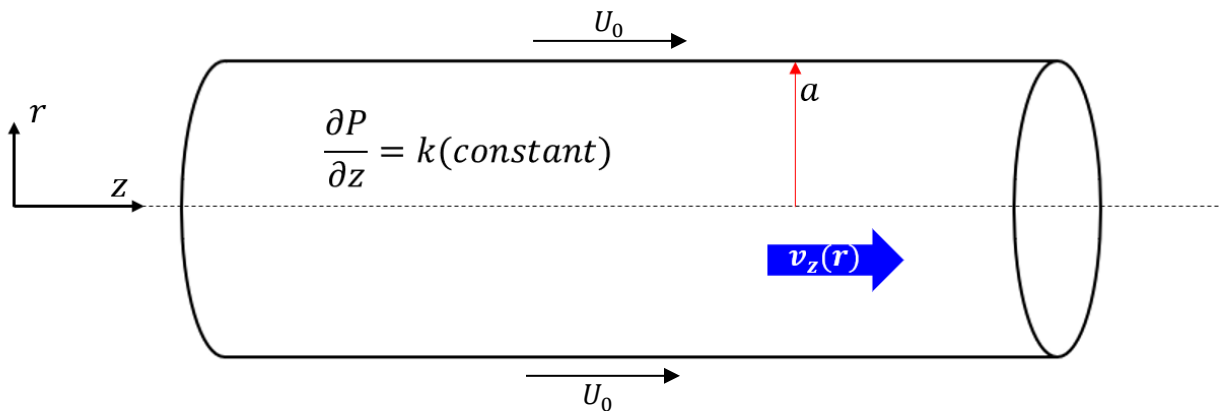


The exam is closed book and closed notes.

The viscous oil in below Figure is set into steady motion by a constant pressure gradient  $\frac{\partial P}{\partial z}$ . The pipe radius is  $a$  and the pipe moves with constant velocity  $U_0$ . Assuming fully developed flow, constant density, circumferentially symmetric flow, and a purely axial fluid motion: (a) Simplify the governing equations (continuity and momentum) with these given conditions. (b) Apply appropriate boundary condition and derive the fluid velocity distribution  $v_z(r)$ . (c) Calculate wall shear stress at pipe wall.



The equations of motion of an incompressible Newtonian fluid with constant density and viscosity in cylindrical coordinates  $(r, \theta, z)$  with velocity components  $(v_r, v_\theta, v_z)$ :

Continuity:

$$\frac{1}{r} \frac{\partial}{\partial r}(r v_r) + \frac{1}{r} \frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0$$

r-momentum:

$$\rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} \right) = \rho g_r - \frac{\partial p}{\partial r} + \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r}(r v_r) \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \right]$$

$\theta$ -momentum:

$$\rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} \right) = \rho g_\theta - \frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r}(r v_\theta) \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right]$$

z-momentum:

$$\rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = \rho g_z - \frac{\partial p}{\partial z} + \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right]$$

Boundary condition Hint

- At the pipe wall, the velocity is  $U_0$
- At the pipe center, the velocity gradient should be zero

Wall shear stress Hint

- $\tau_{wall} = \mu \frac{\partial v_z}{\partial y} \Big|_{y=0} = -\mu \frac{\partial v_z}{\partial r} \Big|_{r=a}$ ,  $y = a - r$  where  $a$ : Radius of pipe

**Solution:**

## ASSUMPTIONS:

1. Steady flow ( $\frac{\partial}{\partial t}=0$ )
2. Incompressible flow ( $\rho=\text{constant}$ )
3. Purely axial flow ( $v_r=v_\theta=0$ ) (+3)
4. Circumferentially symmetric flow, so properties do not vary with  $\theta$  ( $\frac{\partial}{\partial \theta}=0$ )
5. Constant pressure gradient ( $\partial p / \partial z = k$ )
6. Horizontal cylinders ( $g_z=0$ )

(a)

Continuity:

$$\frac{1}{r} \frac{\partial}{\partial r} (r v_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (v_\theta) + \frac{\partial}{\partial z} (v_z) = 0$$

$$0(3) + 0(3) + \frac{\partial v_z}{\partial z} = 0 \quad \text{(+1)}$$

z-momentum:

$$\rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = \rho g_z - \frac{\partial p}{\partial z} + \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] \quad \text{(+1.5)}$$

$$\rho(0(1) + 0(3) + 0(3,4) + 0(\text{continuity})) = 0(6) - k(5) + \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + 0(4) + 0(\text{continuity}) \right]$$

$$\frac{\mu}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) = k \quad \text{(+1)}$$

(b) Integrate

$$\frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) = \frac{k}{\mu} r$$

$$r \frac{\partial v_z}{\partial r} = \frac{k}{2\mu} r^2 + C_1$$

$$\frac{\partial v_z}{\partial r} = \frac{k}{2\mu} r + \frac{C_1}{r}$$

$$\therefore v_z(r) = \frac{k}{4\mu} r^2 + C_1 \ln(r) + C_2 \quad \text{(+1)}$$

Apply two boundary conditions

$$v_z(a) = U_0 \rightarrow \frac{k}{4\mu} a^2 + C_1 \ln(a) + C_2 = U_0 \quad (+0.5)$$

$$\left. \frac{\partial v_z}{\partial r} \right|_{r=0} = 0 \rightarrow \frac{k}{2\mu} (0) + \frac{C_1}{(0)} = 0 \quad (+0.5)$$

$$\therefore C_1 = 0$$

$$\therefore C_2 = U_0 - \frac{k}{4\mu} a^2$$

Hence,

$$\therefore v_z(r) = U_0 + \frac{k}{4\mu} r^2 - \frac{k}{4\mu} a^2 = U_0 + \frac{k}{4\mu} (r^2 - a^2) = U_0 + \frac{1}{4\mu} \frac{\partial P}{\partial z} (r^2 - a^2) \quad (+1)$$

(c) Wall shear stress at the pipe wall

$$\tau_{wall} = -\mu \left. \frac{\partial v_z}{\partial r} \right|_{r=a}$$

$$\mu \frac{\partial v_z}{\partial r} = -\frac{k}{2} r$$

Apply

$$r = a$$

$$\therefore \tau_{wall} = -\frac{ka}{2} = -\frac{a}{2} \frac{\partial P}{\partial z} \quad (+0.5)$$