Viscous Flow in Ducts

Laminar Flow Solutions
Entrance, developing, and fully developed flow

bl theory can predict: x_L, excess Δp over Poiseuille law, and shape of the developing profiles

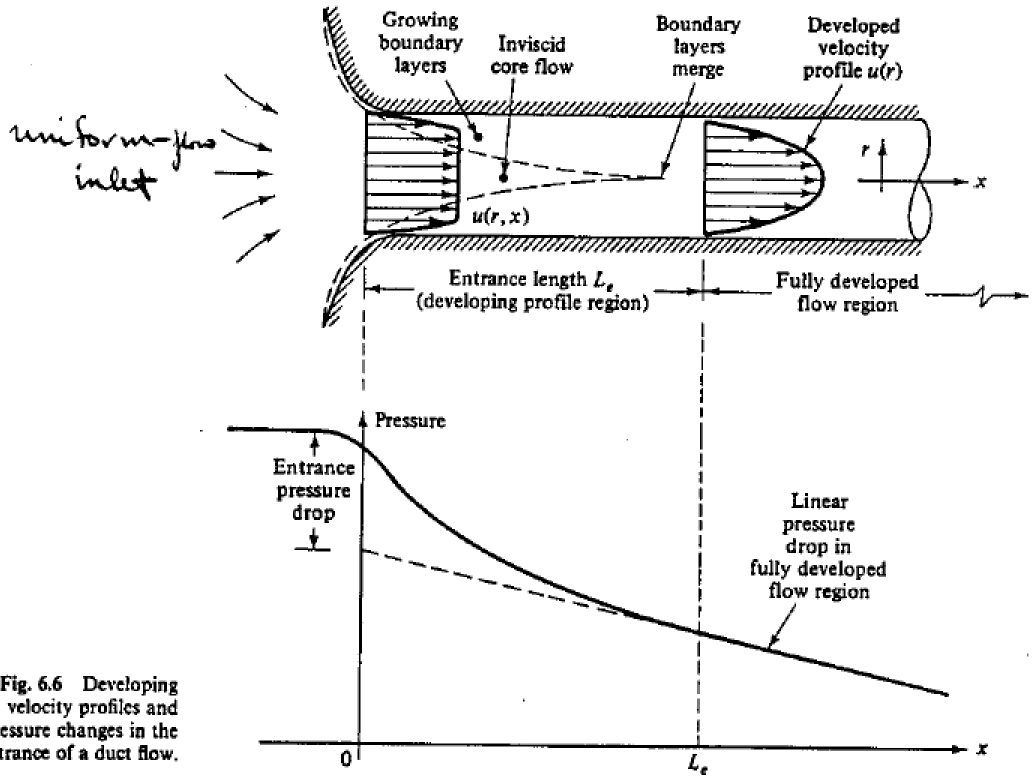


Fig. 6.6 Developing velocity profiles and pressure changes in the entrance of a duct flow.

$$Le = f\,(D, V, \rho, \mu)$$

$$\Pi_i \text{ theorem} \rightarrow L_e/D = f(\text{Re}) \quad f(\text{Re}) \text{ from AFD and EFD}$$

<u>Laminar Flow</u>: $\text{Re}_{crit} \sim 2000$ $\quad$ $\text{Re} < \text{Re}_{crit}$ laminar

$L_e / D \cong .06\,\text{Re}$ $\quad$ $\text{Re} > \text{Re}_{crit}$ unstable

$\text{Re} > \text{Re}_{trans}$ turbulent

$$L_{e\max} = .06\,\text{Re}_{crit}\, D \sim 138D$$

Max L_e for laminar flow

Turbulent flow:

Re	L_e/D
4000	18
10^4	20
10^5	30
10^6	44
10^7	65
10^8	95

$$L_e / D \sim 4.4 \mathrm{Re}^{1/6}$$

(Relatively shorter than for laminar flow)

Laminar vs. Turbulent Flow

Hagen 1839 noted difference in $\Delta p = \Delta p(u)$ but could not explain two regimes

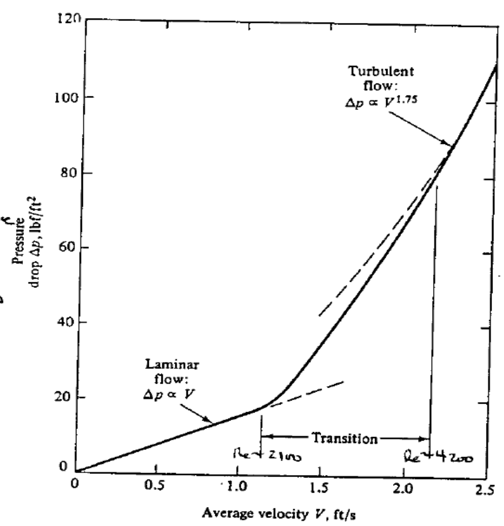
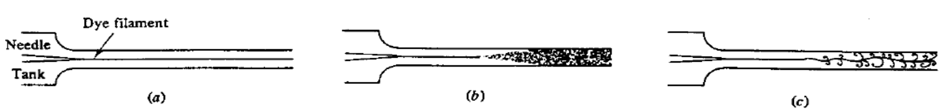


Fig. 6.4 Experimental evidence of transition for water flow in a ½-in smooth pipe 10 ft long.

Hagen 1839 noted difference in $\Delta p=\Delta p(u)$ but could not explain two regimes



laminar turbulent spark photo

Reynolds 1883 showed difference depends on $Re = \frac{VD}{\nu}$

Laminar Turbulent Spark photo

Reynolds 1883 showed that the difference depends on Re = VD/ν

Laminar pipe flow:

1. CV Analysis

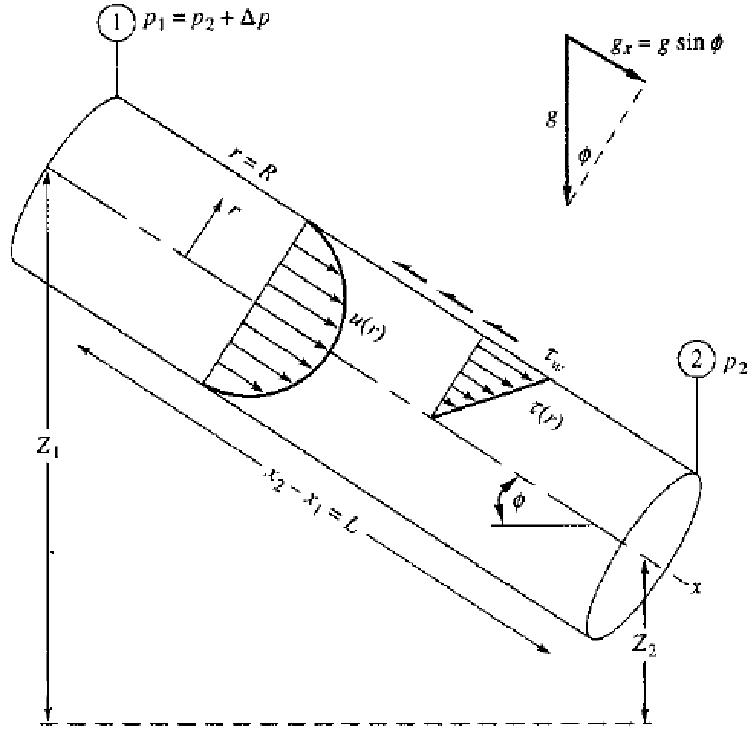


Fig. 6.7 Control volume of steady, fully developed flow between two sections in an inclined pipe.

