Part 5 Relations between 1D and 3D spectra

Note: $R_E(\tau)$ then $\hat{E}_{11}(\omega)$ can be obtained single point measurement (vs $E(\kappa, t)$ requires volume or line measurements) and if Taylor frozen turbulence hypothesis used can be transformed from time to space as approximation for spatial spectra.

Recall, in homogeneous turbulence[1] the velocity(energy)-spectrum tensor and two-point velocity correlation tensor form a Fourier transform pair[2]

$$\mathcal{E}_{ij}(\underline{\kappa}) = \frac{1}{(2\pi)^3}\int_{\forall} \mathcal{R}_{ij}(\underline{r}) e^{i\underline{\kappa}\cdot\underline{r}} \underline{dr} \quad (1)$$

$$\mathcal{R}_{ij}(\underline{r}) = \int_{\forall} \mathcal{E}_{ij}(\underline{\kappa}) e^{-i\underline{\kappa}\cdot\underline{r}} \underline{d\kappa} \quad (2)$$

$$E(\kappa) = \oint \frac{1}{2}\mathcal{E}_{ii}(\underline{\kappa}) dS(\kappa) \quad (3)$$

$$k = \int_0^{\infty} E(\kappa) d\kappa$$

$$\varepsilon = \int_{\forall} 2\nu\kappa^2 E(\kappa) d\kappa$$

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[1] For homogeneous turbulence $\mathcal{R}_{ij}(\underline{r})$ is not f$(\underline{x})$ and the information it contains can be re-expressed in terms of the wave number spectrum.

[2] Note: $\underline{dr}$ and $\underline{d\kappa}$ are not vectors but volumes in physical and wave number space.

Isotropic tensor theory

$$\mathcal{E}_{ij}(\underline{\kappa}) = A(\kappa)\delta_{ij} + B(\kappa)\kappa_i\kappa_j \quad (5)$$

$A(\kappa)$ and $B(\kappa)$ are determined using (1) incompressibility and (2) properties of $\oint dS(\kappa)$ and $\oint \kappa_i\kappa_j dS(\kappa)$.

$$\mathcal{E}_{ij}(\underline{\kappa}) = \frac{E(\kappa)}{4\pi\kappa^2}\left(\delta_{ij} - \frac{\kappa_i\kappa_j}{\kappa^2}\right) = \frac{E(\kappa)}{4\pi\kappa^2}P_{ij}(\kappa) \quad (10)$$

$$\mathcal{R}_{11}(\widehat{e_1}r_1, t) = \overline{u_1^2}f(r_1, t) \quad (11)$$

One-dimensional spectra $E_{ij}(\kappa_1)$ are defined as two times the one-dimensional Fourier transform of $\mathcal{R}_{ij}(\widehat{e_1}r_1)$. $E_{ij}(\kappa_1)$ has units m^3/s^2, i.e., same as $E(\kappa)$.

$$E_{ij}(\kappa_1) = \frac{1}{\pi}\int_{-\infty}^{\infty}\mathcal{R}_{ij}(\widehat{e_1}r_1)e^{-i\kappa_1 r_1}dr_1 \quad (13)$$

$$E_{11}(\kappa_1) = \frac{2}{\pi}\int_0^{\infty}\mathcal{R}_{11}(\widehat{e_1}r_1)\cos(\kappa_1 r_1)\,dr_1 \quad (14)$$

$$\mathcal{R}_{11}(\widehat{e_1}r_1) = \int_0^{\infty}E_{11}(\kappa_1)\cos(\kappa_1 r_1)\,d\kappa_1 \quad (15)$$

(14) and (15) are Fourier Transform Pair

$$\mathcal{R}_{11}(\widehat{e_1}r_1) = \int_{-\infty}^{\infty}\left[\iint_{-\infty}^{\infty}\mathcal{E}_{11}(\underline{\kappa})\,d\kappa_2 d\kappa_3\right]\cos(\kappa_1 r_1)\,d\kappa_1 \quad \text{(16) from (2)}$$

$$E_{11}(\kappa_1) = 2\iint_{-\infty}^{\infty}\mathcal{E}_{11}(\underline{\kappa})\,d\kappa_2 d\kappa_3 \quad \text{(17) from comparing (15) and (16)}$$

$E_{11}(\kappa_1)$ (1D spatial spectra) has contribution from all wavenumbers $\underline{\kappa}$ in the plane $\widehat{e_1}\cdot\underline{\kappa} = \kappa_1$, such that $|\underline{\kappa}| > \kappa_1$, i.e., only greater than κ_1 but Fourier modes contributing to $E_{11}(\kappa_1)$ can be appreciably larger than κ_1.

The one-dimensional spectrum $E_{11}(\kappa_1)$ is related to the longitudinal auto-correlation function by

$$E_{11}(\kappa_1) = \frac{2}{\pi}\overline{u_1^2}\int_0^\infty f(r_1)\cos(\kappa_1 r_1)\,dr_1 \quad \text{(20a)}$$

Which is obtained from the combination of Eq. (11) and (14).

$$E_{11}(\kappa_1) = 2\iint_{-\infty}^{\infty} \frac{E(\kappa)}{4\pi\kappa^2}\left(1 - \frac{{\kappa_1}^2}{\kappa^2}\right) d\kappa_2 d\kappa_3 \quad \text{(18)}$$

Where use is made of Eq. (10) for $i = j = 1$. The integration is over the plane of fixed κ_1, and the integrand is radially symmetric about the κ_1 axis.

