

$$F'(\eta) = \frac{\overline{U}}{\Delta U} = \left[1 - \tanh^2\left(\frac{\sqrt{\alpha R_t}}{2}\eta\right)\right]$$

Appendix A

A.1

$$\frac{\alpha}{2}\left(F'^2 + FF''\right) + R_t^{-1}F''' = 0 \quad (1A)$$

$$F(0) = 0 \quad (2A)$$

$$F'(0) = 1 \quad (3A)$$

$$\lim_{\eta\to\infty} F'(\eta) = 0 \quad (4A)$$

$$\lim_{\eta\to\infty} F''(\eta) = 0 \quad (5A)$$

$$F'^2 + FF'' = \frac{d}{d\eta}(FF')$$

Such that Eq. (1A) becomes:

$$\frac{\alpha}{2}\frac{d}{d\eta}(FF') = -R_t^{-1}F'''$$

Integrating with respect to η:

$$\frac{\alpha}{2}FF' = -R_t^{-1}F'' + C \quad (6A)$$

Application of BCs Eq. (4A) and (5A) into (6A) gives:

$$0 = -R_t^{-1}F''(\infty) + C$$

$$C = R_t^{-1}F''(\infty) = 0$$

The term on the LHS can be rewritten as:

$$FF' = \frac{d}{d\eta}\left(\frac{1}{2}F^2\right)$$

And Eq. (6A) becomes:

$$\frac{\alpha}{2}\frac{d}{d\eta}\left(\frac{1}{2}F^2\right) = -R_t^{-1}F''$$

Integrating with respect to η:

$$\frac{\alpha}{4}F^2 = -R_t^{-1}F' + D$$

$$F^2 = -\frac{4}{\alpha R_t}F' + \frac{4D}{\alpha} \quad (7A)$$

Applying BCs in Eqs. (2A) and (3A) to Eq. (7A) gives:

$$F(0)^2 = -\frac{4}{\alpha R_t}F'(0) + \frac{4D}{\alpha}$$

$$\frac{4}{\alpha R_t} = \frac{4D}{\alpha}$$

$$D = \frac{1}{R_t}$$

$$F^2 + \frac{4}{\alpha R_t}(F' - 1) = 0$$

A.2

We start from the simplified form of the velocity profile:

$$\overline{U} = \Delta U \left(1 - \tanh^2 \eta\right) \tag{18}$$

where the similarity variable is defined as:

$$\eta = \frac{yR_t}{4(x - x_0)}. = \frac{y}{l}.$$

and $l = \frac{4(x-x_0)}{R_t}$ is a local length scale.

The velocity scale ΔU is assumed to decay with downstream distance:

$$\Delta U = C(x - x_0)^{-1/2}$$

Substituting into Eq. (18):

$$\overline{U} = C(x - x_0)^{-1/2} \left(1 - \tanh^2 \eta\right)$$

[T]his shows that the velocity profile depends on both x and η, and that C acts [a]s an amplitude scaling factor.

To relate C to physical quantities, we use the momentum flux M, which is [c]onserved:

$$M = \rho \int_{-\infty}^{\infty} \overline{U}^2 \, dy$$

Using the change of variable $y = l\eta \Rightarrow dy = l \, d\eta$:

$$M = \rho \int_{-\infty}^{\infty} \overline{U}^2 \, l \, d\eta$$

Substitute the expression for $\overline{U}$:

$$\overline{U} = C(x - x_0)^{-1/2}(1 - \tanh^2 \eta) \quad \Rightarrow \quad \overline{U}^2 = C^2(x - x_0)^{-1}(1 - \tanh^2 \eta)^2$$

Now the integral becomes:

$$\begin{aligned} M &= \rho \int_{-\infty}^{\infty} C^2(x - x_0)^{-1}(1 - \tanh^2 \eta)^2 \cdot l \, d\eta \\ &= \rho C^2 (x - x_0)^{-1} \cdot \frac{4(x - x_0)}{R_t} \int_{-\infty}^{\infty} (1 - \tanh^2 \eta)^2 d\eta \\ &= \rho C^2 \cdot \frac{4}{R_t} \cdot \int_{-\infty}^{\infty} \operatorname{sech}^4 \eta \, d\eta \end{aligned}$$

Using the standard integral:

$$\int_{-\infty}^{\infty} \operatorname{sech}^4\eta \, d\eta = \frac{4}{3}.$$

we get:

$$M = \rho C^2 \cdot \frac{4}{R_t} \cdot \frac{4}{3} = \rho C^2 \cdot \frac{16}{3R_t}.$$

Solving for C:

$$C^2 = \frac{3MR_t}{16\rho}. \quad \Rightarrow \quad C = \sqrt{\frac{3MR_t}{16\rho}}.$$

Final Result

$$\overline{U} = C(x - x_0)^{-1/2} \left(1 - \tanh^2 \eta\right)$$
$$C = \sqrt{\frac{3MR_t}{16\rho}}$$