Continuity:

$$0 = \int_{CS} \rho \underline{V} \cdot \underline{dA} \rightarrow \rho Q_1 = \rho Q_2 = const.$$

$$i.e. V_1 = V_2 \quad \text{since} \quad A_1 = A_2, \rho = const., and\ V = V_{ave}$$

Momentum:

$$\sum F_x = \underbrace{(p_1 - p_2)}_{\Delta p} \pi R^2 - \tau_w 2\pi RL + \underbrace{\gamma\pi R^2 L}_{W} \underbrace{\sin\phi}_{\Delta z/L} = \underbrace{\dot{m}(\beta_2 V_2 - \beta_1 V_1)}_{=0}$$

$$\Delta p \pi R^2 - \tau_w 2\pi RL + \gamma\pi R^2 \Delta z = 0$$

$$\Delta p + \gamma\Delta z = \frac{2\tau_w L}{R}$$

$$\Delta h = h_1 - h_2 = \Delta(p/\gamma + z) = \frac{2\tau_w}{\gamma}\frac{L}{R}$$

or

$$\tau_w = \frac{R\gamma}{2}\frac{\Delta h}{L} = -\frac{R\gamma}{2}\frac{dh}{dx} = -\frac{R}{2}\frac{d}{dx}(p + \gamma z)$$

For fluid particle control volume:

$$\tau = -\frac{r}{2}\frac{d}{dx}(p + \gamma z)$$

i.e., shear stress varies linearly in r across pipe, which is valid for either laminar or turbulent flow.

Energy:

$$\frac{p_1}{\gamma} + \frac{\alpha_1}{2g}V_1 + z_1 = \frac{p_2}{\gamma} + \frac{\alpha_2}{2g}V_2 + z_2 + h_L$$

$$\Delta h = h_L = \frac{2\tau_w}{\gamma}\frac{L}{R}$$

$\therefore$ once τ_w is known, we can determine piezometric pressure $\hat{p} = p + \gamma z$ drop, i.e., $\frac{d}{dx}(p + \gamma z)$.

In general,

roughness

$$\tau_w = \tau_w(\rho, V, \mu, D, \varepsilon)$$

Π Theorem

$$\frac{8\tau_w}{\rho V^2} = f = friction\ \ factor = f(\mathrm{Re}_D, \varepsilon / D)$$

where $\mathrm{Re}_D = \dfrac{VD}{\upsilon}$

$$\Delta h = h_L = f\,\frac{L}{D}\frac{V^2}{2g}$$ Darcy-Weisbach Equation

f (Re_D, ε/D) still needs to be determined. For laminar flow, there is an exact solution for *f* since laminar pipe flow has an exact solution. For turbulent flow, approximate solution for *f* using log-law as per Moody diagram and discussed later.

2. Differential Analysis

Continuity:

$$\nabla \cdot \underline{V} = 0 \qquad \nabla = \frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \theta} + \frac{\partial}{\partial z}$$

Use cylindrical coordinates (r, θ, z) where z replaces x in previous CV analysis.

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial v_z}{\partial z} = 0$$

$$\text{where } \underline{V} = v_r \widehat{e_r} + v_\theta \widehat{e_\theta} + v_z \widehat{e_z}$$

Assume $v_\theta = 0$ i.e. no swirl and fully developed flow $\frac{\partial v_z}{\partial z} = 0$, which shows v_r = constant = 0 since $v_r(R)$ =0

$$\therefore \underline{V} = v_z \widehat{e_z} = u(r)\widehat{e_z}$$

Momentum:

$$\rho \frac{D\underline{V}}{Dt} = \rho \frac{\partial \underline{V}}{\partial t} + \rho \underline{V} \cdot \nabla \underline{V} = -\nabla(p + \gamma z) + \mu \nabla^2 \underline{V}$$

Where:

$$\underline{V} \cdot \nabla = v_r \frac{\partial}{\partial r} + v_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z}$$

z equation:

$$\rho\left[\frac{\partial u}{\partial t}+\underline{V}\cdot\nabla\underline{V}\right]=-\frac{\partial}{\partial z}(p+\gamma z)+\mu\nabla^2 u$$

$$\underline{V}\cdot\nabla\underline{V}=v_r\frac{\partial u}{\partial r}+v_\theta\frac{1}{r}\frac{\partial u}{\partial \theta}+v_z\frac{\partial u}{\partial z}=0$$

$$0=\underbrace{-\frac{\partial}{\partial z}(p+\gamma z)}_{f(z)}+\underbrace{\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)}_{f(r)}$$

$\therefore$ both terms must be constant

$$\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)=\frac{\partial \hat{p}}{\partial z}$$

$$\Rightarrow r\frac{\partial u}{\partial r}=\frac{1}{2\mu}\frac{\partial \hat{p}}{\partial z}r^2+A$$

$$\Rightarrow \frac{\partial u}{\partial r}=\frac{1}{2\mu}\frac{\partial \hat{p}}{\partial z}r+A$$

$$\Rightarrow u=\frac{1}{4\mu}\frac{\partial \hat{p}}{\partial z}r^2+A\ln r+B \qquad \hat{p}=p+\gamma z$$

$u(r=0)$ *finite* → $A=0$

$u\ (r=R)=0$ → $B=-\frac{R^2}{4\mu}\frac{d\hat{p}}{dz}$

$u(r)=\frac{r^2-R^2}{4\mu}\frac{d\hat{p}}{dz}=u_{\max}(1-r^2/R^2)$ $\qquad u_{\max}=u(0)=-\frac{R^2}{4\mu}\frac{d\hat{p}}{dz}$

$$\tau_{rz} = \mu\left[\frac{\partial v_r}{\partial z} + \frac{\partial u}{\partial r}\right] = \mu\frac{\partial u}{\partial r} \quad \textit{fluid shear stress}$$

$$= \frac{r}{2}\frac{\partial \hat{p}}{\partial z} \quad \text{where} \quad \frac{\partial u}{\partial r} = \frac{r}{2\mu}\frac{\partial \hat{p}}{\partial z}$$

$$\tau_w = \mu\frac{\partial u}{\partial y}\bigg|_{y=0} = -\mu\frac{\partial u}{\partial r}\bigg|_{r=R} = -\frac{R}{2}\frac{\partial \hat{p}}{\partial z} \quad \text{As per CV analysis}$$

$$y = R - r,\ \frac{du}{dy} = \frac{dr}{dy}\frac{du}{dr} = -\frac{du}{dr}$$

Note: $\tau = \tau_{rz} = \mu\varepsilon_{rz} = -2\mu\omega_\theta$ (see Appendix D) for $\frac{\partial v_r}{\partial z} = 0$, i.e., only one component of vorticity which also varies linearly across the pipe with its maximum at the wall.

$$Q = \int_0^R u(r)2\pi r\, dr = \frac{-\pi R^4}{8\mu}\frac{d\hat{p}}{dz} = \frac{1}{2}u_{\max}\pi R^2$$

