Reynolds Transport Theorem

Preliminary: Leibniz integral theorem = derivative single variable integral having $f(x,t)$ integrand and limits $a(t)$ and $b(t)$.

$$\frac{d}{dt}\int_{x=a(t)}^{x=b(t)} f(x,t)\,dx = \int_a^b \frac{\partial f}{\partial t}\,dx + \frac{db}{dt} f(b,t) - \frac{da}{dt} f(a,t)$$

(1) (2) (3)

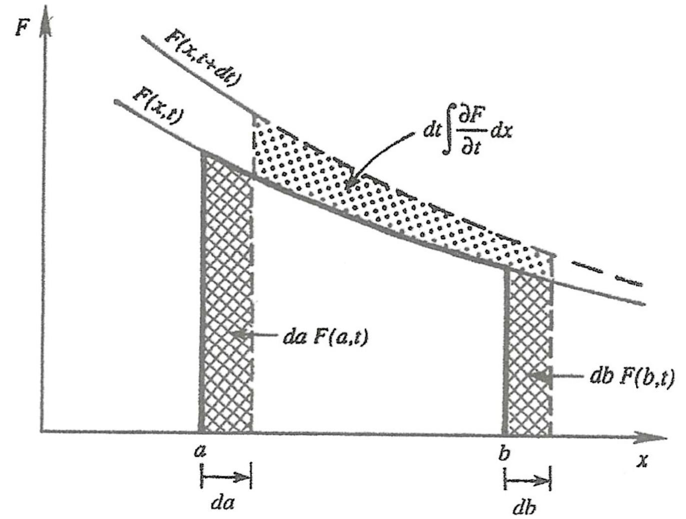

FIGURE 3.19 Graphical illustration of the Liebniz theorem. The three marked areas correspond to the three contributions shown on the right in (3.30). Here da, db, and $\partial F/\partial t$ are all shown as positive.

(1) integral $\left.\frac{\partial f}{\partial t}\right|_a^b$

(2) gain f at upper limit moving at $\frac{db}{dt}$

(3) loss f at lower limit moving at $\frac{da}{dt}$

Total derivative LHS = integral partial derivative $\Big|_a^b$ + terms that account for time dependence of a and b.

Generalization 3D: RTT

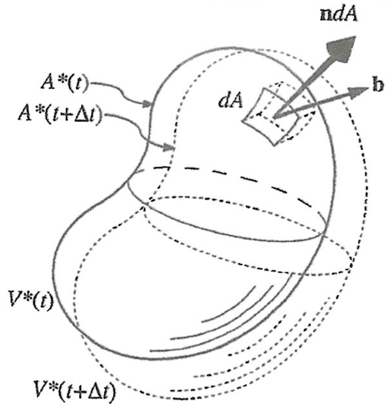

FIGURE 3.20 Geometrical depiction of a control volume $V^*(t)$ having a surface $A^*(t)$ that moves at a nonuniform velocity $\mathbf{b}$ during a small time increment Δt. When Δt is small enough, the volume increment $\Delta V = V^*(t+\Delta t) - V^*(t)$ will be very near $A^*(t)$, so the volume-increment element adjacent to dA will be $(\mathbf{b}\Delta t)\cdot\mathbf{n}dA$ where $\mathbf{n}$ is the outward normal on $A^*(t)$.

V^* = CV bounded by A^* = CS with outward normal $\underline{n}$ and nonuniform, unsteady velocity $\underline{b}$. Assume $F(\underline{x},t)$ is single valued continuous function.

$$\frac{d}{dt}\int_{V^*(t)} F(\underline{x},t)\,dV = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{V^*(t+\Delta t)} F(\underline{x},t+\Delta t)\,dV - \cancel{\int_{V^*(t)} F(\underline{x},t)\,dV}\right]$$

Define $\Delta V = V^*(t+\Delta t) - V^*(t)$

1st order TS $\quad F(\underline{x},t+\Delta t) = F(\underline{x},t) + \frac{\partial F}{\partial t}\Delta t$ for $\Delta t \to 0$

$$\int_{V^*(t+\Delta t)} F(\underline{x},t+\Delta t)\,dV = \cancel{\int_{V^*} F(\underline{x},t)\,dV} + \int_{V^*}\frac{\partial F}{\partial t}\Delta t\,dV + \int_{\Delta V} F(\underline{x},t)\,dV$$

$$+ \cancel{\int_{\Delta V}\frac{\partial F}{\partial t}\Delta t\,dV} \quad \text{higher order}$$

$$\frac{d}{dt}\int_{V^*} F(\underline{x},t)\,dV = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{V^*}\frac{\partial F}{\partial t}\Delta t\,dV + \int_{\Delta V} F(\underline{x},t)\,dV\right]$$

Need relationship ΔV and Δt: $\Delta V = (\underline{b}\,\Delta t)\cdot\underline{n}\,dA^*$

Therefore: $\int_{\Delta V} F(\underline{x},t)\,dV = \int_{A^*} F(\underline{x},t)(\underline{b}\,\Delta t)\cdot\underline{n}\,dA^*$

where all ΔV summed via surface integral

Thus taking $\lim_{\Delta \to 0} \frac{1}{\Delta t}$:

$$\frac{d}{dt}\int_{V^*(t)} F(\underline{x},t)\,dV = \int_{V^*(t)} \frac{\partial F}{\partial t}\,dV + \int_{A^*} F\,\underline{b}\cdot\underline{n}\,dA^* \qquad F = F(\underline{x},t)$$

Inflows/outflows $F(\underline{x},t)$ accounted for via $\underline{b}\cdot\underline{n}$, which monitors whether $A^*(t)$ is advancing $\underline{b}\cdot\underline{n} > 0$ or retreating $\underline{b}\cdot\underline{n} < 0$.

