Chapter 8.4 Method of Images

Another important consequence of the superposition principle for ideal flow (due to the linearity of the Laplace equation) is that it allows boundaries to be built via the method of images. Examples include, e.g., aircraft ground effects during takeoff and landing, wind tunnel walls.

The method of images (or method of mirror images) is a mathematical tool for solving differential equations, in which boundary conditions are satisfied by combining a solution not restricted by the boundary conditions with its possibly weighted mirror image. Generally, original singularities are inside the domain of interest, but the function is made to satisfy boundary conditions by placing additional singularities outside the domain of interest. Typically, the locations of these additional singularities are determined as the virtual location of the original singularities as viewed in a mirror placed at the location of the boundary conditions.

The method can be extended for curved boundaries such as circles and spheres and unsteady flows.

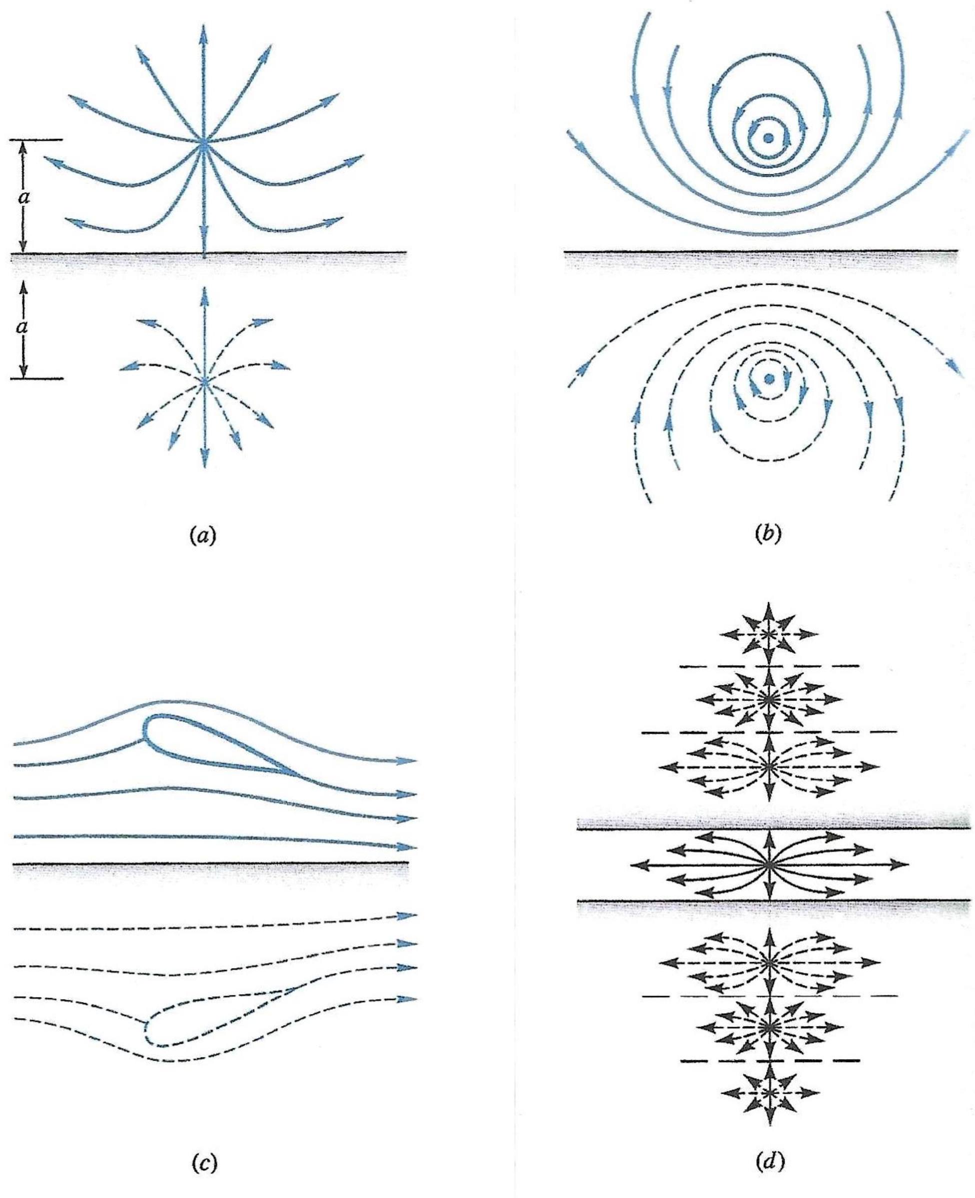

Fig. 8.21 Constraining walls can be created by image flows: (*a*) source near a wall with identical image source; (*b*) vortex near a wall with image vortex of opposite sense; (*c*) airfoil in ground effect with image airfoil of opposite circulation; (*d*) source between two walls requiring an infinite row of images.

