

The exam is closed book and closed notes.

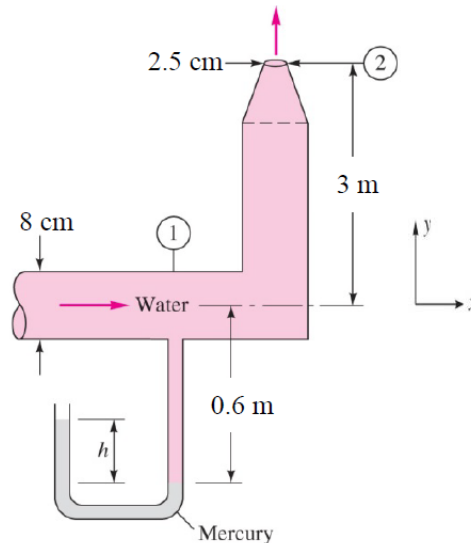
1. In the Figure below the bend is flanged at section 1 (the flange is not shown) and the flow exits to atmosphere at section 2. If $V_1 = 0.5 \text{ m/s}$, $h = 40 \text{ cm}$, $\rho_{\text{water}} = 998 \text{ kg/m}^3$, and $\rho_{\text{mercury}} = 13,550 \text{ kg/m}^3$, neglecting the gravity forces and assuming uniform flows at 1 and 2, find: a) V_2 using continuity equation; b) p_1 using manometry equation; c) the force components on the flange bolts in x and y directions using linear momentum equations; d) the friction head loss between 1 and 2 using energy equation.

Continuity equation: $-\frac{d}{dt} \int_{CV} \rho dV = \int_{CS} \rho \underline{V}_R \cdot \underline{n} dA$

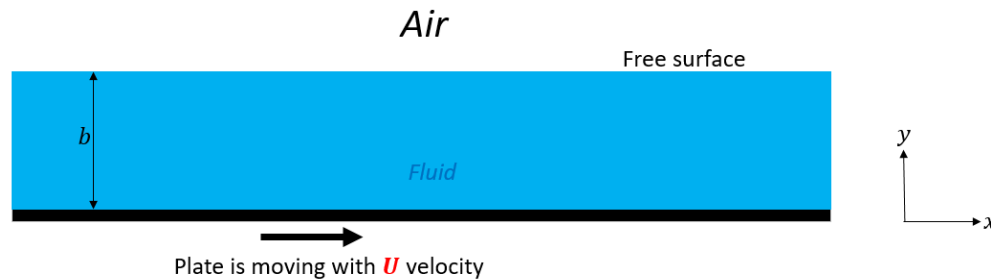
Momentum equation: $\sum \underline{F} = \frac{d}{dt} \int_{CV} \rho \underline{V} dV + \int_{CS} \rho \underline{V} \underline{V}_R \cdot \underline{n} dA$

Energy Equation (for incompressible steady flow):

$$\left(\frac{p}{\rho g} + \frac{\alpha V^2}{2g} + z \right)_1 = \left(\frac{p}{\rho g} + \frac{\alpha V^2}{2g} + z \right)_2 + h_{\text{turbine}} - h_{\text{pump}} + h_{\text{friction}}$$



2. The viscous, incompressible flow between the parallel plates shown in Figure is caused by the motion of the bottom plate. Assuming pressure gradient along the x direction is constant ($\frac{\partial p}{\partial x} = C$), steady, 2D, and parallel flow and using differential analysis: (a) Show that the flow is fully developed using continuity equation; (b) Find the velocity profile $u(y)$ using Navier-Stokes equations with appropriate boundary conditions; (c) Find wall shear stress at bottom wall; and (d) Find the flow rate (hint: $Q = \int \underline{V} \cdot \underline{n} dA$ and assume constant width w). **Explicitly state all assumptions.**



Incompressible Continuity Equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Incompressible Navier-Stokes Equations in Cartesian Coordinates:

$$\rho g_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right)$$

$$\rho g_y - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) = \rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right)$$

$$\rho g_z - \frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) = \rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right)$$

Shear stress (At free surface, shear stress is zero)

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

3. The power P developed by a wind turbine is a function of turbine's diameter D , air density ρ , wind speed V , and rotation rate ω . Namely it can be written as $P = f(D, \rho, V, \omega)$. Viscosity effects are negligible. Rewrite this relationship in dimensionless form.