Note: for given piezometric pressure drop the flow rate is inversely proportional to the viscosity and proportional to the radius to the fourth power such that doubling the pipe radius produces 16-fold increase in the flow rate: Poiseuille's law

$$V_{ave} = \frac{Q}{\pi R^2} = \frac{1}{2}u_{\max} = \frac{-R^2}{8\mu}\frac{d\hat{p}}{dz}$$

vs. $V_{ave} = .53u_{max}$

for $u(r)=u_{max}(1-r/R)^{1/2}$

Substituting $V = V_{ave}$

$$f = \frac{8\tau_w}{\rho V^2}$$

$$\tau_w = -\frac{R}{2}\frac{\partial \hat{p}}{\partial z} = -\frac{R}{2} \times \frac{8\mu V_{ave}}{-R^2} = \frac{4\mu V_{ave}}{R} = \frac{8\mu V}{D}$$

Substituting τ_w into f:

$$\boxed{f = \frac{64\mu}{\rho D V} = \frac{64}{\mathrm{Re}_D}}$$

or

$$\boxed{C_f = \frac{\tau_w}{\frac{1}{2}\rho V^2} = \frac{f}{4} = \frac{16}{\mathrm{Re}_D}}$$

$$\Delta h = h_L = f\,\frac{L}{D}\frac{V^2}{2g} = \frac{64\mu}{\rho D V} \times \frac{L}{D} \times \frac{V^2}{2g} = \frac{32\mu L V}{\rho g D^2} \quad \propto V$$

$$\text{for } \Delta z = 0 \quad \rightarrow \quad \Delta p \propto V$$

Both f and C_f based on V^2 normalization, which is appropriate for turbulent but not laminar flow. The more appropriate case for laminar flow is:

$$\boxed{Poiseuille\,\#\;(P_0)\begin{cases} P_{0_{c_f}} = C_f\,\mathrm{Re} = 16 \\ P_{0_f} = f\,\mathrm{Re} = 64 \end{cases}} \qquad \textit{for pipe flow}$$

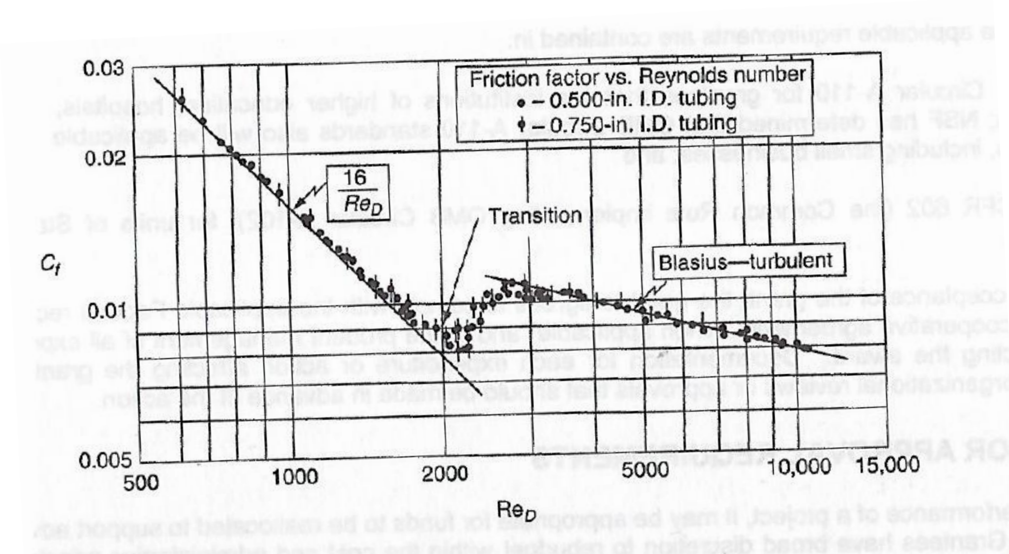
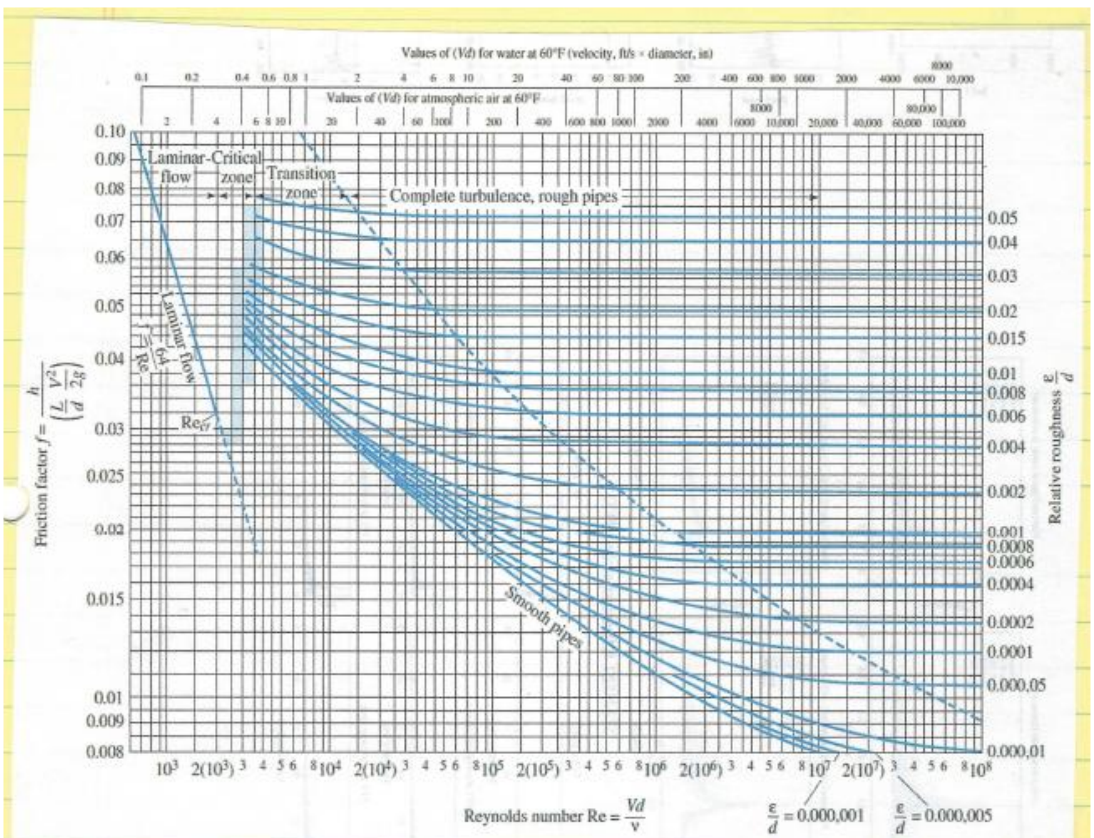


FIGURE 3-7
Comparison of theory and experiment for the friction factor of air flowing in small-bore tubles. [*After Senecal and Rothfus (1953).*]

Blasius power law $C_f = \dfrac{0.0791}{\mathrm{Re}_D^{1/4}}$ (Turbulent flow)



Compare with solution for flow between parallel plates with pressure gradient:

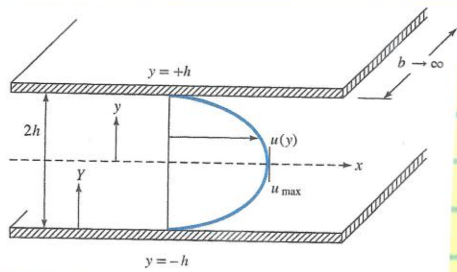


Fig. 6.14 Fully developed flow between parallel plates.