As an example, consider a source of strength q_s a distance +a from the origin with an image source of the same strength and sign a distance -a from the origin. The combination of the two sources creates the appearance of a vertical wall along the y axis through the origin, which can be used for model of source near a vertical wall (1st and fourth quadrants) or flow through a narrow slit at x = a into a right-angled corner (first quadrant).

The stream and potential functions are:

$$\psi = \frac{q_s}{2\pi}\left[\tan^{-1}\left(\frac{y}{x+a}\right) + \tan^{-1}\left(\frac{y}{x-a}\right)\right] \quad \text{and}$$
$$\phi = \frac{q_s}{2\pi}\left[\ln\sqrt{(x+a)^2 + y^2} + \ln\sqrt{(x-a)^2 + y^2}\right].$$

Rearranging and using the tangent two-angle formula provides the equation for the streamlines:

$$x^2 - y^2 - 2xy\cot(2\pi\psi/q_s) = a^2.$$

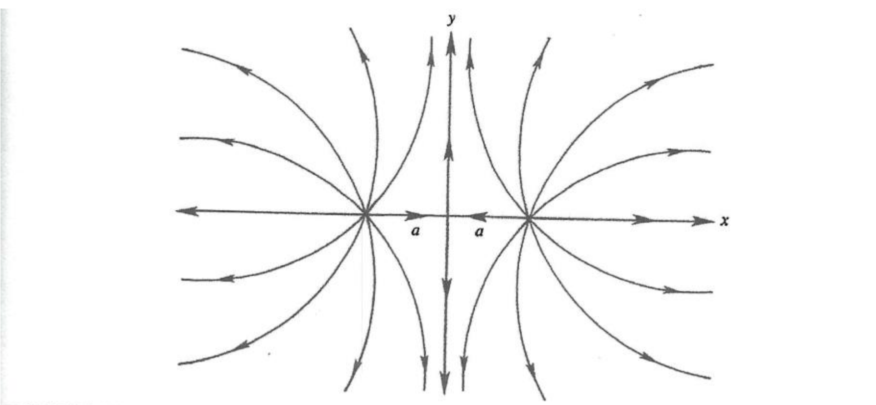

FIGURE 7.16 Ideal flow from two equal sources placed at $x = \pm a$. The origin is a stagnation point. The vertical axis is a streamline and may be replaced by a solid surface. This flow field further illustrates the method of images.

EXAMPLE 14.3

As a simplified means to explain how a flying aircraft's weight is transmitted to the ground, consider two-dimensional ideal flow with density ρ and speed U past an ideal vortex of strength Γ a distance H above an infinite flat surface (see Figure 14.11). Integrate the pressure distribution on the surface to show that it carries a load of $\rho U\Gamma$ when $H \to \infty$.

