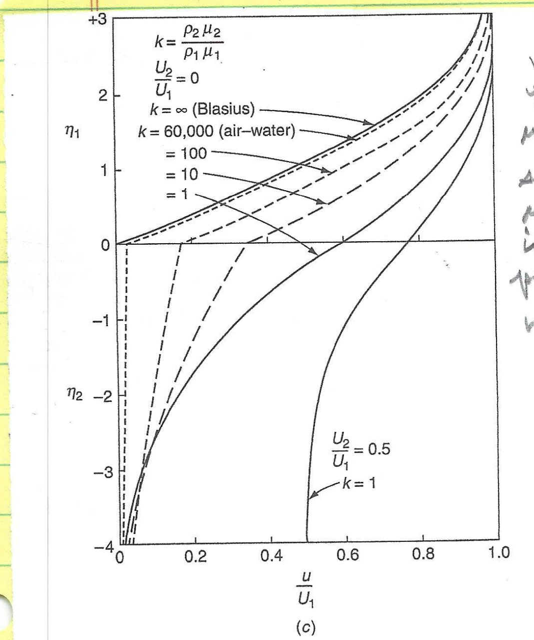
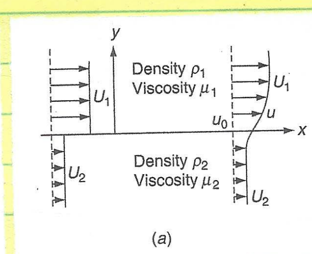
2D Mixing Layer: free shear layer

Two parallel streams, U_1 (upper) & U_2 (lower) meet at $x=0$. Progression downstream smooths discontinuity due to μ into S shaped velocity profile ie free-shear layer.

Assume p = constant upper/lower layer ie $p_x = 0$ such that $y=0$ dividing ψ at BL assumptions ie same eqn as flat plate BL but with different BC. Therefore use Blasius type similarity variable $\eta_\alpha = \left(\frac{u_1}{\nu_\alpha x}\right)^{1/2} y$ and $\psi = (\nu_\alpha u_1 x)^{1/2} f_\alpha(\eta_\alpha)$



Note: at $y=0$ $\mu_1 u_{1y} = \mu_2 u_{2y}$ such that μ itself important physical property not just $\nu = \mu/\rho$



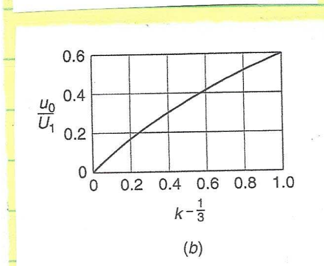


FIGURE 4-17
Velocity distribution between two parallel streams of different properties: (*a*) geometry; (*b*) velocities at the interface ($U_2 = 0$) (*c*) representative velocity profiles. [*After Lock (1951).*] (*By permission of The Clarendon Press, Oxford.*)

Blasius Similarity

$$u u_x + v u_y = \nu u_{yy}$$

$$\psi_\alpha = (\nu_\alpha u_1 x)^{1/2} f_\alpha(\eta_\alpha) = g_\alpha(x) f_\alpha$$

$$\eta_\alpha = [u_1/\nu_\alpha x]^{1/2} y$$
$$= h_\alpha(x)\, y$$

$$u_\alpha = u_1 f'_\alpha = \psi_{\alpha y}$$

$$\eta_{\alpha_x} = y\, h_{\alpha_x} = \frac{\eta_\alpha}{h_\alpha} h_{\alpha_x}$$

$$\psi_{\alpha y} = [\nu_\alpha u_1 x]^{1/2} f'_\alpha \eta_{\alpha y}$$
$$= [\cancel{\nu_\alpha} u_1 \cancel{x}]^{1/2} [u_1/\cancel{\nu_\alpha x}]^{1/2} f'_\alpha \qquad \eta_{\alpha y} = h_\alpha \qquad \eta_{\alpha yy} = 0$$
$$= u_1 f'$$

$$v_\alpha = -\psi_{\alpha_x} = -(g_{\alpha_x} f_\alpha + g_\alpha f_{\alpha_x}) = -(g_{\alpha_x} f_\alpha + g_\alpha f'_\alpha \eta_{\alpha x})$$