Physical Interpretations

(1) $F = 1$: Conservation of volume

Exercise 3.33. Starting from (3.35), set $F = 1$ and derive (3.14) when $\mathbf{b} = \mathbf{u}$ and $V^*(t) = \delta V \to 0$.

Solution 3.33. With $F = 1$, $\mathbf{b} = \mathbf{u}$, and $V^*(t) = \delta V$ with surface δA, (3.35) becomes:

$$\frac{d}{dt}\int_{\delta V} dV = 0 + \int_{\delta A} \mathbf{u}\cdot\mathbf{n}\,dA.$$

The first integral is merely δV. Use Gauss' divergence theorem on the second term to convert it to volume integral.

$$\frac{d}{dt}(\delta V) = \int_{\delta V} \nabla\cdot\mathbf{u}\,dV.$$

As $\delta V \to 0$ the integral reduces to a product of δV and the integrand evaluated at the center point of δV. Divide both sides of the last equation by δV and take the limit as $\delta V \to 0$:

$$\lim_{\delta V\to 0}\frac{1}{\delta V}\frac{d}{dt}(\delta V) = \lim_{\delta V\to 0}\frac{1}{\delta V}\int_{\delta V}\nabla\cdot\mathbf{u}\,dV = \lim_{\delta V\to 0}\frac{1}{\delta V}\left[(\nabla\cdot\mathbf{u})\delta V + \ldots\right] = \nabla\cdot\mathbf{u} = S_{ii}.$$

and this is (3.14).

$$\frac{1}{\delta V}\frac{D(\delta V)}{Dt} = u_x + v_y + w_z$$

$$= u_{i,i}$$

(2) RTT $= \frac{DF}{Dt} = \frac{\partial F}{\partial t} + \underline{u} \cdot \nabla F = \frac{\partial F}{\partial t} + u_i \frac{\partial F}{\partial x_i}$

for $V^*(t) = \delta V \to 0 \wedge \underline{b} = \underline{u}$

3.35. Show that (3.35) reduces to (3.5) when $V^*(t) = \delta V \to 0$ and the control surface velocity $\mathbf{b}$ is equal to the fluid velocity $\mathbf{u}(\mathbf{x},t)$.

Solution 3.35. When $V^*(t) = \delta V$ with surface δA, δV is small, and $\mathbf{b} = \mathbf{u}$, δV represents a fluid particle. Under these conditions (3.35) becomes:

$$\frac{d}{dt}\int_{\delta V} F(\mathbf{x},t)dV = \int_{\delta V} \frac{\partial F(\mathbf{x},t)}{\partial t}dV + \int_{\delta A} F(\mathbf{x},t)\mathbf{u}\cdot\mathbf{n}dA,$$

and the time derivative is evaluated following δV. Use Gauss' divergence theorem on the final term to convert it to a volume integral,

$$\int_{\delta A} F(\mathbf{x},t)\mathbf{u}\cdot\mathbf{n}dA = \int_{\delta V} \nabla\cdot\left(F(\mathbf{x},t)\mathbf{u}\right)dV,$$

so that (3.35) becomes:

$$\frac{d}{dt}\int_{\delta V} F(\mathbf{x},t)dV = \int_{\delta V}\left[\frac{\partial F(\mathbf{x},t)}{\partial t} + \nabla\cdot\left(F(\mathbf{x},t)\mathbf{u}\right)\right]dV = \int_{\delta V}\left[\frac{\partial F(\mathbf{x},t)}{\partial t} + F(\mathbf{x},t)\nabla\cdot\mathbf{u} + (\mathbf{u}\cdot\nabla)F(\mathbf{x},t)\right]dV,$$

where the second equality follows from expanding the divergence of the product $F\mathbf{u}$.