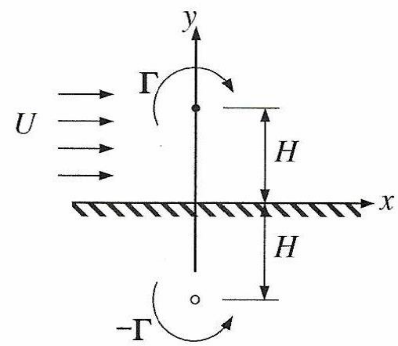

FIGURE 14.11 Two-dimensional ideal flow geometry for Example 14.3. A uniform horizontal stream with speed U passes a vortex (the solid circle) of strength Γ a distance H above a solid surface. The pressure distribution on the solid surface matches the lift load on the vortex. An opposite strength image vortex (the open circle) is located a distance H below the surface to satisfy the no-through-flow boundary condition on the surface.

Stream function for uniform stream in +x direction and clockwise vortex at (0,H) with counter clockwise image vortex at (0,-H):

$$\psi(x,y) = Uy + \frac{\Gamma}{2\pi}\ln\left(\sqrt{x^2+(y-H)^2}\right) - \frac{\Gamma}{2\pi}\ln\left(\sqrt{x^2+(y+H)^2}\right)$$

The velocity components are:

$$u(x,y) = \frac{\partial\psi}{\partial y} = U + \frac{\Gamma}{2\pi}\left(\frac{y-H}{x^2+(y-H)^2} - \frac{y+H}{x^2+(y+H)^2}\right)$$

$$v(x,y) = -\frac{\partial\psi}{\partial x} = -\frac{\Gamma}{2\pi}\left(\frac{x}{x^2+(y-H)^2} - \frac{x}{x^2+(y+H)^2}\right).$$

And pressure via the Bernoulli equation:

$$p(x,y)+\frac{1}{2}\rho\left(u^2(x,y)+v^2(x,y)\right) = p_\infty+\frac{1}{2}\rho U^2, \quad \text{or for} \quad y=0{:}\ p(x,0)-p_\infty = \frac{\rho}{2}\left(U^2-u^2(x,0)\right)$$

Note that v(x,0) = 0 and

$$u(x,0) = U-\frac{\Gamma}{\pi}\left(\frac{H}{x^2+H^2}\right), \quad \text{so} \quad U^2-u^2(x,0) = 2U\frac{\Gamma}{\pi}\left(\frac{H}{x^2+H^2}\right)-\frac{\Gamma^2}{\pi^2}\left(\frac{H}{x^2+H^2}\right)^2$$

Such that the pressure per unit length is:

$$\int_{-\infty}^{+\infty}\left(p(x,0)-p_\infty\right)dx = \frac{\rho U\Gamma}{\pi}\int_{-\infty}^{+\infty}\frac{H}{x^2+H^2}dx-\frac{\rho\Gamma^2}{2\pi^2}\int_{-\infty}^{+\infty}\left(\frac{H}{x^2+H^2}\right)^2dx.$$

Which can be evaluated using the substitution x = Htanx

$$\int_{-\infty}^{+\infty}\left(p(x,0)-p_\infty\right)dx = \frac{\rho U\Gamma}{\pi}[\xi]_{-\pi/2}^{+\pi/2}-\frac{\rho\Gamma^2}{2\pi^2 H}\int_{-\pi/2}^{+\pi/2}\cos^2\xi\, d\xi = \rho U\Gamma-\rho\frac{\Gamma^2}{4\pi H}.$$

The first term balances the lift force on the vortex and is independent of H, which is transmitted to the surface through the combined effects of the uniform stream and vortex induced velocity. The second term is an interference term that is negligible as H increases to ∞ and is more complex for real aircraft modeled using multiple vortices and called ground effect when landing.

$$\underline{F} = -\int_{-\infty}^{\infty} (p - p_\infty)\, \underline{n}\, dx$$ per unit span