$$-v_\alpha = \frac{1}{2}\left[\frac{\nu_\alpha u_1}{x}\right]^{1/2} f_\alpha + [\nu_\alpha u_1]^{1/2} x^{1/2} f'_\alpha \eta_\alpha \left(-\tfrac{1}{2} x^{-1}\right) \qquad g_\alpha(x) = [\nu_\alpha u_1]^{1/2} x^{1/2}$$

$$g_{\alpha_x} = [\nu_\alpha u_1]^{1/2} \tfrac{1}{2} x^{-1/2}$$

$$v_\alpha = \frac{1}{2}\left[\frac{\nu_\alpha u_1}{x}\right]^{1/2} [\eta_\alpha f'_\alpha - f_\alpha] \qquad = \frac{1}{2}\left[\frac{\nu_\alpha u_1}{x}\right]^{1/2}$$

$$u_{\alpha x} = u_1 f''_\alpha \eta_{\alpha x} = u_1 f''_\alpha \left(-\frac{\eta}{2x}\right) \qquad h_\alpha = \left[\frac{u_1}{\nu_\alpha}\right]^{1/2} x^{-1/2}$$

$$u_{\alpha y} = u_1 f''_\alpha \eta_{\alpha y} = u_1 f''_\alpha [u_1/\nu_\alpha x]^{1/2} \qquad h_{\alpha_x} = -\frac{1}{2}\left[\frac{u_1}{\nu_\alpha}\right]^{1/2} x^{-3/2}$$

$$u_{\alpha yy} = u_1 f''' \eta^2_{\alpha y} = u_1 f'''_\alpha [u_1/\nu_\alpha x] \qquad h_{\alpha_x}/h_\alpha = \frac{-\frac{1}{2}\left[\frac{u_1}{\nu_\alpha}\right]^{1/2} x^{-3/2}}{\left[\frac{u_1}{\nu_\alpha}\right]^{1/2} x^{-1/2}}$$

drop α

$$u_1 f' \left[u_1 f'' \left(-\frac{\eta}{2x}\right)\right] + \frac{1}{2}\left[\frac{\nu u_1}{x}\right]^{1/2} [\eta f' - f] u_1 f'' \left[\frac{u_1}{\nu x}\right]^{1/2} \qquad = -\frac{1}{2} x^{-1}$$
$$= \nu u_1 f''' [u_1/\nu x] \qquad \eta_{\alpha x} = \frac{-\eta}{2x}$$

$$\cancel{u_1^2}\left(\frac{-\eta}{2x}\right) f' f'' + \frac{1}{2}\left(\frac{u_1^2}{x}\right)[\eta f' - f] f'' = \frac{u_1^2}{x} f'''$$

$$-\frac{\eta}{2} f' f'' + \frac{1}{2}(\cancel{\eta f' f''} - f f'') = f'''$$

$$2f''' + f f'' = 0$$

ie $\quad 2 f'''_\alpha + f_\alpha f''_\alpha = 0$

ODE solved numerically subject BCs.

BC: $u_1 = u_2 \quad v_x = 0 \quad \mu_1 u_{1y} = \mu_2 u_{2y} \quad y = 0$

$u_1 = U_1 \quad u_2 = U_2 \quad y = \pm\infty$

ie $f_1'(0) = f_2'(0) \quad f_1(0) = f_2(0) = 0 \quad \mu_1 u_1 f_1'' \left[\frac{u_1}{\nu_1 x}\right]^{1/2} = \mu_2 u_1 f_2'' \left[\frac{u_1}{\nu_2 x}\right]^{1/2}$

$f_1'(\infty) = U_1 \quad f_2'(-\infty) = \frac{U_2}{U_1}$ ie $f_1''[\mu_1 \rho_1]^{1/2} = f_2''[\mu_2 \rho_2]^{1/2}$

$f_1''(0) = \left[\frac{\rho_2 \mu_2}{\rho_1 \mu_1}\right]^{1/2} f_2''(0) = A^{1/2} f_2''(0)$

