Reynolds Transport Theorem (RTT)

- RTT transforms the governing differential equations (GDE's) from a system to a control volume (CV):

$$\underbrace{\frac{DB_{\text{sys}}}{Dt}}_{\substack{\text{time rate of change}\\ \text{of } B \text{ for a system}}} = \underbrace{\frac{d}{dt}\int_{\text{CV}} \beta\rho d\mathcal{V}}_{\substack{\text{time rate of change}\\ \text{of } B \text{ in CV}}} + \underbrace{\int_{\text{CS}} \beta\rho \underline{V}_R \cdot d\underline{A}}_{\substack{\text{net flux of } B\\ \text{across CS}}}$$

where, $\beta = \frac{dB}{dm} = \left(1, \underline{V}, e\right)$ for $B = (m, m\underline{V}, E)$

- Fixed a CV,

$$\frac{DB_{\text{sys}}}{Dt} = \frac{d}{dt}\int_{\text{CV}} \beta\rho d\mathcal{V} + \int_{\text{CS}} \beta\rho \underline{V} \cdot d\underline{A}$$

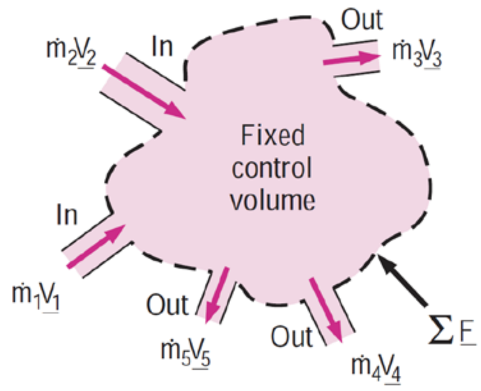
Momentum Equation

- RTT with $B = m\underline{V}$ and $\beta = \underline{V}$,

$$\frac{\partial}{\partial t}\int_{\mathrm{CV}} \underline{V}\rho d\forall + \int_{\mathrm{CS}} \underline{V}\rho\underline{V}\cdot d\underline{A} = \sum \underline{F}$$

- Simplified form:

$$\sum(\dot{m}\underline{V})_{\mathrm{out}} - \sum(\dot{m}\underline{V})_{\mathrm{in}} = \sum\underline{F}$$



or in component forms,

$$\begin{aligned}\sum(\dot{m}u)_{\mathrm{out}} - \sum(\dot{m}u)_{\mathrm{in}} &= \sum F_x\\ \sum(\dot{m}v)_{\mathrm{out}} - \sum(\dot{m}v)_{\mathrm{in}} &= \sum F_y\\ \sum(\dot{m}w)_{\mathrm{out}} - \sum(\dot{m}w)_{\mathrm{in}} &= \sum F_z\end{aligned}$$

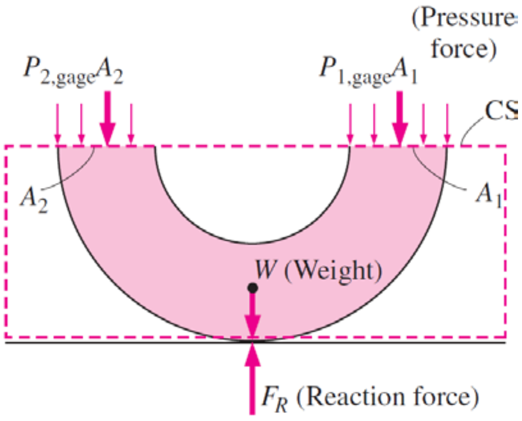
Note: If $\underline{V} = u\hat{\boldsymbol{\imath}} + v\hat{\boldsymbol{\jmath}} + w\hat{\boldsymbol{k}}$ is normal to CS, $\dot{m} = \rho VA$, where $V = |\underline{V}|$.

Momentum Equation – Contd.

- External forces:

$$\sum \underline{F} = \sum \underline{F}_{\text{body}} + \sum \underline{F}_{\text{surface}} + \sum \underline{F}_{\text{other}}$$

  - $\sum \underline{F}_{\text{body}} = \sum \underline{F}_{\text{gravity}}$
    - $\sum \underline{F}_{\text{gravity}}$: gravity force (i.e., weight)
  - $\sum \underline{F}_{\text{Surface}} = \sum \underline{F}_{\text{pressure}} + \sum \underline{F}_{\text{friction}} + \sum \underline{F}_{\text{other}}$
    - $\sum \underline{F}_{\text{pressure}}$: pressure forces normal to CS
    - $\sum \underline{F}_{\text{friction}}$: viscous friction forces tangent to CS
  - $\sum \underline{F}_{\text{other}}$: anchoring forces or reaction forces