$\underline{n} = \hat{j}$

$$F_y = -\rho U \Gamma + \rho \frac{\Gamma^2}{4\pi H}$$

$\underline{n} \cdot \hat{i} = 0$ $F_x = 0$

$\underline{n} \cdot \hat{j} = 1$

equal opposite lift

force on vortex – direction lift is towards ground and reduces lift

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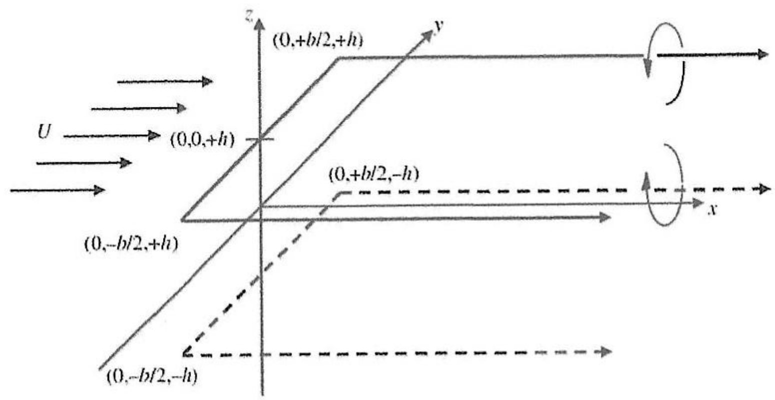
Exercise 14.18. As an airplane lands, the presence of the ground changes the plane's aerodynamic performance. To address the essential features of this situation, consider uniform flow past a horseshoe vortex (heavy solid lines below) with wingspan b located a distance h above a large flat boundary defined by $z = 0$. From the method of images, the presence of the boundary can be accounted for by an image horseshoe vortex (heavy dashed lines below) of opposite strength located a distance h below the boundary.

a) Determine the direction and the magnitude of the induced velocity at $\mathbf{x} = (0, 0, h)$, the center of the wing.

b) Assuming the result of part a) applies along the entire wingspan, estimate L and D_i, the lift and lift-induced drag, respectively, in terms of b, h, Γ, and ρ = fluid density.

c) Compare the result of part b) to that obtained for the horseshoe vortex without a large flat surface: $L = \rho U \Gamma b$ and $D_i = \rho\Gamma^2/\pi$. Which configuration has more lift? Which one has less drag? Why?



Solution 14.18. a) Five vortices will contribute to the induced velocity at $\mathbf{x} = (0,0,h)$. Number these as follows:

1 = starboard wingtip vortex
2 = starboard wingtip image vortex
3 = port wingtip vortex
4 = port wingtip image vortex
5 = main wing image vortex

The geometrical layout ensures that the induced velocities will not point in the same direction, so work through each one individually using the Biot-Savart induced velocity law from exercise 4.10:

$$\mathbf{u}_i = \frac{\mathbf{e}_\omega \times \mathbf{e}_R \Gamma}{4\pi}\left(\cos\theta_1 - \cos\theta_2\right)$$

where i is the vortex number (not a coordinate index), $\mathbf{e}_\omega$ points along the vortex direction, $\mathbf{e}_R$ points along a line perpendicular to the vortex that intersects the point of interest, and θ_1 & θ_2 are the polar angles between the $\mathbf{e}_\omega$ and the line connecting the ends of the vortex to the point of interest. Here the first four vortices extend from $x = 0$ to $+\infty$, so the cosine terms above are: $\cos(\pi/2) - \cos(\pi) = +1$

$$\mathbf{u}_1 = -\mathbf{e}_z\Gamma/\left(4\pi(b/2)\right) = -\mathbf{e}_z\Gamma/\left(2\pi b\right)$$

$$\mathbf{u}_2 = \frac{\Gamma}{4\pi\sqrt{(b/2)^2 + 4h^2}}\left[\mathbf{e}_z \frac{b/2}{\sqrt{(b/2)^2 + 4h^2}} + \mathbf{e}_y \frac{2h}{\sqrt{(b/2)^2 + 4h^2}}\right]$$

$$\mathbf{u}_3 = -\mathbf{e}_z\Gamma/(4\pi(b/2)) = -\mathbf{e}_z\Gamma/(2\pi b)$$

$$\mathbf{u}_4 = \frac{\Gamma}{4\pi\sqrt{(b/2)^2 + 4h^2}}\left[\mathbf{e}_z \frac{b/2}{\sqrt{(b/2)^2 + 4h^2}} - \mathbf{e}_y \frac{2h}{\sqrt{(b/2)^2 + 4h^2}}\right]$$