Quantity	Symbol	Dimensions	
		$MLT\Theta$	$FLT\Theta$
Length	L	L	L
Area	A	L^2	L^2
Volume	$\mathcal{V}$	L^3	L^3
Velocity	V	LT^{-1}	LT^{-1}
Acceleration	dV/dt	LT^{-2}	LT^{-2}
Speed of sound	a	LT^{-1}	LT^{-1}
Volume flow	Q	L^3T^{-1}	L^3T^{-1}
Mass flow	$\dot{m}$	MT^{-1}	FTL^{-1}
Pressure, stress	p, σ, τ	$ML^{-1}T^{-2}$	FL^{-2}
Strain rate	$\dot{\epsilon}$	T^{-1}	T^{-1}
Angle	θ	None	None
Angular velocity	ω, Ω	T^{-1}	T^{-1}
Viscosity	μ	$ML^{-1}T^{-1}$	FTL^{-2}
Kinematic viscosity	ν	L^2T^{-1}	L^2T^{-1}
Surface tension	Υ	MT^{-2}	FL^{-1}
Force	F	MLT^{-2}	F
Moment, torque	M	ML^2T^{-2}	FL
Power	P	ML^2T^{-3}	FLT^{-1}
Work, energy	W, E	ML^2T^{-2}	FL
Density	ρ	ML^{-3}	FT^2L^{-4}
Temperature	T	Θ	Θ
Specific heat	c_p, c_v	$L^2T^{-2}\Theta^{-1}$	$L^2T^{-2}\Theta^{-1}$
Specific weight	γ	$ML^{-2}T^{-2}$	FL^{-3}
Thermal conductivity	k	$MLT^{-3}\Theta^{-1}$	$FT^{-1}\Theta^{-1}$
Thermal expansion coefficient	β	Θ^{-1}	Θ^{-1}

Solution 1:

a) Continuity:

$$A_1 V_1 = A_2 V_2$$

$$V_2 = \frac{A_1 V_1}{A_2} = \frac{(\pi/4) D_1^2 V_1}{(\pi/4) D_2^2} = \frac{D_1^2 V_1}{D_2^2} = \frac{(0.08)^2 (0.5)}{(0.025)^2} = 5.12 \text{ m/s}$$

+3.5

b) Manometry:

$$p_1 + \rho_{\text{water}} g (0.6) - \rho_{\text{mercury}} g h = p_a = 0$$

$$p_1 = \rho_{\text{mercury}} g h - \rho_{\text{water}} g (0.6)$$

$$= (13550)(9.81)(0.4) - (998)(9.81)(0.6) = 47,296 \text{ pa (gage)}$$

+3

c) x – momentum:

$$\sum F_x = \dot{m} u_2 - \dot{m} u_1$$

$$-F_{x, \text{bolts}} + p_1 A_1 = \rho Q (0 - V_1)$$

$$F_{x, \text{bolts}} = p_1 A_1 + \rho A_1 V_1 (V_1) = A_1 (p_1 + \rho V_1^2)$$

$$F_{x, \text{bolts}} = \left[\frac{\pi}{4} (0.08)^2 \right] [(47296) + (998)(0.5)^2] = 239 \text{ N}$$

+2

y – momentum:

$$\sum F_y = \dot{m} v_2 - \dot{m} v_1$$

$$-F_{y, \text{bolts}} = \rho Q (V_2 - 0)$$

$$F_{y, \text{bolts}} = -\rho A_2 V_2^2 = -(998) \left[\frac{\pi}{4} (0.025)^2 \right] (5.12)^2 = -12.84 \text{ N}$$

+1

d) Energy equation:

$$\frac{p_1}{\rho g} + \frac{\alpha_1 V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{\alpha_2 V_2^2}{2g} + z_2 + h_f$$

$$\frac{(47296)}{(998)(9.81)} + \frac{(1.0)(0.5)^2}{2(9.81)} + 0 = 0 + \frac{(1.0)(5.12)^2}{2(9.81)} + (3.0) + h_f$$

$$h_f = 0.51 \text{ m}$$

+0.5

Solution 2:KNOWN: *Flow condition, Boundary condition*

FIND: (2)

- Show that the flow is fully developed
- velocity profile $u(y)$
- Find wall shear stress at bottom wall
- Find the flow rate

ASSUMPTIONS:

- steady: $\frac{\partial}{\partial t} = 0$
- 2D flow: $w = 0$; $\frac{\partial}{\partial z} = 0$
- Parallel flow: $v = w = 0$
- $\frac{\partial p}{\partial x} = \text{constant}$

ANALYSIS:

(a) Continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} + 0 + 0 = 0 \rightarrow \frac{\partial u}{\partial x} = 0 \rightarrow \text{fully developed} \quad (2)$$

