

Part 2: Laminar Boundary Layer: $Re_{crit} = 91,000$; $Re_{trans} = 5 \times 10^5 - 3 \times 10^6$.

Similarity solutions (2D, steady, incompressible): method of reducing PDE to ODE by appropriate similarity transformation; also, because of transformation at least one coordinate lacks origin such that the solution collapses to same form at all length or time scales.

$$u_x + v_y = 0$$

$$uu_x + vu_y = UU_x + vv_{yy}$$

BCs: $u(x, 0) = v(x, 0) = 0$ no slip
 $u(x, \infty) = U(x)$ matching outer flow
 + inlet condition

For Similarity $\frac{u(x,y)}{U(x)} = F\left(\frac{y}{g(x)}\right)$ expect $g(x)$ related to $\delta(x)$

Or in terms of stream function $\Psi : u = \psi_y \quad v = -\psi_x$

For similarity $\psi = U(x)g(x)f(\eta) \quad \eta = y/g(x)$

$$u = \psi_y = Uf' \quad v = -\psi_x = -(U_xgf + Ug_xf - Ug_x\eta f')$$

BC:

$$u(x,0) = 0 \Rightarrow U(x)f'(0) = 0 \Rightarrow f'(0) = 0$$

$$v(x,0) = 0 \Rightarrow U_x(x)g(x)f(0) + U(x)g_x(x)f(0)$$

$$-U(x)g_x(x) \times 0 \times f'(0) = 0$$

$$\Rightarrow (U_x(x)g(x) + U(x)g_x(x))f(0) = 0$$

$$\Rightarrow f(0) = 0$$

$$u(x, \infty) = U(x) \Rightarrow U(x)f'(\infty) = U(x) \Rightarrow f'(\infty) = 1$$

Write boundary layer equations in terms of ψ

$$\psi_y \psi_{yx} - \psi_x \psi_{yy} = UU_x + \vartheta \psi_{yyy}$$

Substitute

$$\psi_{yy} = Uf'' / g$$

$$\psi_{yyy} = Uf''' / g^2$$

$$\psi_{xy} = U_x f' - Uf'' \eta g_x / g$$

Assemble them together:

$$\begin{aligned} (Uf') \left(U_x f' - Uf'' \frac{g_x}{g} \right) - (U_x g f + U g_x f - U g_x \eta f') (Uf'' / g) \\ = UU_x + \nu (Uf''' / g^2) \end{aligned}$$

$$UU_x f'^2 - UU_x f f'' - (U^2 g_x / g) f f'' = UU_x + \nu \frac{U}{g^2} f'''$$

$$UU_x f'^2 - \frac{U}{g} (Ug)_x f f'' = UU_x + \nu \frac{U}{g^2} f'''$$

$$f''' + \underbrace{\frac{g}{\nu} (Ug)_x}_{C_1} f f'' + \underbrace{\frac{g^2}{\nu} U_x}_{C_2} (1 - f'^2) = 0$$

Where for similarity C_1 and C_2 are constant or function η only

- i.e. for a chosen pair of C_1 and $C_2 \rightarrow U(x), g(x)$ can be found, i.e., potential flow is NOT known a priori.
- Then solution of $f''' + C_1 f f'' + C_2 (1 - f'^2) = 0$ gives $f(\eta) \rightarrow u(x, y), \tau_w = \mu \frac{\partial u}{\partial y} \Big|_w = \frac{\mu U f''(0)}{g}, \delta, \delta^*, \theta, H, C_f, C_D$

The Blasius Solution for Flat-Plate Flow

$$U = \text{constant} \rightarrow U_x = 0 \rightarrow C_2 = 0$$

$$\text{Then } C_1 = \frac{U}{\nu} g g_x \neq \text{function}(x)$$

$$\frac{d}{dx}(g^2) = \frac{2C_1\nu}{U} \implies g(x) = [2C_1\nu x/U]^{1/2}$$

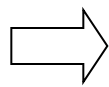
$$\text{Let } C_1 = 1, \text{ then } g(x) = \sqrt{\frac{2\nu x}{U}} \implies \eta = y \sqrt{\frac{U}{2\nu x}} = \frac{y}{\sqrt{\frac{2\nu x}{U}}} \propto \frac{y}{\delta}$$

$$\text{Note } \frac{\delta}{x} = \frac{5}{\sqrt{Re_x}}, \text{ i.e., } \delta = \frac{5x}{\sqrt{\frac{Ux}{\nu}}} = 5 \sqrt{\frac{\nu x}{U}}$$

$$\psi = U[2\nu x/U]^{1/2} f\left(y \sqrt{\frac{U}{2\nu x}}\right) = \sqrt{2\nu U x} f(\eta)$$