As $\delta V \to 0$ the various integrals reduce to a product of δV and the integrand evaluated at the center point of δV. Divide both sides of the prior equation by δV and take the limit as $\delta V \to 0$ to find:

$$\lim_{\delta V\to 0}\frac{1}{\delta V}\frac{d}{dt}\int_{\delta V} F(\mathbf{x},t)dV = \lim_{\delta V\to 0}\frac{1}{\delta V}\int_{\delta V}\left[\frac{\partial F(\mathbf{x},t)}{\partial t} + F(\mathbf{x},t)\nabla\cdot\mathbf{u} + (\mathbf{u}\cdot\nabla)F(\mathbf{x},t)\right]dV,$$

$$\lim_{\delta V\to 0}\frac{1}{\delta V}\frac{d}{dt}\left[F(\mathbf{x},t)\delta V + ...\right] = \lim_{\delta V\to 0}\frac{1}{\delta V}\left[\left(\frac{\partial F(\mathbf{x},t)}{\partial t} + F(\mathbf{x},t)\nabla\cdot\mathbf{u} + (\mathbf{u}\cdot\nabla)F(\mathbf{x},t)\right)\delta V + ...\right], \text{ or}$$

$$\frac{d}{dt}F(\mathbf{x},t) + F(\mathbf{x},t)\lim_{\delta V\to 0}\frac{1}{\delta V}\frac{d}{dt}(\delta V) = \frac{\partial F(\mathbf{x},t)}{\partial t} + F(\mathbf{x},t)\nabla\cdot\mathbf{u} + (\mathbf{u}\cdot\nabla)F(\mathbf{x},t),$$

where the product rule for derivative has been used on product $F\delta V$ in [.]-braces on the left. From (3.14) or Exercise 3.33: $\lim_{\delta V\to 0}\frac{1}{\delta V}\frac{d}{dt}(\delta V) = \nabla\cdot\mathbf{u}$, so the second terms on both sides of the last equation are equal and may be subtracted out leaving:

$$\frac{d}{dt}F(\mathbf{x},t) = \frac{\partial F(\mathbf{x},t)}{\partial t} + (\mathbf{u}\cdot\nabla)F(\mathbf{x},t),$$

and this is (3.5) when the identification $D/Dt \equiv d/dt$ is made.

Relationship CV and material volume = MV

(1) MV : $\frac{d}{dt}\int_{V(t)} F(\underline{x},t)\,dV = \int_{V(t)} \frac{\partial F}{\partial t}\,dV + \int_{A^*} F\,\underline{u}\cdot\underline{n}\,dA$

$V(t)$ = MV $A(t)$ = MV boundary with local $\underline{n}$ and moving at nonuniform unsteady $\underline{u}(\underline{x},t)$

Greens theorem : $\int_V \nabla\cdot\underline{b}\,dV = \int_S \underline{b}\cdot\underline{n}\,dA$

$\frac{d}{dt}\int_{MV} F\,dV = \int_{MV}\left[\frac{\partial F}{\partial t} + \nabla\cdot(F\underline{u})\right]dV$

$F = B\rho$

lim MV → 0 $\frac{d}{dt}\int_{MV} F\,dV = \frac{\partial F}{\partial t} + \nabla\cdot(F\underline{u})$ $LHS = \frac{dB_{sys}}{dt}$

(2) Assume at time t MV and CV coincide

$\frac{d}{dt}\int_{MV} F\,dV = \int_{V^*}\frac{\partial F}{\partial t}\,dV + \int_{A^*} F\,\underline{u}\cdot\underline{n}\,dA^*$

$t + \Delta t$

However from RTT : $\int_{V^*}\frac{\partial F}{\partial t}\,dV = \frac{d}{dt}\int_{V^*} F\,dV - \int_{A^*} F\,\underline{u_s}\cdot\underline{n}\,dA^*$

Therefore : $\frac{d}{dt}\int_{MV} F\,dV = \frac{d}{dt}\int_{V^*} F\,dV + \int_{A^*} F\,\underbrace{(\underline{u}-\underline{u_s})}_{\underline{u_R}}\cdot\underline{n}\,dA^*$

Provides relationship $\frac{d}{dt}\int_{MV} F\,dV$ and RHS which represent equivalent changes for CV

Application CV GDE

$F = \beta \rho \quad \beta = \frac{dB}{dm} \quad B = \int \beta \, dm = \int_V \beta \rho \, dV$

$\frac{dB_{sys}}{dt} = \frac{d}{dt}(m, m\underline{u}, E) = RHS = (0, \Sigma \underline{F}, \dot{Q} - \dot{W})$

$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{MV} \beta \rho \, dV = \frac{d}{dt}\int_{CV} \beta \rho \, dV + \int_{CS} \beta \rho \, \underline{u_R} \cdot \underline{n} \, dA$

$\beta = (1, \underline{u}, e)$ and $\underline{u_R} = \underline{u} - \underline{u_{cs}}$

Specific CV cases depending on $\underline{V_s}(\underline{x_{CS}}(t), t)$.

1) Deforming CV:

   (a) $\underline{V_s} = \underline{V_s}(\underline{x_{CS}}(t), t)$ non-uniform/accelerating velocity
   (b) $\underline{V_s} = \underline{V_s}(\underline{x_{CS}}(t))$ uniform/constant velocity (steady moving)
   (c) $\int_{CS} \underline{V_s}(\underline{x_{CS}}, t) \cdot \underline{n} dA = 0$ as a whole at rest (stationary)

2) Non deforming CV:

   (a) $\underline{V_s} = \underline{V_s}(t)$ accelerating velocity
   (b) $\underline{V_s} =$ constant velocity, i.e., relative inertial coordinates (steady moving)
   (c) $\underline{V_s} = 0$ at rest (stationary)