Chapter 7.5 Turbulent Boundary Layer

Introduction: Transition to Turbulence

The transition process can be described as a succession of Tollmien-Schlichting waves, development of Λ – structures (hairpin vortices), vortex decay (viscous diffusion and dissipation) and formation of turbulent spots as preliminary stages to fully turbulent boundary-layer flow.

The phenomena observed during the transition process are similar for the flat plate boundary layer and for the plane channel flow, as shown in the following figure based on measurements by M. Nishioka et al. (1975). Periodic initial perturbations were generated in the BL using an oscillating cord.

For typical commercial surfaces transition occurs at $Re_{x,tr} \approx 5\times10^5$. However, one can delay the transition to $Re_{x,tr} \approx 3\times10^6$ with care in polishing the wall.

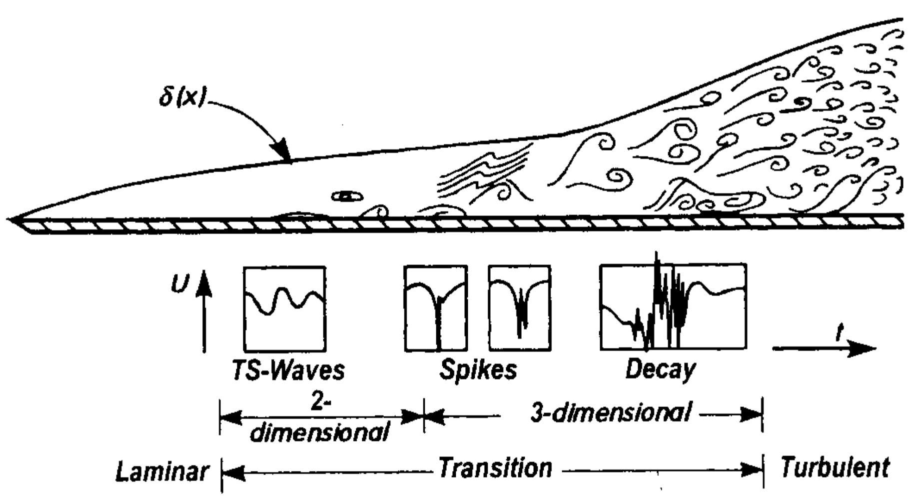


Fig. 15.38. Signals found at different regions in the transition at a plate at zero incidence, after M. Nishioka et al. (1975, 1990)

Reynolds Average of 2D boundary layer equations

$$u = \bar{u} + u'; \quad v = \bar{v} + v'; \quad w = \bar{w} + w'; \quad p = \bar{p} + p';$$

Substituting *u*, *v* and *w* into continuity equation and taking the time average we obtain,

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0 \qquad \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

Similarly, for the momentum equations and using continuity (neglecting g),

$$\rho \frac{D\bar{V}}{Dt} = -\nabla \bar{p} + \nabla \cdot \tau_{ij}$$

Where:

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \rho \overline{u_i' u_j'}$$

Laminar Turbulent

Assume

$$\delta(x) << x \text{ which means } \bar{v} << \bar{u}, \quad \frac{\partial}{\partial x} << \frac{\partial}{\partial y}$$

mean flow structure is two-dimensional: $\bar{w} = 0$, $\frac{\partial}{\partial z} = 0$

Note the mean lateral turbulence is not zero, $\overline{w'^2} \neq 0$, but its z derivative is assumed to vanish.