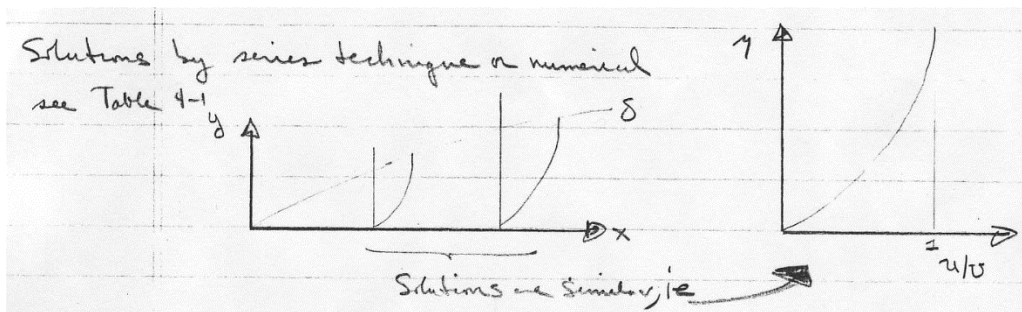
$$u = \psi_y = U f'$$

$$v = -\psi_x = U g_x (\eta f' - f) = [U\nu/2x]^{1/2} (\eta f' - f) = \frac{U(\eta f' - f)}{\sqrt{2Re_x}}$$



$f''' + ff'' = 0$ $f(0) = f'(0) = 0, f'(\infty) = 1$	Blasius equations for Flat Plate Boundary Layer
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Solutions by series or numerical methods