Here the extra factor in [,] brackets for the second and fourth vortex is merely a unit vector rotated to the induced velocity direction. The fifth, vortex induces a velocity that opposes the on-coming free stream and its angular extent is different from the four other vortices.

$$\mathbf{u}_5 = \frac{-\Gamma\mathbf{e}_x}{4\pi(2h)}\left[\frac{b/2}{\sqrt{(b/2)^2 + 4h^2}} - \frac{-b/2}{\sqrt{(b/2)^2 + 4h^2}}\right] = \frac{-\Gamma\mathbf{e}_x}{2\pi(2h)} \frac{b/2}{\sqrt{(b/2)^2 + 4h^2}}$$

The sum of these five terms is:

$$\mathbf{u}_{total} = -\frac{\Gamma\mathbf{e}_z}{\pi b} + \frac{\Gamma}{2\pi\sqrt{(b/2)^2 + 4h^2}}\left[\mathbf{e}_z \frac{b/2}{\sqrt{(b/2)^2 + 4h^2}}\right] - \frac{\Gamma\mathbf{e}_x}{2\pi(2h)} \frac{b/2}{\sqrt{(b/2)^2 + 4h^2}}$$

$$\mathbf{u}_{total} = -\frac{\Gamma}{\pi b}\left\{\frac{4h^2}{(b/2)^2 + 4h^2}\right\}\mathbf{e}_z - \frac{\Gamma}{4\pi h} \frac{b/2}{\sqrt{(b/2)^2 + 4h^2}}\mathbf{e}_x = -u_z\mathbf{e}_z - u_x\mathbf{e}_x$$

b) The lift force will be: $L = \rho(U - w_x)\Gamma b$, and the induced drag force will be:

$$D_i = L\varepsilon = \rho(U - w_x)\Gamma b \frac{w_z}{U - w_x} = \rho\Gamma b w_z = \rho\frac{\Gamma^2}{\pi}\left\{\frac{4h^2}{(b/2)^2 + 4h^2}\right\}$$

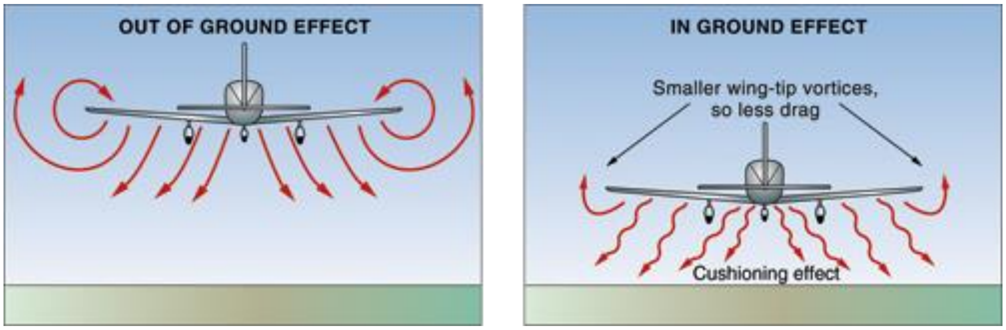
Note, that as $h/b \to 0$, the induced drag disappears.

c) For constant speed flight close to the ground surface, there is less lift and less drag for a fixed value of Γ. The lowering of the lift occurs because the induced velocity from the main wing's image vortex slows the on-coming stream at the location of the real wing. The induced drag is lower because the image tip vortices produce upwash that partially counter acts the downwash from the actual wingtip vortices.

For actual aircraft, there are two effects that more than offset the apparent reduction in lift found in part b) and mentioned here.

(i) Wing circulation Γ increases as the aircraft approaches the ground because the downwash decreases, and less downwash means an increase in the wing's angle of attack.