Then, we get the following BL equations for incompressible steady flow:

$$\boxed{\frac{\partial \overline{u}}{\partial x}+\frac{\partial \overline{v}}{\partial y}=0}$$ Continuity

$$\boxed{\overline{u}\frac{\partial \overline{u}}{\partial x}+\overline{v}\frac{\partial \overline{u}}{\partial y}\approx U_e\frac{dU_e}{dx}+\frac{1}{\rho}\frac{\partial \tau}{\partial y}}$$ x-momentum

$$\boxed{\frac{\partial p}{\partial y}\approx -\rho\frac{\partial v'^2}{\partial y}}$$ y-momentum

Where U_e is the free-stream velocity and

$$\tau=\mu\frac{\partial \overline{u}}{\partial y}-\rho\overline{u'v'}$$

Note:

- The equations are solved for the time averages $\overline{u}$ and $\overline{v}$
- The shear stress now consists of two parts: 1. first part is due to the molecular exchange and is computed from the time-averaged field as in the laminar case; 2. The second part appears additionally and is due to turbulent motions.
- The additional term is a new unknown for which a relation with the average velocity must be constructed via a turbulence model.

Integrate y- momentum equation across the boundary layer

$$p\approx p_e(x)-\rho\overline{v'^2}$$

So, unlike laminar BL, there is a slight variation of pressure across the turbulent BL due to velocity fluctuations normal to the wall, which is often no more than 4% of the stream-wise velocity and thus can be neglected.

The Bernoulli relation is assumed to hold in the inviscid free stream:

$$dp_e / dx \approx -\rho U_e dU_e / dx$$

Assume the free stream conditions, $U_e(x)$ is known. The boundary conditions are:

No slip: $\bar{u}(x,0) = \bar{v}(x,0) = 0$

Free stream matching: $\bar{u}(x,\delta) = U_e(x)$

<u>Flat plate boundary layer (zero pressure gradient)</u>

$Re_t = 5\times10^5 \sim 3\times10^6$ for a flat plate boundary layer

$Re_{crit} \sim 100{,}000$

$$\frac{c_f}{2} = \frac{d\theta}{dx}$$

As was done for the approximate laminar flat plate boundary-layer analysis, solve by expressing $c_f = c_f(\delta)$ and $\theta = \theta(\delta)$ and integrate, i.e., assume that the log-law valid across entire turbulent boundary-layer

$$\frac{u}{u^*} = \frac{1}{\kappa} \ln \frac{yu^*}{\nu} + B$$

neglect laminar sub and buffer layer and velocity defect regions at $y = \delta$, $u = U$

$$\frac{U}{u^*} = \frac{1}{\kappa}\ln\frac{\delta u^*}{\nu} + B$$

$$\frac{\delta u^*}{\nu} = Re_\delta\left(\frac{c_f}{2}\right)^{1/2}$$

or $$\left(\frac{2}{c_f}\right)^{1/2} = 2.44\ln\left[Re_\delta\left(\frac{c_f}{2}\right)^{1/2}\right] + 5$$

$$Re_\delta = \frac{U\delta}{\nu} \qquad c_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{2u^{*2}}{U^2} \qquad u^* = \sqrt{\frac{\tau_w}{\rho}}$$

$$\frac{U}{u^*} = \left(\frac{2}{c_f}\right)^{1/2}$$

Re_δ	10^4	10^5	10^6	10^7
	.00493	.00315	.00217	.00158

$$\frac{\delta u^*}{\nu} = \frac{U\delta}{\nu} \times \frac{u^*}{U} = Re_\delta\left(\frac{c_f}{2}\right)^{1/2}$$

$c_f(\delta) \cong .02\, Re_\delta^{-1/6}$ power-law fit

Next, evaluate

$$\frac{d\theta}{dx} = \frac{d}{dx}\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)dy$$

can use log-law or more simply a power law fit

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7}$$

Note: cannot be used to obtain cf (δ) since $\tau_w \to \infty$

$$\theta = \frac{7}{72}\delta = \theta(\delta)$$

$$u = U\,(y/\delta)^{1/7} \qquad \frac{d}{dx}(x^n) = n x^{n-1}\frac{dy}{dx}$$

$$u_y = U\,\frac{1}{7}(y/\delta)^{\frac{1}{7}-1}\delta^{-1} = \frac{U}{7\delta}\left(\frac{y}{\delta}\right)^{-6/7}\Bigg|_{y=0} = \infty$$