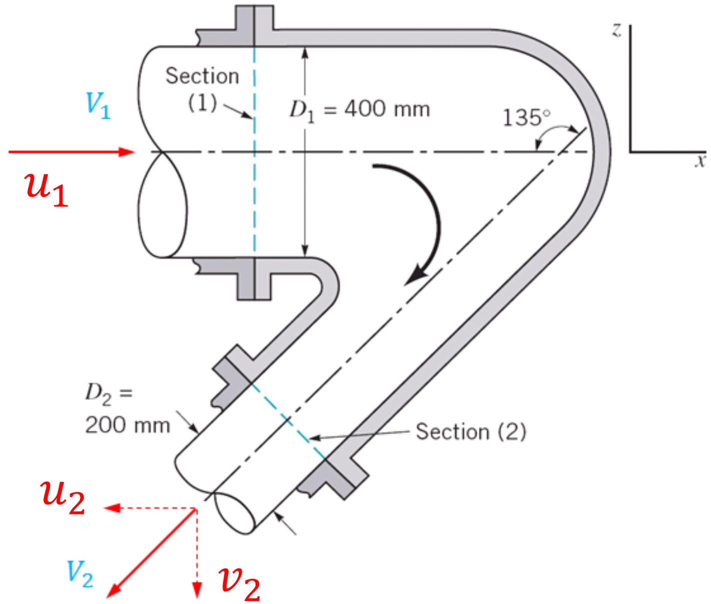


An 180° elbow supported by the ground

In most flow systems, the force $\vec{F}$ consists of weights, pressure forces, and reaction forces. Gage pressures are used here since atmospheric pressure cancels out on all sides of the control surface.

Note: Surface forces arise as the CV is isolated from its surroundings, similarly to drawing a free-body diagram. A well-chosen CV exposes only the forces that are to be determined and a minimum number of other forces

Example (Bend)



Inlet (1):

$$\dot{m}_1 = \rho V_1 A_1$$
$$u_1 = V_1$$
$$v_1 = 0$$

Outlet (2):

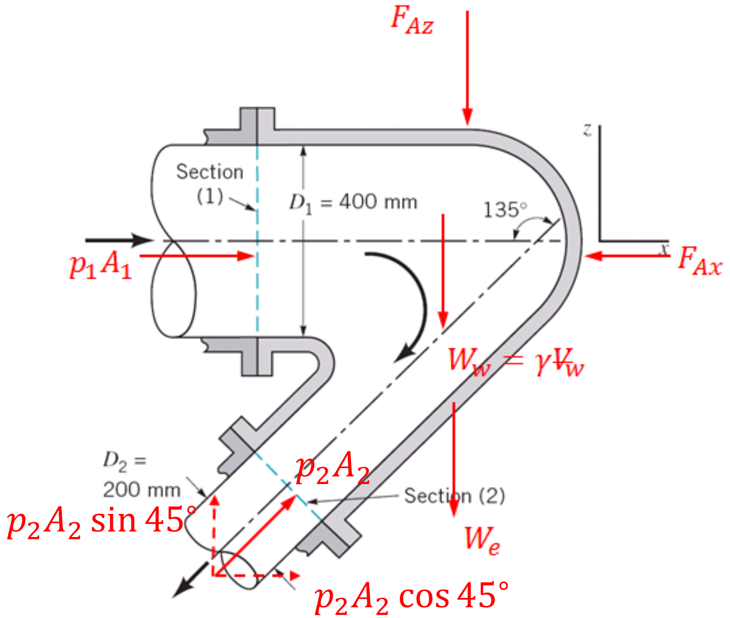
$$\dot{m}_2 = \rho V_2 A_2$$
$$u_2 = -V_2 \cos 45^\circ$$
$$v_2 = -V_2 \sin 45^\circ$$

$$(\dot{m}u)_{\text{out}} - (\dot{m}u)_{\text{in}} = (\rho V_2 A_2)(-V_2 \cos 45^\circ) - (\rho V_1 A_1)(V_1)$$
$$(\dot{m}v)_{\text{out}} - (\dot{m}v)_{\text{in}} = (\rho V_2 A_2)(-V_2 \sin 45^\circ) - (\rho V_1 A_1)(0)$$

Since $\rho V_1 A_1 = \rho V_2 A_2$,

$$(\dot{m}u)_{\text{out}} - (\dot{m}u)_{\text{in}} = -(\rho V_2 A_2)(V_2 \cos 45^\circ + V_1)$$
$$(\dot{m}v)_{\text{out}} - (\dot{m}v)_{\text{in}} = -\rho V_2^2 A_2 \sin 45^\circ$$

Example – Contd.



$\sum F_x$:

1) Body force = 0
2) Pressure force = $p_1A_1 + p_2A_2 \cos 45°$
3) Anchoring force = $-F_{Ax}$

$\sum F_y$:

1) Body force = $-W_w - W_e$
2) Pressure force = $p_2A_2 \sin 45°$
3) Anchoring force = $-F_{Az}$

Thus,

$$-(\rho V_2 A_2)(V_2 \cos 45° + V_1) = p_1A_1 + p_2A_2 \cos 45° - F_{Ax}$$
$$-\rho V_2^2 A_2 \sin 45° = -\gamma V_w - W_e + p_2A_2 \sin 45° - F_{Az}$$

$$\therefore F_{Ax} = (\rho V_2 A_2)(V_2 \cos 45° + V_1) + p_1A_1 + p_2A_2 \cos 45°$$
$$F_{Az} = \rho V_2^2 A_2 \sin 45° - \gamma V_w - W_e + p_2A_2 \sin 45°$$