5.47 The differential equation for small-amplitude vibrations $y(x, t)$ of a simple beam is given by

$$\rho A\frac{\partial^2 y}{\partial t^2}+EI\frac{\partial^4 y}{\partial x^4}=0$$

where ρ = beam material density
A = cross-sectional area
I = area moment of inertia
E = Young's modulus

Use only the quantities ρ, E, and A to nondimensionalize y, x, and t, and rewrite the differential equation in dimensionless form. Do any parameters remain? Could they be removed by further manipulation of the variables?

Solution: The appropriate dimensionless variables are

$$y^* = \frac{y}{\sqrt{A}};\quad t^* = t\sqrt{\frac{E}{\rho A}};\quad x^* = \frac{x}{\sqrt{A}}$$

$$y = \sqrt{A}\,y^*$$

$$t = \frac{E}{\rho A}\,t^* \quad (a)$$

$$x = \sqrt{A}\,x^*$$

Substitution into the PDE above yields a dimensionless equation with *one* parameter:

$$\frac{\partial^2 y^*}{\partial t^{*2}}+\left(\frac{I}{A^2}\right)\frac{\partial^4 y^*}{\partial x^{*4}}=0;\quad \text{One geometric parameter: } \frac{I}{A^2} \quad \textit{Ans.}$$

We could *remove* (I/A^2) completely by redefining $\mathbf{x^* = x/I^{1/4}}$. *Ans.*

$$\frac{\partial}{\partial x} = \frac{\partial x^*}{\partial x}\frac{\partial}{\partial x^*} = \frac{1}{\sqrt{A}}\frac{\partial}{\partial x^*} \qquad \frac{\partial}{\partial t} = \frac{\partial t^*}{\partial t}\frac{\partial}{\partial t^*} = a\frac{\partial}{\partial t^*}$$

$$\frac{\partial}{\partial y} = \frac{1}{\sqrt{A}}\frac{\partial}{\partial y^*}$$

$$\frac{E}{\rho A} = \frac{\frac{N}{m^2}}{\frac{kg}{m^3}\times m^2} = \frac{N}{m\,kg} = \frac{1}{s^2}$$

$$N = kg\frac{m}{s^2}$$

$$\rho A\, a^2 \frac{\partial^2 \sqrt{A}\,y^*}{\partial t^{*2}} + EI\frac{1}{A^2}\frac{\partial^4 \sqrt{A}\,y^*}{\partial x^{*4}}$$

$$\rho A \frac{E}{\rho A}\sqrt{A}\frac{\partial^2 y^*}{\partial t^{*2}} + EI\frac{A^{1/2}}{A^2}\frac{\partial^4 y^*}{\partial x^{*4}} = 0$$

$$\frac{\partial^2 y^*}{\partial t^{*2}} + \frac{I}{A^2}\frac{\partial^4 y^*}{\partial x^{*4}}$$