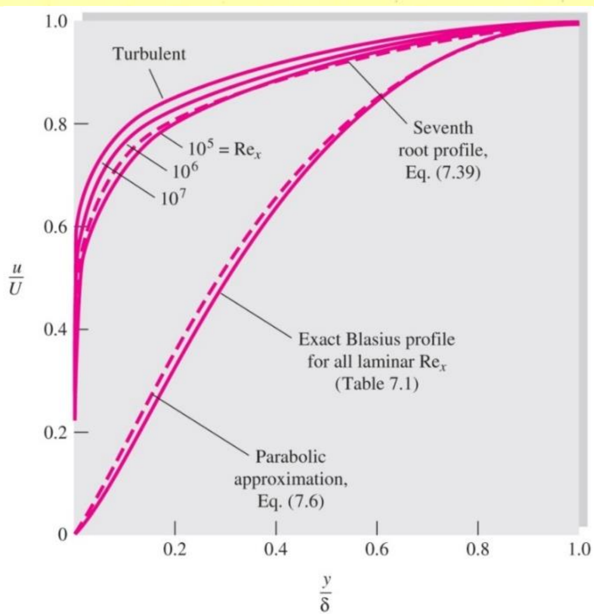
$$\theta = \int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy = \int_0^1 \left[y'^{1/7} - y'^{2/7}\right]\delta\,dy' \qquad y' = y/\delta,\quad dy' = dy/\delta,\quad dy = \delta\,dy'$$

$$\delta\left[\left(\frac{y}{\delta}\right)^{8/7}\left(\frac{7}{8}\right) - \left(\frac{y}{\delta}\right)^{9/7}\left(\frac{7}{9}\right)\right]_0^\delta$$

$$\delta\left[\frac{7}{8} - \frac{7}{9}\right] = \frac{7}{72}\delta$$

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\theta = \frac{7}{72}\delta$$



Comparison of dimensionless laminar and turbulent flat-plate velocity profiles

$$\Rightarrow \qquad \tau_w = c_f \frac{1}{2}\rho U^2 = \rho U^2 \frac{d\theta}{dx} = \frac{7}{72}\rho U^2 \frac{d\delta}{dx}$$

$$Re_\delta^{-1/6} = 9.72 \frac{d\delta}{dx}$$

or $$\frac{\delta}{x} = 0.16\, Re_x^{-1/7}$$

$$\delta \propto x^{6/7} \quad \text{almost linear}$$

i.e., much faster growth rate than laminar boundary layer

$$c_f = \frac{0.027}{Re_x^{1/7}}$$

$$\tau_{w,turb} = \frac{0.0135\mu^{1/7}\rho^{6/7}U^{13/7}}{x^{1/7}}$$

$\tau_{w,turb}$ decreases slowly with x, increases with ρ and U^2 and insensitive to μ

$c_f/2 = \Theta_x$ or $c_f = 2\Theta_x$ $\qquad c_f = \frac{2\tau_w}{\rho U^2}$

$$.02\, Re_\delta^{-1/6} = 2 \frac{d}{dx}\left(\frac{7}{72}\delta\right)$$

$$Re_\delta^{-1/6} = 9.72\, \delta_x = 9.72 \frac{dRe_\delta}{dRe_x}$$

$Re_\delta = \frac{U\delta}{\nu}$, $\quad Re_x = \frac{Ux}{\nu}$, $\quad \frac{Re_\delta}{Re_x} = \frac{\delta}{x}$