(ii) With a constant engine-throttle setting, the loss of induced drag as $h/b \to 0$ causes the aircraft to mildly accelerate, thereby increasing U. The aircraft's passengers may even feel this mild acceleration, and it may lead to a steady-state speed that prevents the aircraft from descending further and touching down. When this happens the aircraft is said to be flying *with* or *in ground effect*. Once the aircraft is low enough for ground effect to be apparent, the pilot must typically reduce the wing's C_L by using its spoilers or by lowering the engine throttle setting in a controlled fashion to achieve a safe smooth landing. Observant airline passengers will notice that commercial airliners sometimes fly at very low altitudes over the landing runway for an unexpectedly long period of time before touching down. This apparent delay in touching down is merely the time necessary for pilot to adjust the aircraft's trim to continue its decent while flying in ground effect. [Spoilers are flaps on the top of the wing that *spoil* the airflow on the suction side of the wing; they are used to increase form drag and reduce lift in a controlled manner.]



Ground effect (aerodynamics)

From Wikipedia, the free encyclopedia

For fixed-wing aircraft, **ground effect** is the reduced aerodynamic drag that an aircraft's wings generate when they are close to a fixed surface.[1] During takeoff, ground effect can cause the aircraft to "float" while below the recommended climb speed. The pilot can then fly just above the runway while the aircraft accelerates in ground effect until a safe climb speed is reached.[2]

For rotorcraft, ground effect results in less drag on the rotor during hovering close to the ground. At high weights this sometimes allows the rotorcraft to lift off while stationary in ground effect but does not allow it to transition to flight out of ground effect. Helicopter pilots are provided with performance charts which show the limitations for hovering their helicopter in ground effect (IGE) and out of ground effect (OGE). The charts show the added lift benefit produced by ground effect.[3]
For fan- and jet-powered vertical take-off and landing (VTOL) aircraft, ground effect when hovering can cause suckdown and fountain lift on the airframe and loss in hovering thrust if the engine sucks in its own exhaust gas, which is known as hot gas ingestion (HGI).[4][5]

Explanations

Fixed-wing aircraft

When an aircraft flies at or below approximately half the length of the aircraft's wingspan above the ground or water there occurs an often-noticeable *ground effect*. The result is lower induced drag on the aircraft. This is caused primarily by the ground or water obstructing the creation of wingtip vortices and interrupting downwash behind the wing.[6][7]
A wing generates lift by deflecting the oncoming airmass (relative wind) downward.[8] The deflected or "turned" flow of air creates a resultant force on the wing in the opposite direction (Newton's 3rd law). The resultant force is identified as lift. Flying close to a surface increases air pressure on the lower wing surface, nicknamed the "ram" or "cushion" effect, and thereby improves the aircraft lift-to-drag ratio. The lower/nearer the wing is to the ground, the more pronounced the ground effect becomes. While in the ground effect,

the wing requires a lower angle of attack to produce the same amount of lift. In wind tunnel tests, in which the angle of attack and airspeed remain constant, an increase in the lift coefficient ensues,[9] which accounts for the "floating" effect. Ground effect also alters thrust versus velocity, where reduced induced drag requires less thrust in order to maintain the same velocity.[9]

Low winged aircraft are more affected by ground effect than high wing aircraft.[10] Due to the change in up-wash, down-wash, and wingtip vortices, there may be errors in the airspeed system while in ground effect due to changes in the local pressure at the static source.[9]

Rotorcraft

When a hovering rotor is near the ground the downward flow of air through the rotor is reduced to zero at the ground. This condition is transferred up to the disc through pressure changes in the wake which decreases the inflow to the rotor for a given disc loading, which is rotor thrust for each square foot of its area. This gives a thrust increase for a particular blade pitch angle, or, alternatively, the power required for a thrust is reduced. For an overloaded helicopter that can only hover IGE it may be possible to climb away from the ground by translating to forward flight first while in ground effect.[11] The ground-effect benefit disappears rapidly with speed but the induced power decreases rapidly as well to allow a safe climb.[12] Some early underpowered helicopters could only hover close to the ground.[13] Ground effect is at its maximum over a firm, smooth surface.[14]