2D constant property flow between parallel plates due pressure gradient dp/dx = constant:

$u = u(y)$, $v = w = \frac{\partial}{\partial z} = 0$; Also steady flow $\frac{\partial}{\partial t} = 0$.

$u_x + v_y + w_z = 0 \Rightarrow u_x = 0$ i.e. fully developed flow

x $\rho(uu_x + vu_y) = -p_x + \mu(u_{xx} + u_{yy})$

y $0 = -p_y$

z $0 = -p_z$ $\Rightarrow p = p(x)$

two BC: $u(\pm h) = 0$

$$\mu u_{yy} = p_x$$

$$\frac{\partial}{\partial y}(u_y) = p_x/\mu$$

$$u_y = \frac{p_x}{\mu} y + c_1$$

$$u(y) = \frac{p_x}{2\mu} y^2 + c_1 y + c_2$$

$$u(-h) = \frac{p_x}{2\mu} h^2 - c_1 h + c_2 = 0$$

$$u(h) = \frac{p_x}{2\mu} h^2 + c_1 h + c_2 = 0$$

add $c_2 = -p_x \frac{h^2}{2\mu}$ $\Rightarrow u(y) = \underbrace{\frac{-dp}{dx}\frac{h^2}{2\mu}}_{u_{max}}\left(1 - \frac{y^2}{h^2}\right)$

subtract $c_1 = 0$

① $\tau_w = \tau_{xy}\Big|_{\substack{wall\\ y=\pm h}} = \mu(u_y + v_x) = -\mu \frac{dp}{dx}\frac{h^2}{2\mu}\left(-\frac{2y}{h^2}\right)\Big|_{y=\pm h}$

$= \pm \frac{dp}{dx} h = \mp \frac{2\mu u_{max}}{h}$

$|\tau_w| = \frac{dp}{dx} h =$ same $y = \pm h$, but + upper and − lower wall

② $u = \psi_y = u_{max}(1 - y^2/h^2)$ $v = -\psi_x = 0$

$\psi = u_{max}\left(y - \frac{y^3}{3h^2}\right)$ $\psi = 0$ $y = 0$

$\psi = \pm 2 u_{max} h/3$ $y = \pm h$

③ $\omega_z = v_x - u_y = \frac{2u_{max}}{h^2} y$ $\nabla \times \underline{V} \neq 0$ ∴ ϕ does not exist i.e. $\underline{V} \neq \nabla\phi$

④ $Q = \int \underline{V}\cdot\underline{n}\, dA = \int u\, dA = \int_{-h}^{h} u_{max}\left(1 - \frac{y^2}{h^2}\right) b\, dy$

$b u_{max}\left[\int_{-h}^{h} dy - \frac{1}{h^2}\int_{-h}^{h} y^2 dy = y - \frac{y^3}{3h^2}\Big|_{-h}^{h}\right.$ $= \frac{4}{3} b h u_{max}$

$b u_{max}\left[\frac{2}{3}h + \frac{2}{3}h\right] = \frac{4}{3} b h u_{max}$ $V_{ave} = Q/A = Q/2hb = \frac{2}{3} u_{max}$

or $Q = \psi_u - \psi_l = \frac{4}{3} u_{max} h$ per unit width

$V_{ave} = Q/A_{b=1} = \frac{2}{3} u_{max}$

Summary:

$$u = u_{max}\left(1 - y^2/h^2\right) \qquad u_{max} = -\frac{dp}{dx}\frac{h^2}{2\mu} \qquad -\frac{dp}{dx} = \frac{\Delta p}{L}$$

$$\Delta p = p_1 - p_2$$

$$Q = \frac{2bh^3}{3\mu}\frac{\Delta p}{L} \qquad\qquad = \frac{h^2}{2\mu}\frac{\Delta p}{L}$$

$$V_{ave} = Q/A = \frac{h^2}{3\mu}\frac{\Delta p}{L} = \frac{2}{3}u_{max}$$

$$\tau_w = \mu\left|\frac{du}{dy}\right|_{y=h} = \frac{\Delta p h}{L} = \frac{3\mu V_{ave}}{h} \qquad V_{ave} = V$$

$$h_f = \frac{\Delta p}{\rho g} = \frac{3\mu V L}{\rho g h^2} \qquad D_h = \frac{4A}{P} = \lim_{b\to\infty}\frac{4(2hb)}{2b+4h}$$

$$= f\frac{L}{D_h}\frac{V^2}{2g} \qquad\qquad = 4h$$

$$f = \frac{h_f}{(L/D_h)\frac{V^2}{2g}} = \frac{h_f D_h 2g}{V^2 L} = \frac{\frac{3\mu VL}{\rho g h^2} D_h 2g}{V^2 L}$$

$$f = 96/Re_{D_h} \qquad\qquad = \frac{3\mu \cancel{VL}\,\overset{4h}{D_h}\,2g}{\rho g h^2 V^2 L}$$

$$c_f = f/4 = 24/Re_{D_h}$$

$$= 6\mu 4h/\rho h^2 V$$

$$Po_{c_f} = Re_{D_h} c_f = 24$$

$$= 24\mu/\rho h V$$

$$Po_f = Re_{D_h} f = 96$$

$$= 96\mu/\rho 4h V$$

Same pipe other than constants $\frac{Po_{pipe}}{Po_{channel}} = 2/3$ $\qquad f = 96/Re_{D_h}$

Concept of hydraulic diameter for noncircular ducts

For noncircular ducts, τ_w= f(perimeter); thus, new definitions of $f = \frac{8\tau_w}{\rho V^2}$ and $C_f = \frac{2\tau_w}{\rho V^2}$ are required.

Define average wall shear stress

$$\bar{\tau}_w = \frac{1}{P}\int_0^P \tau_w \, ds \qquad \text{ds = arc length, P = perimeter}$$

Momentum:

$$\Delta p A - \bar{\tau}_w P L + \underbrace{\gamma A L}_{W}\left(\frac{\Delta z}{L}\right) = 0$$

$$\Delta h = \Delta\left(p/\gamma + z\right) = \frac{\bar{\tau}_w L}{\gamma A / P}$$

A/P =R_h= Hydraulic radius (=R/2 for circular pipe and $\Delta h = \frac{\tau_w L}{\gamma R/2}$)

Energy:

$$\Delta h = h_L = \frac{\bar{\tau}_w L}{\gamma A / P}$$

$$\bar{\tau}_w = \frac{A}{P}\frac{\Delta h \gamma}{L} = \frac{-A\gamma}{P}\frac{dh}{dx} = \frac{-A}{P}\frac{d(p+\gamma z)}{dx} = \frac{A}{P}\left(-\frac{d\hat{p}}{dx}\right)$$ non-circular duct

Recall for circular pipe:

$$\tau_w = -\frac{R}{2}\frac{d\hat{p}}{dx} = -\frac{D}{4}\frac{d\hat{p}}{dx}$$

In analogy to circular pipe:

$$\bar{\tau}_w = \frac{A}{P}\left(-\frac{d\hat{p}}{dx}\right) = \frac{D_h}{4}\left(-\frac{d\hat{p}}{dx}\right) \Rightarrow \frac{A}{P} = \frac{D_h}{4} \Rightarrow D_h = \frac{4A}{P}$$ Hydraulic diameter