$$dRe_x = 9.72\, Re_\delta^{1/6}\, dRe_\delta$$

$$Re_x = 9.72 \frac{Re_\delta^{7/6}}{7/6}$$

$$Re_x = 8.3314\, Re_\delta^{7/6}$$

$$Re_\delta = Re_x^{6/7}/(8.3314)^{6/7} \qquad 6/7 = .8571$$

$$= .162\, Re_x^{6/7} \qquad 6.1539$$

$$= .162\, Re_x\, Re_x^{-1/7}$$

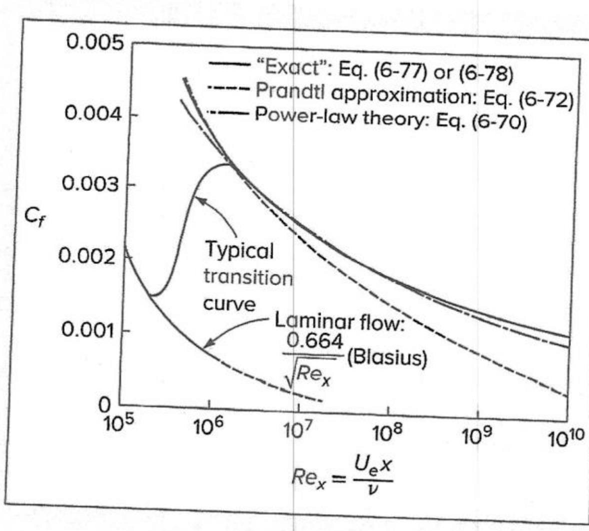
$$\delta/x = .162\, Re_x^{-1/7} \qquad \delta \propto x^{6/7}$$

nearly linear vs $x^{1/2}$ for laminar flow

$$c_f = .02\, Re_\delta^{-1/6} = .02\,(.162\, Re_x^{6/7})^{-1/6}$$

$$c_f = .02\,(.162)^{-1/6}\, Re_x^{-1/7}$$

$$c_f = .027\, Re_x^{-1/7} \qquad .1667$$



"Exact" = more sophisticated turbulence modeling

$C_f = .455/\ln^2(.06\,Re_x)$

Prandtl $C_f = .058/Re_x^{1/5}$

Power-law $C_f = .027/Re_x^{1/7}$

FIGURE 6-20
Local skin friction on a smooth flat plate for turbulent flow, showing several theories.

$$C_D = \frac{0.031}{Re_L^{1/7}} = \frac{7}{6} c_f(L) \qquad \delta^* = \frac{1}{8}\delta \qquad H = \frac{\delta^*}{\theta} = 1.3$$

$$D = \int_0^L \tau_w b\, dx$$

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 b L} = L^{-1}\int_0^L c_f\, dx = L^{-1}\int_0^L .027\, U^{-1/7} x^{-1/7} \nu^{1/7} dx$$

$$= \frac{.027\, U^{-1/7}\nu^{1/7}}{L}\int_0^L x^{-1/7} dx$$

$c_f = .027\, Re_x^{-1/7}$

$c_f(L) = .027\, Re_L^{-1/7}$

$$= \frac{.027\, U^{-1/7}\nu^{1/7}}{L} \frac{1}{6/7} x^{6/7}\Big|_0^L$$

$$= .0315\, U^{-1/7}\nu^{1/7} L^{-1/7}$$

$$= .0315\, Re_L^{-1/7}$$

C_D is 17% $> c_f(L)$ $\quad = 7/6\, c_f(L)$

$$\delta^* = \int_0^\delta \left[1 - \left(\frac{y}{\delta}\right)^{1/7}\right] dy \qquad y' = y/\delta, \quad dy' = dy/\delta$$

$$= \delta\int_0^1 \left[1 - y'^{1/7}\right] dy' = \delta\left[y' - y'^{8/7}\left(\tfrac{7}{8}\right)\right]_0^1$$

$$= \delta/8$$

$$H = (\delta/8)/(7\delta/72) = 1.3$$

These formulas are for a fully turbulent flow over a smooth flat plate from the leading edge; in general, give better results for sufficiently large Reynolds number $Re_L > 10^7$.