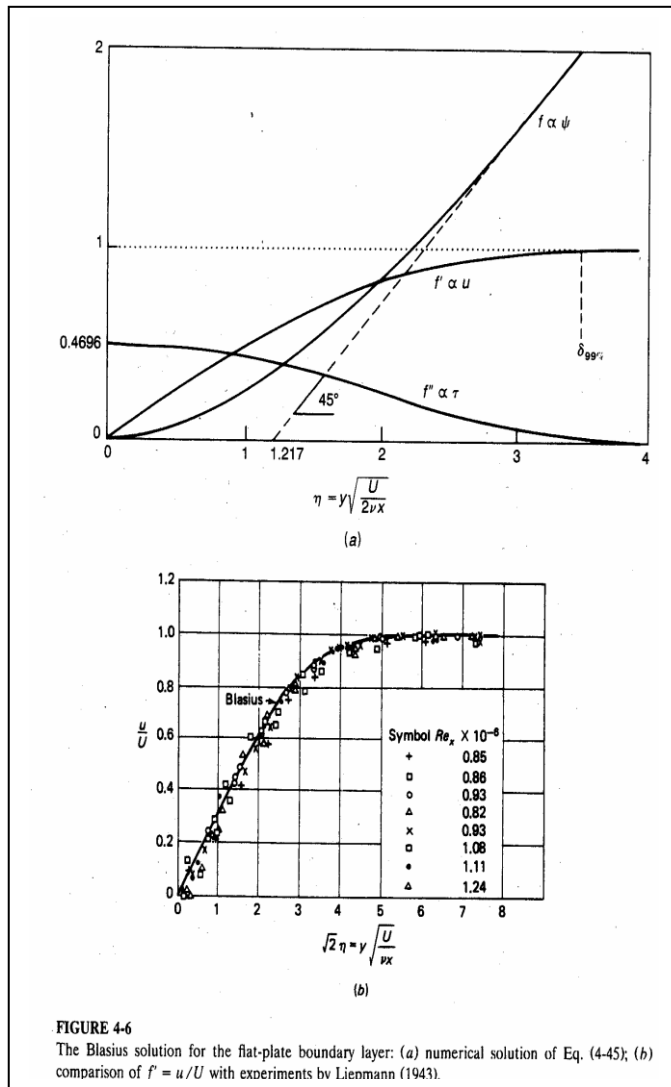


Fig. 7.10. Local coefficient of skin friction on a flat plate at zero incidence in incompressible flow, determined from direct measurement of shearing stress by Liepmann and Dhawan [6, 18]
 Theory: laminar from eqn. (7.32); turbulent from eqn. (21.12)

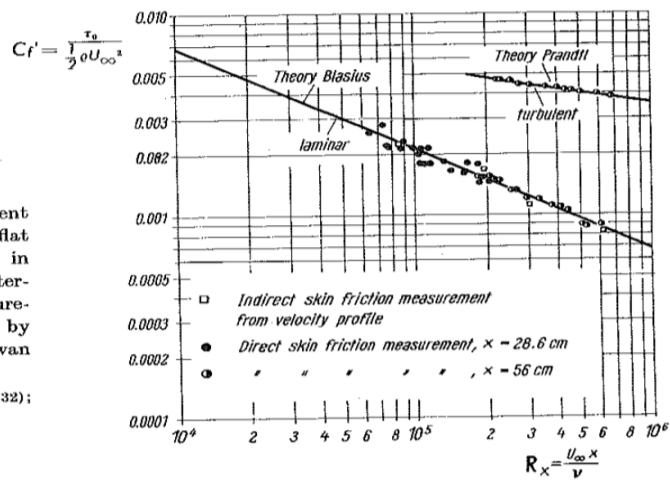


TABLE 4-3

Highly resolved numerical solution of the Blasius equation for flow over a flat plate, Eq. (4-60)

η	$\eta/\eta_{99\%}$	$f(\eta)$	$f'(\eta)$	$f''(\eta)$	$f'''(\eta)$
0.0	0.00000	0.00000	0.00000	0.46960	0.00000
0.0	0.05761	0.00939	0.09391	0.46931	-0.00441
0.4	0.11521	0.03755	0.18761	0.46725	-0.01755
0.6	0.17282	0.08439	0.28058	0.46173	-0.03896
0.8	0.23042	0.14967	0.37196	0.45119	-0.06753
1.0	0.28803	0.23299	0.46063	0.43438	-0.10121
1.2	0.34563	0.33366	0.54525	0.41057	-0.13699
1.4	0.40324	0.45072	0.62439	0.37969	-0.17114
1.6	0.46084	0.58296	0.69670	0.34249	-0.19966
1.8	0.51845	0.72887	0.76106	0.30045	-0.21899
2.0	0.57606	0.88680	0.81669	0.25567	-0.22673
2.2	0.63366	1.05495	0.86330	0.21058	-0.22215
2.4	0.69127	1.23153	0.90107	0.16756	-0.20636
2.6	0.74887	1.41482	0.93060	0.12861	-0.18196
2.8	0.80648	1.60328	0.95288	0.09511	-0.15249
3.0	0.86408	1.79557	0.96905	0.06771	-0.12158
3.2	0.92169	1.99058	0.98036	0.04637	-0.09230
3.3	0.95049	2.08883	0.98456	0.03781	-0.07899
3.4	0.97929	2.18747	0.98797	0.03054	-0.06679
3.47 [†]	1.00000	2.25856	0.99000	0.02603	-0.05878
3.5	1.00810	2.28641	0.99071	0.02441	-0.05582
3.6	1.03690	2.38559	0.99289	0.01933	-0.04611
3.8	1.09451	2.58450	0.99594	0.01176	-0.03039
4.0	1.15211	2.78389	0.99777	0.00687	-0.01914
4.2	1.20972	2.98356	0.99882	0.00386	-0.01152
4.4	1.26732	3.18338	0.99940	0.00208	-0.00663
4.6	1.32493	3.38330	0.99970	0.00108	-0.00366
4.8	1.38253	3.58325	0.99986	0.00054	-0.00193
5.0	1.44014	3.78323	0.99994	0.00026	-0.00098
5.2	1.49774	3.98323	0.99997	0.00012	-0.00047
5.4	1.55535	4.18322	0.99999	0.00005	-0.00022
5.6	1.61296	4.38322	1.00000	0.00002	-0.00010
5.8	1.67056	4.58322	1.00000	0.00001	-0.00004
6.0	1.72817	4.78322	1.00000	0.00000	-0.00002