For multiple surfaces such as concentric annulus P and A based on wetted perimeter and area

$$\bar{f} = \frac{8\bar{\tau}_w}{\rho V^2} = \bar{f}(Re_{D_h}, \varepsilon / D_h) \qquad Re_{D_h} = \frac{V D_h}{\upsilon}$$

$$\Delta h = h_L = \frac{\bar{\tau}_w L}{\gamma R_h} = \frac{\rho V^2 \bar{f}}{8}\frac{L}{\gamma R_h} = \bar{f}\frac{L}{D_h}\frac{V^2}{2g}$$

However, accuracy not good for laminar flow $\bar{f} = 64/Re_{D_h}$ (about 40% error) and marginal turbulent flow $\bar{f}(Re_{D_h}, \varepsilon/D_h)$ (about 15% error).

a. *Accuracy for laminar flow (smooth non-circular pipe)*

Recall for pipe flow:

$$\boxed{Poiseuille\,\#\,(P_0)\begin{cases} P_{0c_f} = C_f \,\mathrm{Re} = 16 \\ P_{0f} = f\,\mathrm{Re} = 64 \end{cases}}$$

Recall for channel flow:

$$\boxed{f = \frac{24\mu}{\rho V h} = \frac{48}{\mathrm{Re}_{2h}} = \frac{96}{\underbrace{\mathrm{Re}_{4h}}_{\mathrm{Re}_{D_h}}}}$$

$$\boxed{\begin{aligned} &C_f = f/4 \Rightarrow \\ &C_f = \frac{6\mu}{\rho V h} = \frac{12}{\mathrm{Re}_{2h}} = \frac{24}{\underbrace{\mathrm{Re}_{4h}}_{\mathrm{Re}_{D_h}}} \end{aligned}}$$

$$\boxed{Poiseuille\,\#\,(P_0)\begin{cases} P_{0c_f} = C_f \,\mathrm{Re}_{D_h} = 24 \\ P_{0f} = f\,\mathrm{Re}_{D_h} = 96 \end{cases}}$$

Therefore:

$$\boxed{\frac{P_{0c_f\ pipe}}{P_{0c_f\ channel\,based\,on\,D_h}} = \frac{P_{0\,f\ pipe}}{P_{0\,f\ channel\,based\,on\,D_h}} = \frac{16}{24} = \frac{64}{96} = \frac{2}{3}}$$

Thus, if we could not work out the laminar theory and chose to use the approximation $f\,\mathrm{Re}_{D_h} \approx 64$ or $C_f\,\mathrm{Re}_{D_h} \approx 16$, we would be 33 percent low for channel flow.

b. Accuracy for turbulent flow (smooth non-circular pipe)

For turbulent flow, D_h works much better especially if combined with "effective diameter" concept based on ratio of exact laminar circular and noncircular duct P_0 numbers, i.e., $16/\overline{P}_{0C_f}$ or $64/\overline{P}_{0f}$.

First recall turbulent circular pipe solution and compare with turbulent channel flow solution using log-law in both cases

Channel Flow

$$V = \frac{1}{h}\int_0^h u^* \left[\frac{1}{\kappa} \ln \frac{(h-y)u^*}{\upsilon} + B \right] dY \quad \text{Y=h-y} \quad \text{wall coordinate}$$

$$= u^* \left(\frac{1}{\kappa} \ln \frac{hu^*}{\upsilon} + B - \frac{1}{\kappa} \right)$$

$$D_h = \frac{4A}{P} = \lim_{B\to\infty} \frac{4(2hB)}{2B+4h} = 4h \quad \text{h= half width}$$

Define $\mathrm{Re}_{D_h} = \dfrac{VD_h}{\upsilon} = \dfrac{V4h}{\upsilon}$

$$f^{-1/2} = 2\log\left(\mathrm{Re}_{D_h} f^{1/2}\right) - 1.19 \text{ (Using } D_h\text{)}$$

Very nearly the same as circular pipe
7% to large at Re = 10^5
4% to large at Re = 10^8

Therefore, error in D_h concept relatively smaller for turbulent flow.

Note $$f^{-1/2}(channel) = 2\log\left(0.64\,\mathrm{Re}_{D_h} f^{1/2}\right) - 0.8$$

Rewriting such that exact agreement pipe flow with Re_D replaced by $0.64\mathrm{Re}_{Dh}$

Define $D_{effective}$ $$= 0.64 D_h \sim \underbrace{\frac{P_{0f}(circle) = 16}{P_{0f}(channel) = 24}}_{\text{Laminar solution}} D_h$$

(therefore, improvement on D_h is)

$$\mathrm{Re}_{D_{eff}} = \frac{VD_{eff}}{\upsilon}$$

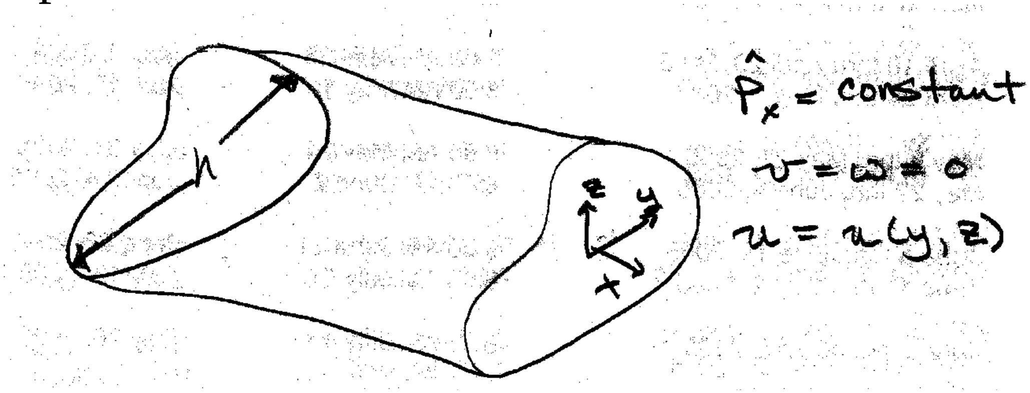
$$D_{eff} = \frac{P_{0f}(circle)}{P_{0f}(non-circular)} D_h = \frac{P_{0C_f}(circle)}{P_{0C_f}(non-circular)} D_h$$

Or

$$\boxed{D_{eff} = \frac{64}{P_{0f}(non-circular)} D_h = \frac{16}{P_{0C_f}(non-circular)} D_h}$$

From exact laminar solution

Non-Circular Ducts: Exact laminar solutions are available for any "arbitrary" cross section for laminar steady fully developed duct flow.



$$u_x = 0$$

$$0 = -\hat{p}_x + \mu(u_{YY} + u_{ZZ})$$

$$u(h) = 0$$

Re only enters through stability and transition

$$y^* = y/h \qquad z^* = z/h \qquad u^* = u/U \qquad U = \frac{h^2}{\mu}(-\hat{p}_x)$$

Related u_{max}

$$\nabla^2 u = -1 \qquad \textit{Poisson equation}$$

$$u(1) = 0 \qquad \textit{Dirichlet boundary condition}$$

Note: No characteristic velocity and length scale for fully developed flow therefore use characteristic duct width h and U with units' L/T formed from μ, h and $\hat{p}_x$ using dimensional analysis. The pressure force/∀ $(-\hat{p}_x)$ is balanced by the net viscous force/∀ $(\mu U/h^2)$ and their nondimensional ratio provides measure for U = u_{max}.