Alternate forms by using the same velocity profile $u/U = (y/\delta)^{1/7}$ assumption but using an experimentally determined shear stress formula $\tau_w = 0.0225\rho U^2(\nu/U\delta)^{1/4}$ are:

$$\frac{\delta}{x} = 0.37Re_x^{-1/5} \qquad c_f = \frac{0.058}{Re_x^{1/5}} \qquad C_D = \frac{0.072}{Re_L^{1/5}}$$

$$\text{shear stress:} \quad \tau_w = \frac{0.029\rho U^2}{Re_x^{1/5}}$$

These formulas are valid only in the range of the experimental data, which covers $Re_L = 5 \times 10^5 \sim 10^7$ for smooth flat plates.

Other empirical formulas are by using the logarithmic velocity-profile instead of the 1/7-power law:

$$\frac{\delta}{L} = c_f(0.98\log Re_L - 0.732)$$

$$c_f = (2\log Re_x - 0.65)^{-2.3}$$

$$C_D = \frac{0.455}{(\log_{10} Re_L)^{2.58}}$$

These formulas are also called as the *Prandtl-Schlichting skin-friction formula* and valid in the whole range of $Re_L \leq 10^9$.

For these experimental/empirical formulas, the boundary layer is usually "tripped" by some roughness or leading-edge disturbance, to make the boundary layer turbulent from the leading edge.

No definitive values for turbulent conditions since depend on empirical data and turbulence modeling.

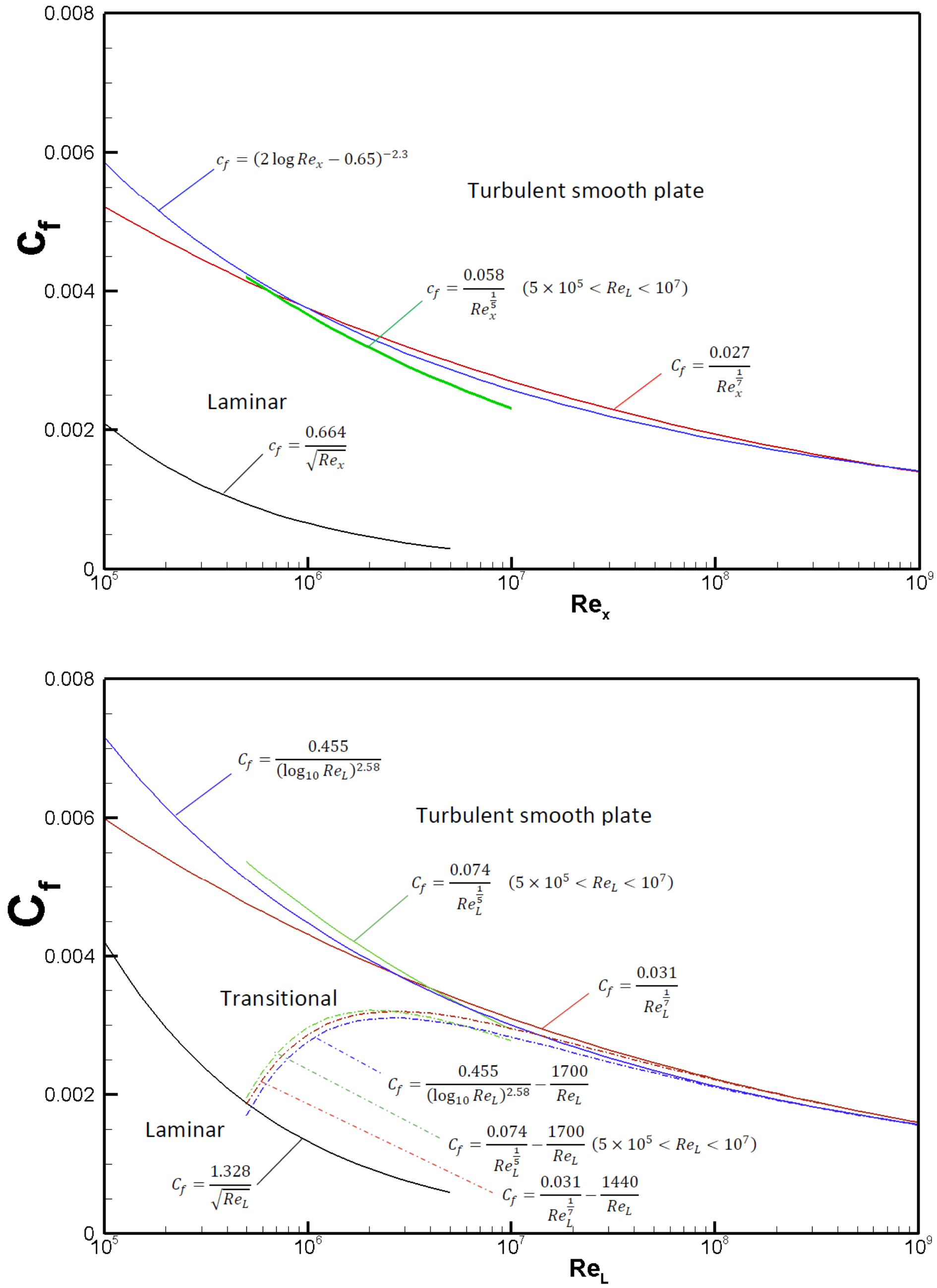
Finally, composite formulas that consider both the initial laminar boundary layer and subsequent turbulent boundary layer, i.e., in the transition region ($5 \times 10^5 < Re_L < 8 \times 10^7$) where the laminar drag at the leading edge is an appreciable fraction of the total drag:

$$C_D = \frac{0.031}{Re_L^{\frac{1}{7}}} - \frac{1440}{Re_L} \qquad (Re_{trans} = 5 \text{ x } 10^5)$$

$$C_D = \frac{0.031}{Re_L^{\frac{1}{7}}} - \frac{8700}{Re_L} \qquad (Re_{trans} = 3 \text{ x } 10^6)$$

$$C_D = \frac{0.074}{Re_L^{\frac{1}{5}}} - \frac{1700}{Re_L} \qquad (Re_{trans} = 5 \text{ x } 10^5)$$

$$C_D = \frac{0.455}{(\log_{10} Re_L)^{2.58}} - \frac{1700}{Re_L} \qquad (Re_{trans} = 5 \text{ x } 10^5)$$



Local friction coefficient c_f (top) and friction drag coefficient $C_D = C_f$ (bottom) for a flat plate parallel to the upstream flow. Lower case for skin friction and upper case for drag coefficient.

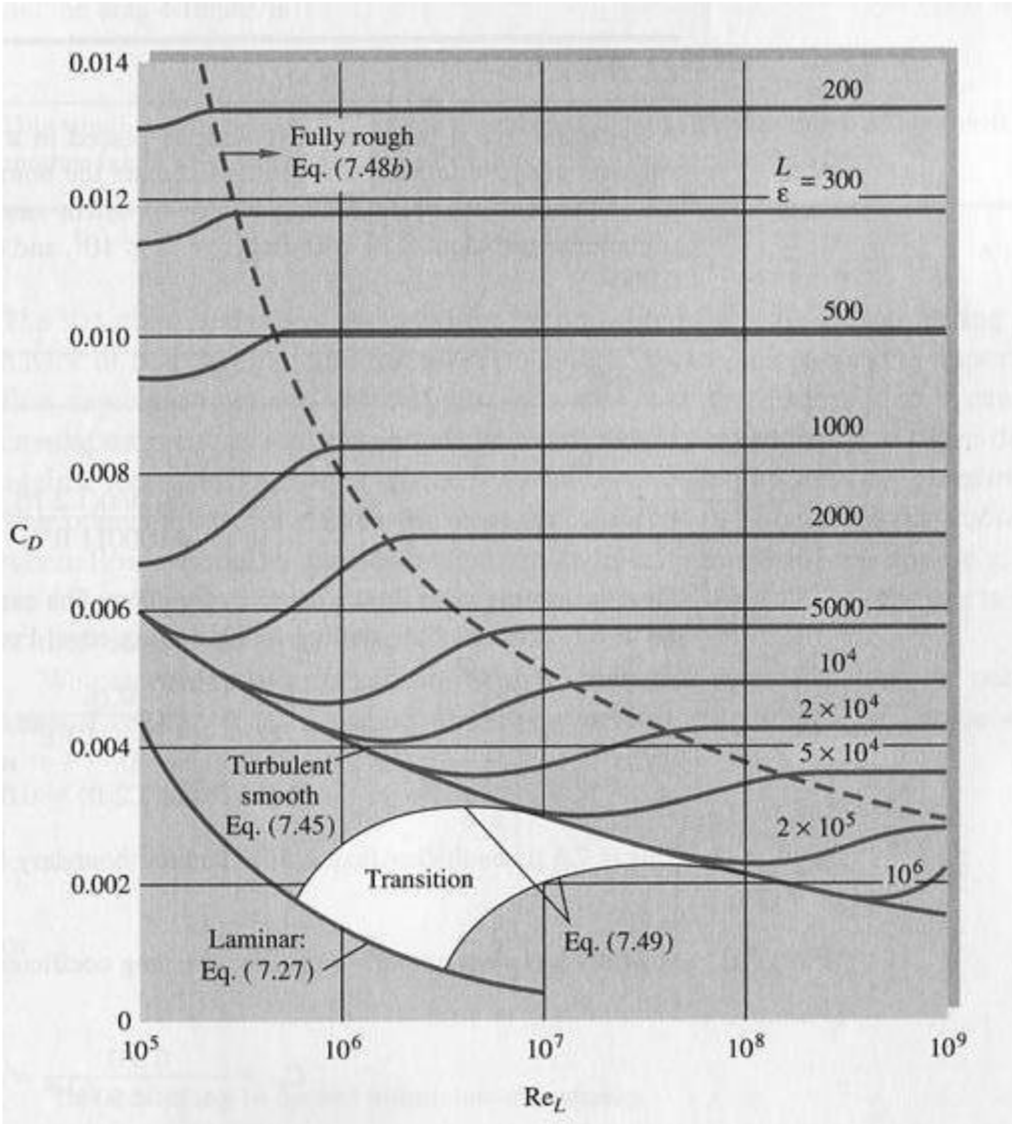


Fig. 7.6 Drag coefficient of laminar and turbulent boundary layers on smooth and rough flat plates.

$$c_f = (2.87 + 1.58\, log\frac{x}{\varepsilon})^{-2.5}$$
$$C_D = (1.89 + 1.62\, log\frac{L}{\varepsilon})^{-2.5}$$

} Fully rough flow

Again, shown on Fig. 7.6. along with transition region curves developed by Schlichting which depend on $Re_t = \begin{cases} 5\times10^5 \\ 3\times10^6 \end{cases}$

Momentum Integral Equations valid for BL solutions

The momentum integral equation has the identical form as the laminar-flow relation:

$$\frac{d\theta}{dx} + (2 + H)\frac{\theta}{U_e}\frac{dU_e}{dx} = \frac{\tau_w}{\rho U_e^{\ 2}} = \frac{C_f}{2}$$

For laminar flow:

(C_f, H, θ) are correlated in terms of simple parameter $\lambda = \frac{\theta^2}{\upsilon}\frac{dU_e}{dx}$

For Turbulent flow:

(C_f, H, θ) cannot be correlated in terms of a single parameter. Additional parameters and relationships are required that model the influence of the turbulent fluctuations. There are many possibilities all of which require a certain amount of empirical data. As an example, we will review the $\pi - \beta$ method.