[†]Actual value to 16 significant digits: $\eta_{\delta} = 3.471886880405967$.

$$\frac{u}{U} = 0.99 \text{ when } \eta = 3.5 \rightarrow \frac{\delta}{x} = \frac{5}{\sqrt{Re_x}} \quad Re_x = \frac{Ux}{\nu}$$

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy = \int_0^\infty (1 - f') d\eta \sqrt{\frac{2\nu x}{U}} \rightarrow \frac{\delta^*}{x} = \frac{1.7208}{\sqrt{Re_x}}$$

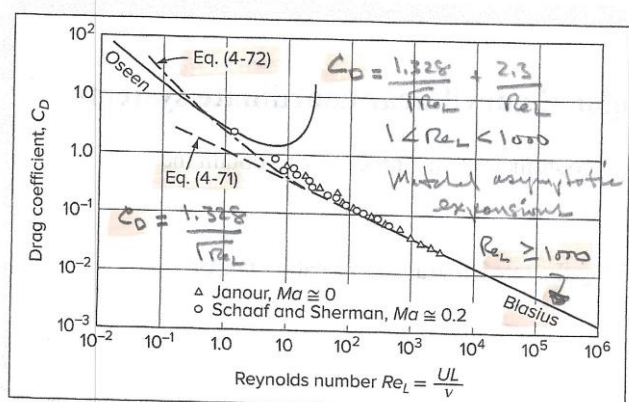
$$\theta = \int_0^\infty \left(1 - \frac{u}{U}\right) \frac{u}{U} dy = \int_0^\infty (1 - f') f' \sqrt{\frac{2\nu x}{U}} d\eta \rightarrow \frac{\theta}{x} = \frac{0.664}{\sqrt{Re_x}}$$

$$\frac{\delta^*}{\theta} = H = \text{shape parameter } 2.59$$

$$\tau_w = \mu \left. \frac{\partial u}{\partial y} \right|_w = \frac{\mu U f''(0)}{\sqrt{\frac{2\nu x}{U}}} \rightarrow C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.664}{\sqrt{Re_x}} = \frac{\theta}{x}$$

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 L} = \int_0^L C_f \frac{dx}{L} = \frac{1.328}{\sqrt{Re_L}} \quad Re_L = \frac{UL}{\nu}$$

$$\frac{v}{U} = \frac{\eta f' - f}{\sqrt{2Re_x}} \ll 1 \quad \text{for } Re_x \gg 1$$



Stokes flow: Oseen
 Improvement, i.e.,
 laminar
 convection
 $C_D = \frac{4\pi}{Re_L [1 - \gamma + \ln(16/Re_L)]}$
 $\gamma = 0.577216 = \text{Euler constant}$
 $Re_L \ll 1$

FIGURE 4-11 Theoretical and experimental drag of a flat plate.

