Sudden Valve Closure in a Pipe
(Application of the Unsteady Momentum Equation)

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Problem statement

Water flows steadily through a horizontal pipe of diameter $D = 0.1\,\mathrm{m}$ with average velocity $U = 2\,\mathrm{m/s}$. At time $t = 0$ a valve at the downstream end of the pipe begins to close, reducing the volumetric flow rate *linearly* to zero over a period $\Delta t = 2\,\mathrm{s}$. Assume incompressible flow, constant density $\rho = 1000\,\mathrm{kg/m^3}$, negligible friction, and a uniform velocity profile across the section.

1. Write the integral form of the unsteady momentum equation for a control volume that encloses a straight section of pipe of length L ending at the valve.
2. Determine the time-dependent force exerted by the fluid on the valve during closure (for $0 \le t \le \Delta t$).
3. Compute the numerical force-time curve and the maximum force on the valve for the representative choice $L = 1\,\mathrm{m}$.

Assumptions and modeling choices

- Incompressible fluid with constant density $\rho = 1000\,\mathrm{kg/m^3}$.
- Uniform velocity profile.
- The upstream reservoir is large enough that the upstream (inlet) velocity stays at U during the closure; the downstream (exit) velocity at the valve decreases linearly from U to 0 in time Δt.
- Neglect wall friction and pressure variations.
- Cross-sectional area: $A = \pi D^2/4$.

Mathematical formulation

Choose a control volume (CV) consisting of the fluid inside a straight pipe segment of length L that extends from the upstream section (at $x = -L$) to the valve at $x = 0$. The control surfaces are the inlet section (at $x = -L$) and the outlet section (the valve) at $x = 0$. Take the x-axis positive to the right (from upstream toward the valve).

The integral form of the linear momentum equation (one-dimensional, x-direction) for the CV is

$$\sum F_x = \frac{d}{dt}\int_{CV} \rho u \, dV + \sum_{CS} \int \rho u (\mathbf{u} \cdot \mathbf{n}) \, dA, \tag{1}$$

where $\sum F_x$ is the net external force acting on the CV in the x-direction (positive to the right), and the surface integrals are the net momentum fluxes through the control surface (positive when momentum leaves the CV in the positive x-direction).

Kinematics / boundary conditions

The cross-sectional area is

$$A = \frac{\pi D^2}{4} = \frac{\pi (0.1)^2}{4} = 7.853\,98 \times 10^{-3}\,\mathrm{m}^2.$$

The upstream (inlet) velocity is held constant at

$$u_{in} = U = 2\,\mathrm{m/s}.$$

The exit (valve) velocity is prescribed to decrease linearly to zero over Δt:

$$u_{out}(t) = U\Big(1 - \frac{t}{\Delta t}\Big), \qquad 0 \le t \le \Delta t. \tag{2}$$

With $\Delta t = 2\,\mathrm{s}$ this simplifies to

$$u_{out}(t) = 2 - t \quad (\mathrm{m/s}), \qquad 0 \le t \le 2.$$

Its time derivative is constant:

$$\frac{du_{out}}{dt} = -\frac{U}{\Delta t} = -1\,\mathrm{m/s}^2.$$

We model the velocity inside the CV as spatially uniform and equal to $u_{cv}(t) = u_{out}(t)$ (this is a simplifying assumption so that the unsteady term can be represented easily). The mass of fluid in the CV is $m = \rho A L$ and the momentum inside the CV is $m u_{cv}(t)$.

Applying the momentum equation

We consider forces acting on the CV in the x-direction. The important external force for this problem is the force that the valve exerts on the fluid, which we denote by $F_v(t)$ and take positive to the right (so a positive F_v corresponds to the valve pushing the fluid to the right).

The left-hand side of (1) is therefore

$$\sum F_x = F_v(t).$$

The time derivative of momentum inside the CV is

$$\frac{d}{dt}\int_{CV} \rho u\, dV = \frac{d}{dt}(\rho A L\, u_{cv}(t)) = \rho A L\,\frac{du_{out}}{dt}.$$

The net momentum flux term (sum over control surfaces) is the algebraic sum of momentum leaving the CV (positive when leaving in the positive x-direction). With two sections,

$$\sum_{CS}\int \rho u(\mathbf{u}\cdot\mathbf{n})\, dA = \big[\rho A u_{out}^2(t)\big]_{\text{out}} - \big[\rho A u_{in}^2\big]_{\text{in}} = \rho A\big(u_{out}^2(t) - U^2\big).$$