π-β Method

As mentioned earlier, the momentum integral equation for turbulent flow has the identical form as the laminar-flow relation:

$$\frac{d\theta}{dx} = \frac{C_f}{2} - (2+H)\frac{\theta}{U_e}\frac{dU_e}{dx} \qquad \text{(I)}$$

With U(x) assumed known, there are three unknown C_f, H, θ for turbulent flow. Thus, at least two additional relations are needed to find unknowns. There are many possibilities for additional relations all of which require a certain amount of empirical data. As an example we will review the π−β method.

Cole's law of the wake:

By adding the wake to the log-law, the velocity profile for both overlap and outer layers can be written as:

$$u^+ = \frac{1}{\kappa}\ln y^+ + B + \frac{2\Pi}{\kappa} f(\eta)$$

where

$$\eta = y/\delta$$

$$f(\eta) = \sin^2(\frac{\pi}{2}\eta) = 3\eta^2 - 2\eta^3$$

$$\Pi = \kappa A/2$$

The quantity Π is called Coles' wake parameter.

By integrating wall-wake law across the boundary layer:

$$\lambda = a(\Pi)\frac{H}{H-1}$$

$$a(\Pi) = \frac{2 + 3.179\Pi + 1.5\Pi^2}{\kappa(1+\Pi)}$$

$$\mathrm{Re}_\theta = \frac{U\theta}{\upsilon} = \frac{1+\Pi}{\kappa H}\exp(\kappa\lambda - \kappa B - 2\Pi)$$

If we eliminate Π between these formulas, we obtain a unique relation among $C_f = 2/\lambda^2$, H and θ:

$$\begin{cases} C_f = 2/\lambda^2 = 2/[a(\Pi)\dfrac{H}{H-1}]^2 \\ a(\Pi) = \dfrac{2+3.179\Pi+1.5\Pi^2}{\kappa(1+\Pi)} \\ \mathrm{Re}_\theta = \dfrac{U\theta}{\upsilon} = \dfrac{1+\Pi}{\kappa H}\exp(\kappa\lambda-\kappa B-2\Pi) \end{cases} \qquad \text{(II)}$$

<u>Clauser's equilibrium parameter β:</u>

For outer layer,

$$U_e - \bar{u} = f(\tau_w, \rho, y, \delta, \frac{dp}{dx})$$

Using dimensional analysis:

$$\frac{U_e - \bar{u}}{(\tau_w/\rho)^{1/2}} = g(\frac{y}{\delta}, \frac{\delta}{\tau_w}\frac{dp}{dx})$$

Clauser (1954) replaced δ by displacement thickness δ^*:

$$\frac{U_e - \bar{u}}{(\tau_w/\rho)^{1/2}} = g(\frac{y}{\delta}, \beta)$$

$$\beta = \frac{\delta^*}{\tau_w}\frac{dp}{dx} = -\lambda^2 H \frac{\theta}{U_e}\frac{dU_e}{dx}$$

β is called Clauser's equilibrium parameter.

Das (1987) showed that EFD data points fit into the following polynomial correlation:

$$\beta = -0.4 + 0.76\Pi + 0.42\Pi^2$$

Therefore:

$$-\lambda^2 H \frac{\theta}{U_e}\frac{dU_e}{dx} = -0.4 + 0.76\Pi + 0.42\Pi^2 \qquad \text{(III)}$$

If we eliminate Π using that $\mathrm{Re}_\theta = \frac{U\theta}{\upsilon} = \frac{1+\Pi}{\kappa H}\exp(\kappa\lambda - \kappa B - 2\Pi)$, we obtain another relation among $C_f = 2/\lambda^2$, H and θ.

Equations (I), (II), and (III) can be solved simultaneously using say a Runge-Kutta method to find C_f, H, θ. Equations are solved with initial condition for $\theta(x_0)$ and integrated to $x=x_0+\Delta x$ iteratively. Estimated θ gives Re_θ and Π, β gives H. Lastly C_f is evaluated using Re_θ and H. Iterations required until all relations satisfied and then proceed to next Δx.