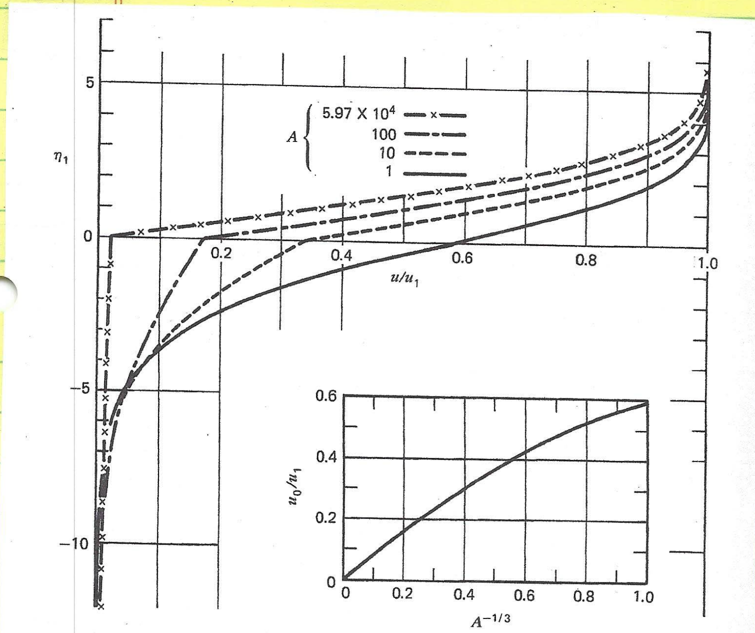


Figure 20.23 Velocity profiles between shear layers. $A = \rho_1\mu_1/\rho_2\mu_2$. Inset shows centerline velocity as a function of A. Reprinted by permission from Lock (1951) Oxford University Press.

two uniform flows arrive $x=0$ and sudden merge.
p = constant everywhere
$u_1 > 0$ whereas $u_2 \geq 0$
$y = 0$ dividing x

$A = 5.97 \times 10^4 = \frac{\text{water}}{\text{air}}$
shown with $u_2 = 0$
Also other A values down to 1 for which $u_1/u_2 = .58$ at $y = 0$

Discontinuity smoothed by μ as move downstream to S shaped free shear layer
Assume no mass transfer between fluids ie immiscible
Most practical cases: air/water and $\rho_2\mu_2 = \rho_1\mu_1$ ie same fluid
$A \uparrow$ lower fluid moves more slowly; air-water slow wind driven motion ② $f_2(-\infty)/\sqrt{2} = -.619$ = flat plate at $-\infty$ with BL shown η_0
① $A = 1$, $U_2 = 0$: asymmetric, $u/u_1(0) > .5$ due differences convection between upper/lower

Additional discussion

(1) For $\rho_2\mu_2 \neq \rho_1\mu_1$ jump fluid properties
20°C 1atm and discontinuity at interface

	ρ	μ	ν
air	1.205	1.8×10^{-5}	1.5×10^{-5}
water	1000	1.003×10^{-3}	1.005×10^{-6}
	kg/m^3	Ns/m^2 kg/ms	m^2/s

	ρ	μ	ν	$\rho\mu$	$(\rho\mu)^{1/2}$
water/air	832	56	.067	46592	216

∴ water/air jump condition mainly due ρ, which is reason sharp interface methods often only consider ρ jump and smooth μ.

(2) $v_2 = \frac{1}{2}\left[\frac{\nu_2 u_1}{x}\right]^{1/2}\left[\eta_2 f_2' - f_2\right]$ $\quad \eta_2 = \left[u_1/\nu_2 x\right]^{1/2} y$

$v_2(-\infty) = \frac{1}{2}\left[\frac{\nu_2 u_1}{x}\right]^{1/2}\left[(u_1/\nu_2 x)^{1/2}(-\infty)\frac{u_2}{u_1} - f_2(-\infty)\right]$

(3) Interesting compare diffusion vortex sheet

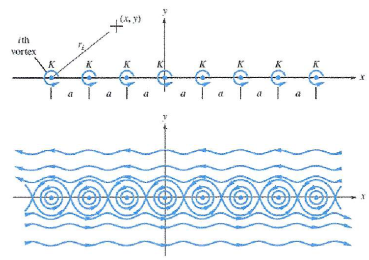
$u_t = \nu u_{yy}$ $\quad u(y,0) = U\,\mathrm{sgn}(y)$ $\quad u(\infty,t) = U$ $\quad u(-\infty,t) = -U$