Combine terms into (1):

$$F_v(t) = \rho A L\,\frac{du_{out}}{dt} + \rho A\big(u_{out}^2(t) - U^2\big). \tag{3}$$

This expression gives the net force (positive to the right) exerted on the CV by external actors. By Newton's third law, the force exerted by the fluid on the valve (which acts to the left) is

$$F_{\text{fluid on valve}}(t) = -F_v(t). \tag{4}$$

Evaluate the force-time history

Insert $u_{out}(t) = U(1 - t/\Delta t)$ and $du_{out}/dt = -U/\Delta t$ into (3):

$$\begin{aligned} F_v(t) &= \rho A L\left(-\frac{U}{\Delta t}\right) + \rho A\Big(U^2\Big(1 - \frac{t}{\Delta t}\Big)^2 - U^2\Big) \\ &= -\rho A\frac{UL}{\Delta t} + \rho A U^2\left(\Big(1 - \frac{t}{\Delta t}\Big)^2 - 1\right) \\ &= -\rho A\frac{UL}{\Delta t} + \rho A U^2\left(-\frac{2t}{\Delta t} + \frac{t^2}{\Delta t^2}\right) \\ &= \rho A\left(-\frac{UL}{\Delta t} - \frac{2U^2 t}{\Delta t} + \frac{U^2 t^2}{\Delta t^2}\right). \end{aligned}$$

Thus the force exerted by the *fluid on the valve* (to the left) is

$$F_{\text{fluid on valve}}(t) = -F_v(t) = \rho A\left(\frac{UL}{\Delta t} + \frac{2U^2 t}{\Delta t} - \frac{U^2 t^2}{\Delta t^2}\right). \tag{5}$$

Note that the terms represent: the first is the contribution from decelerating the mass stored in the CV, the remaining terms come from the change in momentum flux out minus in.

Numerical evaluation for $L = 1\,\mathrm{m}$

Insert numeric values:

$$\begin{aligned} D &= 0.1\,\mathrm{m}, & A &= \frac{\pi D^2}{4} = 7.853\,98 \times 10^{-3}\,\mathrm{m}^2, \\ U &= 2\,\mathrm{m/s}, & \Delta t &= 2\,\mathrm{s}, \\ \rho &= 1000\,\mathrm{kg/m^3}, & L &= 1\,\mathrm{m}. \end{aligned}$$

Compute the constant ρA:

$$\rho A = 1000 \times 7.85398 \times 10^{-3} = 7.853\,98\,\mathrm{kg/s}.$$

Using (5) and simplifying for the given numbers (recall $U/\Delta t = 1\,\mathrm{s}^{-1}$ and $U^2/\Delta t^2 = 1\,\mathrm{s}^{-2}$):

$$F_{\text{fluid on valve}}(t) = 7.853\,98\,\big(L + 2t - t^2\big) \quad (\text{N, with } L = 1 \text{ m}).$$

For $L = 1$ m this reduces to

$$F_{\text{fluid on valve}}(t) = 7.853\,98\,\big(1 + 2t - t^2\big) \quad \text{N}.$$

Evaluate at $t = 0$ and $t = \Delta t = 2$:
$t = 0 \quad F = 7.853\,98\,\mathrm{N}$
$t = 2 \quad F = 7.853\,98\,(1 + 4 - 4) = 39.2699\,\mathrm{N}$

The quadratic $1 + 2t - t^2$ achieves its maximum on $[0, 2]$ at the endpoint $t = 2$ (its derivative $2 - 2t$ vanishes at $t = 1$, but that gives a local saddle; checking endpoints yields the maximum at $t = 2$), so the maximum force in the closure interval is

$$F_{\max} = 39.27\,\mathrm{N} \quad (\text{for } L = 1\,\mathrm{m}).$$

Remarks and limitations

- The model makes a number of simplifying assumptions (uniform velocity profile, upstream velocity held constant, neglect of compressibility and friction). In a real pipe the closure will generate pressure waves (water hammer) and spatial variations of velocity, and compressibility would likely be important for very fast closures. Including compressibility and wave propagation would convert this into a transient hyperbolic (water-hammer) problem.
- The numerical result depends on the chosen CV length L. A larger CV contains more fluid to decelerate, increasing the inertial contribution $\rho AL\, du_{out}/dt$. If instead one intended to model the whole pipe length explicitly, set L to the pipe length.
- If the inlet does not supply a constant velocity (i.e., the upstream reservoir responds dynamically), the boundary conditions must be adjusted and the momentum balance re-evaluated.