(b) x-momentum equation

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = \rho g_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

$$0 = -\frac{\partial P}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu} \frac{\partial p}{\partial x} \quad (1.5)$$

Integrate twice

$$u(y) = \frac{1}{2\mu} \frac{\partial p}{\partial x} y^2 + c_1 y + c_2 \quad (1.5)$$

Apply boundary conditions

$$y = 0 \rightarrow u = U \rightarrow c_2 = U$$

$$y = b \rightarrow \tau = 0 \rightarrow \mu \left(\frac{\partial u}{\partial y} \right) = 0 \rightarrow \mu \left(\frac{1}{\mu} \left(\frac{\partial p}{\partial x} \right) b + c_1 \right) = 0 \quad (2)$$

$$c_1 = -\frac{1}{\mu} \left(\frac{\partial p}{\partial x} \right) b$$

Therefore,

$$u(y) = \frac{1}{2\mu} \frac{\partial p}{\partial x} y^2 - \frac{b}{\mu} \left(\frac{\partial p}{\partial x} \right) y + U$$

$$u(y) = \frac{1}{2\mu} \left(\frac{\partial p}{\partial x} \right) (y^2 - 2by) + U$$

(c) Shear stress at bottom wall

$$\tau = \mu \frac{du}{dy} = \mu \left\{ \frac{1}{2\mu} \left(\frac{\partial p}{\partial x} \right) (2y - 2b) \right\} = \left(\frac{\partial p}{\partial x} \right) (y - b)$$

$$\tau_{wall} = \tau(0) = -b \left(\frac{\partial p}{\partial x} \right) \quad (0.5)$$

(d) Flow rate

$$Q = \int \underline{V} \cdot \underline{n} dA = w \int_0^b u(y) dy = \frac{w}{2\mu} \left(\frac{\partial p}{\partial x} \right) \left(\frac{1}{3} y^3 - by^2 \right) \Big|_0^b + wU(y) \Big|_0^b$$

$$Q = \frac{w}{2\mu} \left(\frac{\partial p}{\partial x} \right) \left(\frac{1}{3} b^3 - b^3 \right) + wUb = -\frac{wb^3}{3\mu} \left(\frac{\partial p}{\partial x} \right) + wUb \quad (0.5)$$

Solution 3:

ASSUMPTIONS: the problem is only a function of the above dimensional variables

ANALYSIS:

$$P = f(D, \rho, V, \omega).$$

$$n = 5$$

$$P = \left\{ \frac{ML^2}{T^3} \right\}, \quad D = \{L\}, \quad \rho = \left\{ \frac{M}{L^3} \right\}, \quad V = \left\{ \frac{L}{T} \right\}, \quad \omega = \left\{ \frac{1}{T} \right\}$$

$$j = 3 \rightarrow k = n - j = 5 - 3 = 2 \quad (4)$$

The repeating variables are D, ρ, V

Calculate first non-dimensional group

$$\Pi_0 = PD^a \rho^b V^c$$

$$\text{unit} [\Pi_0] \rightarrow \left\{ \frac{ML^2}{T^3} \right\} \{L^a\} \left\{ \frac{M^b}{L^{3b}} \right\} \left\{ \frac{L^c}{T^c} \right\} \quad (2)$$

Above one should be non-dimensional

$$\begin{aligned} [M] &\rightarrow 1 + b = 0 \\ [L] &\rightarrow 2 + a - 3b + c = 0 \\ [T] &\rightarrow -3 - c = 0 \\ \therefore b &= -1, c = -3, a = -2 \end{aligned}$$

$$\therefore \Pi_0 = PD^{-2} \rho^{-1} V^{-3} = \frac{P}{\rho D^2 V^3} \quad (1)$$

Calculate second non-dimensional group

$$\Pi_1 = \omega D^a \rho^b V^c$$

$$\text{unit} [\Pi_1] \rightarrow \left\{ \frac{1}{T} \right\} \{L^a\} \left\{ \frac{M^b}{L^{3b}} \right\} \left\{ \frac{L^c}{T^c} \right\} \quad (2)$$

$$\begin{aligned} [M] &\rightarrow b = 0 \\ [L] &\rightarrow a - 3b + c = 0 \\ [T] &\rightarrow -1 - c = 0 \end{aligned}$$

$$\therefore b = 0, c = -1, a = 1$$

$$\therefore \Pi_1 = \omega D^1 \rho^0 V^{-1} = \frac{\omega D}{V}$$

As a result,

$$\Pi_0 = f_{nc}(\Pi_1) \rightarrow \frac{P}{\rho D^2 V^3} = f_{nc} \left(\frac{\omega D}{V} \right) \quad (1)$$