BVP can be solved by many methods such as complex variables and conformal mapping, transformation into Laplace equation by redefinition of dependent variables, and numerical methods.

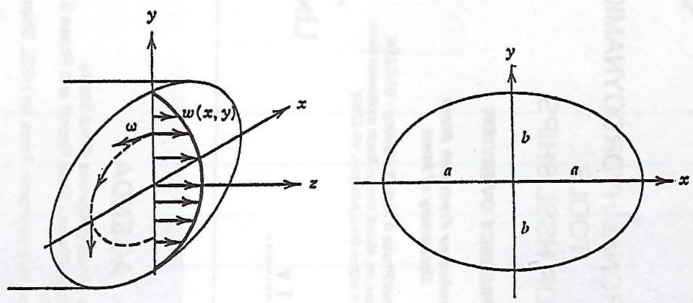
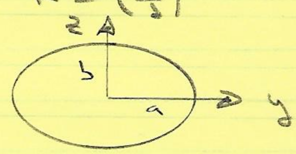


Figure 11.1 Parallel flow in an elliptical tube.

Elliptical Cross Section

The equation for the wall location is

$$y_w^2 + K z_w^2 = 1 \qquad K = \left(\frac{a}{b}\right)^2$$



& $h = a$, i.e., $y^* = y/a$ etc.
* in previous equations

One approach to the solution of Poisson's equation is to transform it into the Laplace equation by redefining the dependent variables, i.e.

$$v = u + c_1 y^2 + c_2 z^2$$

$$\nabla^2 v = \nabla^2 u + 2(c_1 + c_2)$$

$\therefore$ choose $2(c_1 + c_2) = 1$ recall $\nabla^2 u = -1$

The values of c_1 + c_2 are fixed by considering the b.c.

$$v(\text{wall}) = c_1 y_w^2 + c_2 z_w^2 = c_1\left[y_w^2 + \frac{c_2}{c_1} z_w^2\right]$$

$$\upsilon(\text{wall}) = \text{Constant} = c_1$$

if $c_2/c_1 = K$ (by comparison $y_w^2 + Kz_w^2 = 1$)

$$\Rightarrow \upsilon(\text{wall}) = c_1 \quad \& \quad c_1 = \frac{1}{2(1+K)} \quad c_2 = \frac{K}{2(1+K)}$$

$2(c_1+c_2)=1$
$c_2/c_1 = K$

$$\left.\begin{array}{l}\nabla^2\upsilon = 0\\ \upsilon(\text{wall}) = c_1\end{array}\right\}\ \text{new problem to be solved}$$

Since, the maximum and the minimum values of the solution of the Laplace equation must occur on the boundary

$\alpha \le \upsilon \le \beta$
α & β must be on bdry
$\upsilon = c_1 =$ Const on wall
so $\alpha=\beta=c_1=\upsilon$
ie everywhere

$$\upsilon = c_1 \qquad \upsilon = u + c_1y^2 + c_2z^2 = c_1$$

& definitions c_1 & c_2

$$u = \frac{1}{2(1+K)}\left(1 - y^2 - Kz^2\right)$$

The isovels are ellipses which are confocal with the wall ellipse. The vorticity components are

$$\omega_z = \frac{1}{K+1}y \qquad \omega_y = -\frac{K}{K+1}z$$

$$|\omega| = \frac{1}{K+1}\left(y^2 + Kz^2\right)^{1/2} = \text{constant on ellipses}$$

confocal with the wall, ie, vortex lines are ellipses

$$\frac{Q}{-\frac{a^4}{\mu}\hat{p}_x} = \frac{\pi}{4}\,\frac{1}{K^{1/2}(K+1)}$$

Note the not parameter of K

All duct flow have $Q = C a^4(-d\hat{p}/dz)$ flow rate pressure drop relation where C depends on cross section shape. For circular pipe $K=1$ & $C = \pi/8 = .3926$

Other solutions are given in the text for rectangular, equilateral triangle, circular sector, and concentric annulus sections.

← formula for Q used in viscometry to calculate μ

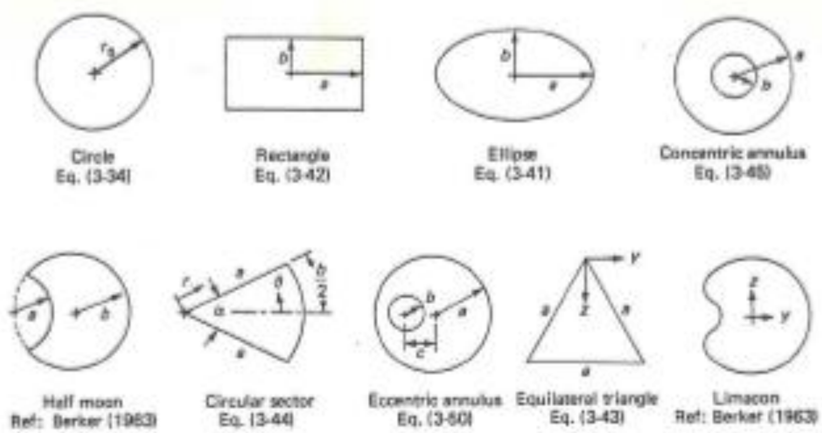


FIGURE 3-7
Some cross sections for which fully developed flow solutions are known; for still more, consult Berker (1963, pp. 67ff.).

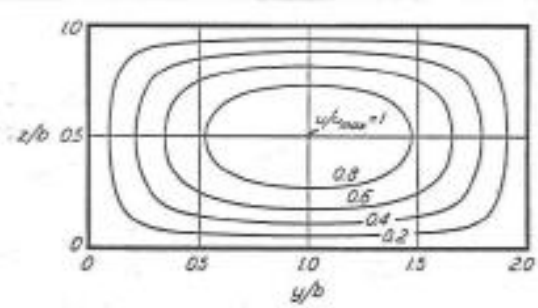


FIG. 77. Velocity distribution in a rectangular conduit.