Double Integrals in Polar Coordinates: $dS = d\kappa_2 d\kappa_3 = \kappa_r d\kappa_r d\theta$ Since $\kappa_r^2 = \kappa^2 - \kappa_1^2$, changing variables for integration over κ instead of κ_r, and letting $\kappa_r = 0$ such that $\kappa = \kappa_1$ *and* $\kappa_r d\kappa_r = \kappa d\kappa$. Then Eq. (18) becomes:

$$E_{11}(\kappa_1) = \int_{\kappa_1}^{\infty} \frac{E(\kappa)}{\kappa}\left(1 - \frac{\kappa_1^2}{\kappa^2}\right) d\kappa \quad \text{(19)}$$

Therefore, $E_{11}(\kappa_1)$ contains contribution from $\kappa > \kappa_1$, which is a phenomenon called aliasing.[3] Furthermore, $E_{11}(\kappa_1)$ is a monotonically decreasing function of κ_1, so that E_{11} is maximum at zero wavenumber, irrespective of the shape of $E(\kappa)$.

Note that both (17) [or (18)] and (19) are equivalent, i.e., representations in different coordinate systems and (19) is radially symmetric about the κ_1 axis. A statistically homogenous field is statistically invariant under translation. If the field is also statistically invariant under rotations and reflections of the coordinate system, then it is also statistically isotropic. Therefore, (19) is equivalent to (20a). Furthermore, inverting Eq. (19) to get Eq. (21), shows E_{11} also obeys a -5/3 law in the inertial range with different Kolmogorov constant.

[3] Aliasing in sampling is a type of measurement error that occurs when a signal is sampled at an insufficient rate, which results in a false lower frequency component, or alias, in the sampled data. In present context, κ_1 is affected by wave number > κ_1.

Inverting Eq. (19), gives the relation

$$E(\kappa) = \frac{1}{2}\kappa^3 \frac{d}{d\kappa}\left(\frac{1}{\kappa}\frac{dE_{11}(\kappa)}{d\kappa}\right) \quad (21)$$

The following animation represents the domains of integration (volumes, surfaces, and lines) for the different representations of $\mathcal{R}_{11}$ and $E_{11}(\kappa_1)$. The two functions can take arbitrary values at each point in space, but the relationship between the different integration domains is independent of their specific expressions.

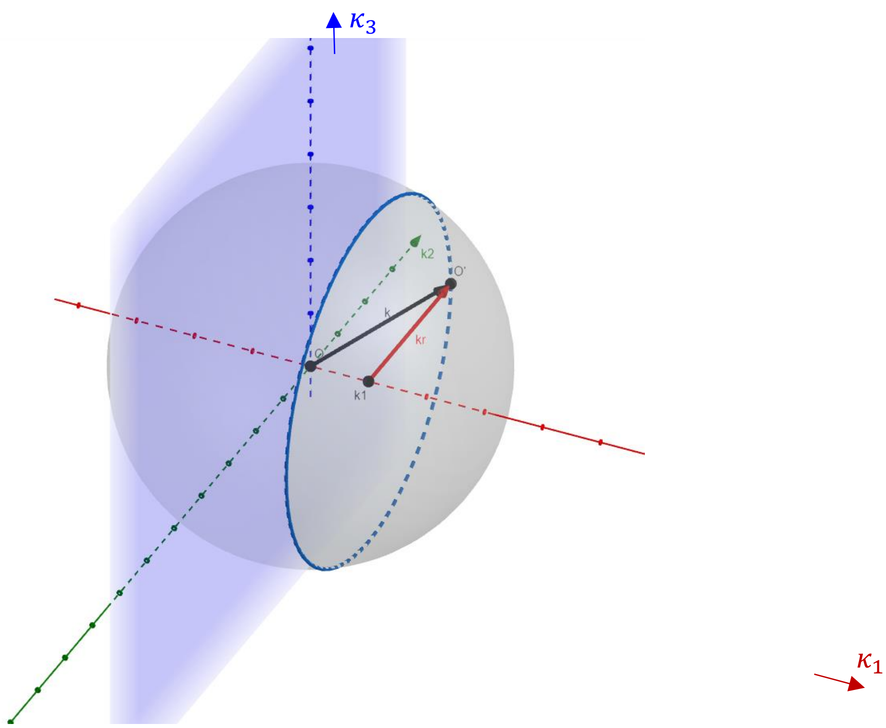

Figure 1: Representation in the wave number space of the different integrals:

Animation Play the animation for the parameter r to visualize

- (16) integral over $\mathbb{R}^3$, equivalent to integration over sphere as $\kappa \to \infty$.
- (17) and (18) integral over $d\kappa_2 d\kappa_3$ plane intersecting κ_1, equivalent to integration over intersection between sphere and κ_1 plane as $\kappa \to \infty$.
- (19) integral over κ (black vector), when the radius of the sphere κ is $<$ κ_1, the integral is not defined (i.e., no intersection of the radius of the sphere and the $d\kappa_2 d\kappa_3$ plane intersecting κ_1); once $\kappa = \kappa_1$ the integral is defined from κ_1 to ∞.
- (20a) integral over κ_1 axis from 0 to ∞.