$\delta = \pm 5.52\sqrt{\nu t}$ for $u = \pm .95U$ S shaped profile but with $\pm U$

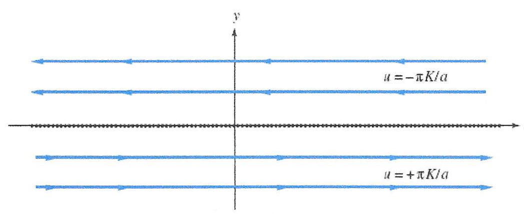
(4) Stability: linearized stability parallel viscous flow: $Re_{crit} = 0$

Also related Kelvin-Helmholtz interfacial instability for horizontal interface dividing two ideal fluids with different U and ρ ie ΔU and $\Delta\rho$ jump at interface ⇒ vortex sheet

Potential flow solution for vortex sheet: Superposition infinite row equally spaced vortices of equal strength.



From afar (i.e. $y \geq a$) looks like a thin sheet with velocity discontinuity.



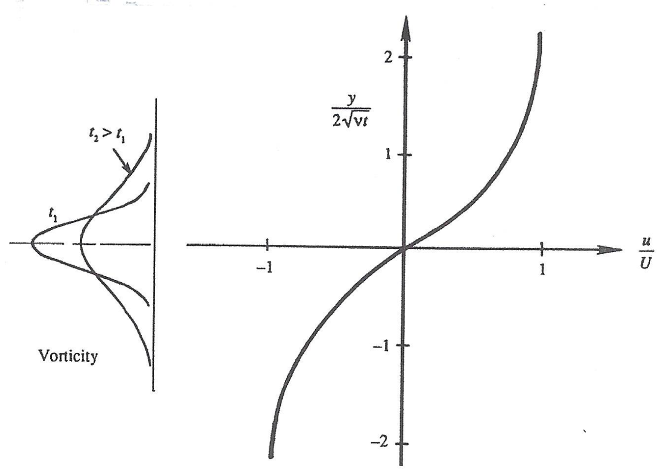


FIGURE 9.14 Viscous thickening of a vortex sheet. The left panel indicates the vorticity distribution at two times, while the right panel shows the velocity field solution in similarity coordinates. The upper half of this flow is equivalent to a temporally developing boundary layer.

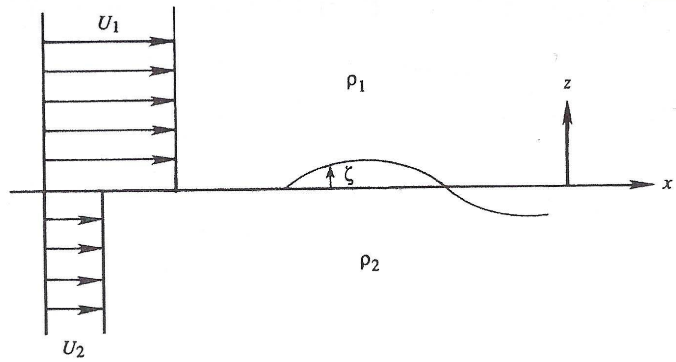


FIGURE 11.2 Basic flow configuration leading to the Kelvin-Helmholtz instability. Here the velocity and density profiles are discontinuous across an interface nominally located at $z = 0$. If the small vertical perturbation $\zeta(x,t)$ to this interface grows, then the flow is unstable.

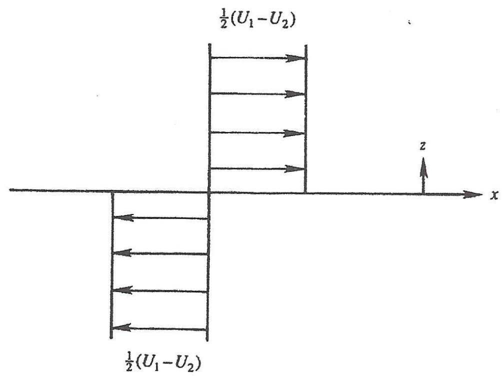


FIGURE 11.3 Background velocity field for the Kelvin-Helmholtz instability as seen by an observer traveling at the average velocity $(U_1 + U_2)/2$ of the two layers. When the densities of the two layers are equal, a disturbance to the interface will be stationary in this frame of reference.