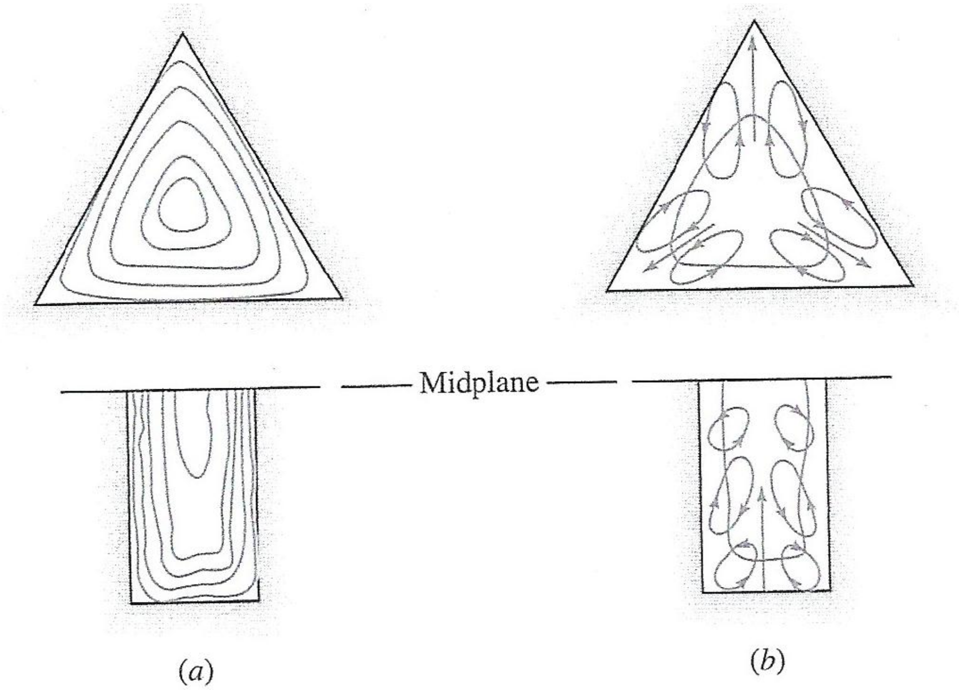


Fig. 6.16 Illustration of secondary turbulent flow in noncircular ducts: (*a*) axial mean velocity contours; (*b*) secondary flow in-plane cellular motions. *(After J. Nikuradse, dissertation, Göttingen, 1926.)*

For rectangular and triangular ducts, for laminar flow τ_w largest mid-points of the sides and zero in corners, whereas for turbulent flow τ_w nearly constant along the sides and falls sharply to zero in the corners due to secondary flows induced by the turbulence anisotropy. For laminar flows in straight ducts secondary flows are absent. As a result the hydraulic diameter concept works poorly for laminar vs. turbulent flow.

Elliptical section: $y^2/a^2 + z^2/b^2 \leq 1$:

$$u(y,z) = \frac{1}{2\mu}\left(-\frac{d\hat{p}}{dx}\right)\frac{a^2b^2}{a^2+b^2}\left(1-\frac{y^2}{a^2}-\frac{z^2}{b^2}\right)$$

$$Q = \frac{\pi}{4\mu}\left(-\frac{d\hat{p}}{dx}\right)\frac{a^3b^3}{a^2+b^2} \tag{3-47}$$

Rectangular section: $-a \leq y \leq a, -b \leq z \leq b$:

$$u(y,z) = \frac{16a^2}{\mu\pi^3}\left(-\frac{d\hat{p}}{dx}\right)\sum_{i=1,3,5,\ldots}^{\infty}(-1)^{(i-1)/2}\left[1-\frac{\cosh(i\pi z/2a)}{\cosh(i\pi b/2a)}\right] \times \frac{\cos(i\pi y/2a)}{i^3} \tag{3-48}$$

$$Q = \frac{4ba^3}{3\mu}\left(-\frac{d\hat{p}}{dx}\right)\left[1-\frac{192a}{\pi^5 b}\sum_{i=1,3,5,\ldots}^{\infty}\frac{\tanh(i\pi b/2a)}{i^5}\right]$$

Equilateral triangle of side a: coordinates in Fig. 3-9:

$$u(y,z) = \frac{-d\hat{p}/dx}{2\sqrt{3}\,a\mu}\left(z-\frac{1}{2}a\sqrt{3}\right)(3y^2-z^2)$$

$$Q = \frac{a^4\sqrt{3}}{320\mu}\left(-\frac{d\hat{p}}{dx}\right) \tag{3-49}$$

Circular sector: $-\frac{1}{2}\alpha \leq \theta \leq +\frac{1}{2}\alpha, 0 \leq r \leq a$:

$$u(r,\theta) = \frac{d\hat{p}/dx}{4\mu}\left[r^2\left(1-\frac{\cos 2\theta}{\cos\alpha}\right)-\frac{16a^2\alpha^2}{\pi^3} \times \sum_{i=1,3,5,\ldots}^{\infty}(-1)^{(i+1)/2}\left(\frac{r}{a}\right)^i\frac{\cos(i\pi\theta/\alpha)}{i(i+2\alpha/\pi)(i-2\alpha/\pi)}\right]$$

$$Q = \frac{a^4}{4\mu}\left(-\frac{d\hat{p}}{dx}\right) \times \left[\frac{\tan\alpha-\alpha}{4}-\frac{32\alpha^4}{\pi^5}\sum_{i=1,3,5,\ldots}^{\infty}\frac{1}{i^2(i+2\alpha/\pi)^2(i-2\alpha/\pi)}\right] \tag{3-50}$$

Concentric circular annulus: $b \le r \le a$:

$$u(r) = \frac{-d\hat{p}/dx}{4\mu}\left[a^2 - r^2 + (a^2 - b^2)\frac{\ln(a/r)}{\ln(b/a)}\right]$$

$$Q = \frac{\pi}{8\mu}\left(-\frac{d\hat{p}}{dx}\right)\left[a^4 - b^4 - \frac{(a^2 - b^2)^2}{\ln(a/b)}\right] \tag{3-51}$$

This is but a sample of the wealth of solutions available. The formula for a concentric annulus is important in viscometry, with a measured Q being used to calculate μ. To increase the pressure drop, the clearance $(a - b)$ is held small, in which case Eq. (3-51) for Q becomes the difference between two nearly equal numbers. However, if we expand the bracketed term [ ] in a series, the result is

$$(a^4 - b^4) - \frac{(a^2 - b^2)^2}{\ln(a/b)} = \frac{4}{3}b(a - b)^3 + \frac{2}{3}(a - b)^4 + \cdots + O(a - b)^5$$

so that Q for small clearance is seen to be cubic in $(a - b)$.

The eccentric annulus in Fig. 3-9 has practical applications, for example, when a needle valve becomes misaligned. The solution was given by Piercy et al. (1933), using an elegant complex-variable method which transformed the geometry to a concentric annulus, for which the solution was already known, Eq. (3-51). We reproduce here only their expression for volume rate of flow:

$$Q = \frac{\pi}{8\mu}\left(-\frac{d\hat{p}}{dx}\right)\left[a^4 - b^4 - \frac{4c^2M^2}{\beta - \alpha} - 8c^2M^2\sum_{n=1}^{\infty}\frac{ne^{-n(\beta+\alpha)}}{\sinh(n\beta - n\alpha)}\right] \tag{3-52}$$

where $M = (F^2 - a^2)^{1/2}$ $\quad F = \dfrac{a^2 - b^2 + c^2}{2c}$

$$\alpha = \frac{1}{2}\ln\frac{F + M}{F - M} \qquad \beta = \frac{1}{2}\ln\frac{F - c + M}{F - c - M}$$

Flow rates computed from this formula are compared in Fig. 3-10 to the concentric result $Q_{c=0}$ from Eq. (3-51). It is seen that eccentricity substantially increases the flow rate, the maximum ratio of $Q/Q_{c=0}$ being 2.5 for a narrow annulus of maximum eccentricity. The curve for $b/a = 1$ can be derived from lubrication theory:

Narrow annulus:
$$\frac{Q}{Q_{c=0}} = 1 + \frac{3}{2}\left(\frac{c}{a - b}\right)^2 \tag{3-53}$$

The reason for the increase in Q is that the fluid tends to bulge through the wider side. This is illustrated for one case in Fig. 3-11, where the wide side develops a set of closed high-velocity streamlines. This effect is well known to piping engineers, who have long noted the drastic leakage that occurs when a nearly closed valve binds to one side.

The eccentric annulus

A solution method using complex variables is outlined in the text. Also, the result for the volume flow rate is given

$$Q = Q(a, b, c)$$

← eccentricity

$$\frac{Q(b/a=1)}{Q_{c=0}} = 1 + \frac{3}{2}\left(\frac{c}{a-b}\right)^2$$

← narrow annulus solved using lubrication theory

↑ concentric flow rate

* eccentricity can increase Q considerably

* even tiny core (b/a = .01) can increase Q by 28% if it is pushed over against the outer wall c = a − b

* data for turbulent flow less dependent on b/a and are close to b/a = .01 result in Fig 3-10

FIGURE 3-9
Constant-velocity lines for an eccentric annulus, $b/a = c/a = \frac{1}{4}$. [*After Piercy et al.* (1933).]

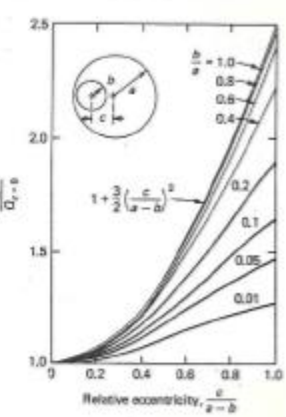


FIGURE 3-8
Volume flow through an eccentric annulus as a function of eccentricity, Eq. (3-50).

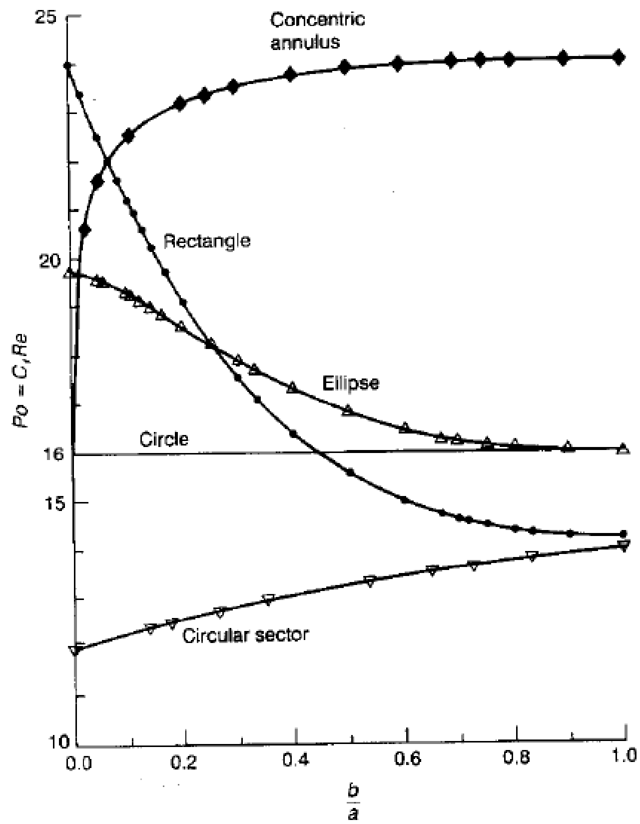


FIGURE 3-13
Comparison of Poiseuille numbers for various duct cross sections when Reynolds number is scaled by the hydraulic diameter. [*Numerical data taken from Shah and London (1978).*]

For laminar flow, $\overline{P}_0$ varies greatly, therefore it is better to use the exact solution vs. D_h as discussed next.

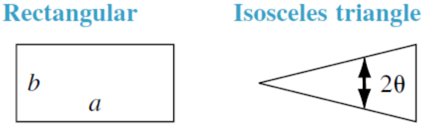
Table 6.3 Laminar Friction Factors for a Concentric Annulus

b/a	$f\,Re_{D_h}$	$D_{eff}/D_h = 1/\zeta$
0.0	64.0	1.000
0.00001	70.09	0.913
0.0001	71.78	0.892
0.001	74.68	0.857
0.01	80.11	0.799
0.05	86.27	0.742
0.1	89.37	0.716
0.2	92.35	0.693
0.4	94.71	0.676
0.6	95.59	0.670
0.8	95.92	0.667
1.0	96.0	0.667

$\tau_{wi} > \tau_{wo}$

Table 6.4 Laminar Friction Constants f Re for Rectangular and Triangular Ducts

Rectangular		Isosceles triangle	



b/a	$f\,Re_{D_h}$	θ, deg	$f\,Re_{D_h}$
0.0	96.00	0	48.0
0.05	89.91	10	51.6
0.1	84.68	20	52.9
0.125	82.34	30	53.3
0.167	78.81	40	52.9
0.25	72.93	50	52.0
0.4	65.47	60	51.1
0.5	62.19	70	49.5
0.75	57.89	80	48.3
1.0	56.91	90	48.0

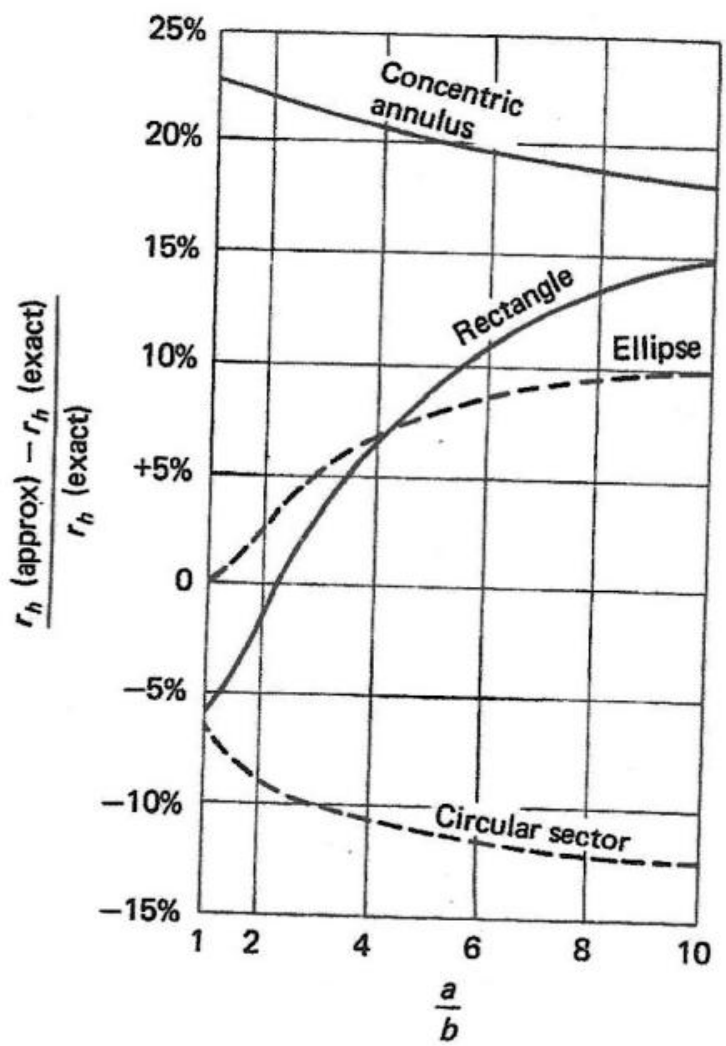


FIGURE 3-11
Percent error in the approximate hydraulic radius, Eq. (3-55), compared to the exact laminar-flow expression, Eq. (3-58).