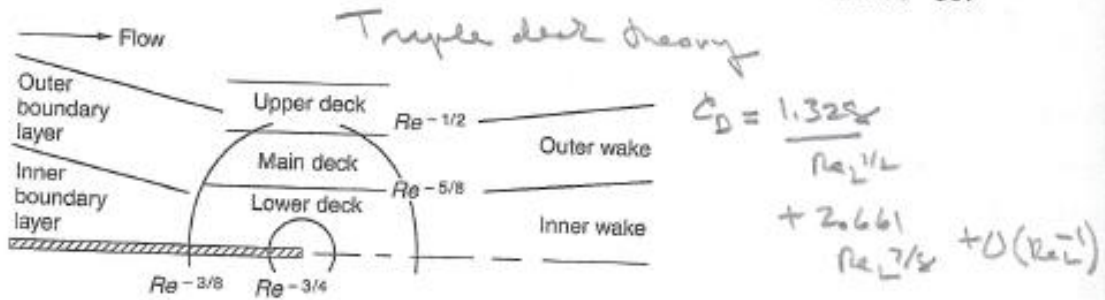
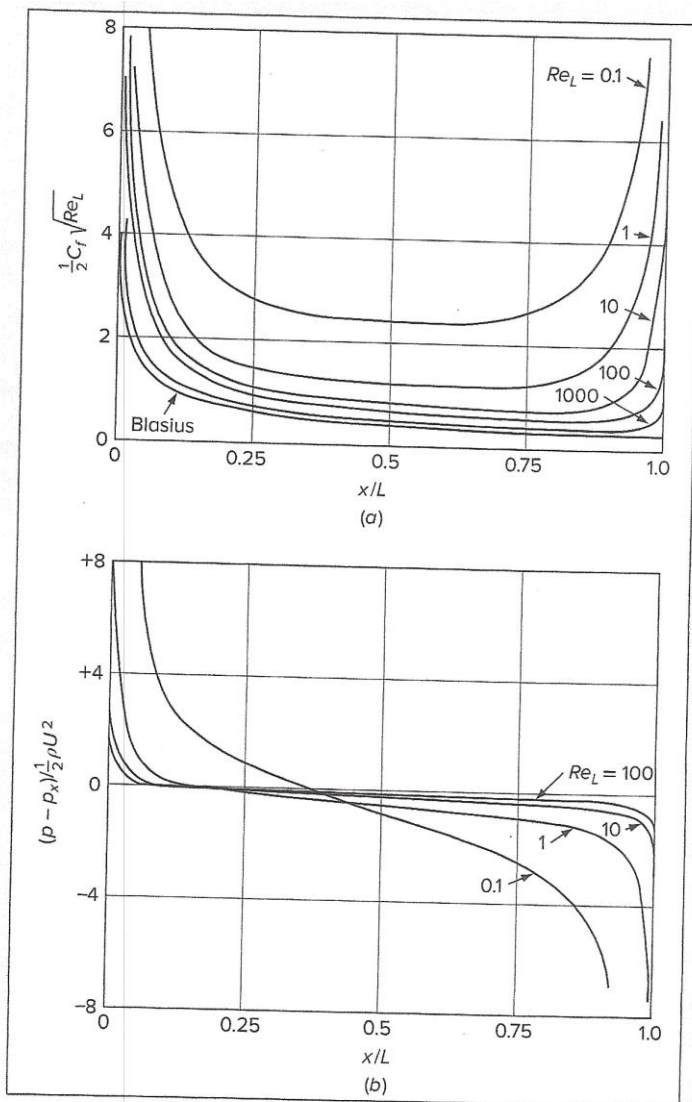


FIGURE 4-40
 Sketch of the triple-deck region at the trailing edge of a flat plate, merging into two-layer upstream and downstream regions. [After Stewartson (1969) and Messiter (1970).]



WS CFD - 1 < Re_L < 1000
low Re large LE & TE effect
BL approximation Re_L > 1000

FIGURE 4-12
 Numerical solution of the full Navier-Stokes equations for flat-plate flow at moderate Reynolds numbers: (a) local friction coefficient; (b) local surface pressure. [After Dennis and Dunwoody (1966).]

Falkner-Skan Wedge Flows

$$f''' + C_1 f f'' + C_2 (1 - f'^2) = 0 \quad \begin{array}{l} f = f(\eta) \\ \eta = y/g(x) \\ u/U = f'(\eta) \end{array}$$

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1$$

$$C_1 = \frac{g}{v} (Ug)_x \quad C_2 = \frac{g^2}{v} U_x \quad (\text{Blasius Solution: } C_2=0, C_1=1)$$

} Similarity form of BL equations

Consider $(Ug^2)_x = 2Ug g_x + g^2 U_x$

$$= 2Ug g_x + 2g^2 U_x - g^2 U_x = 2g(Ug_x + gU_x) - g^2 U_x$$

$$= 2g(Ug)_x - g^2 U_x$$

$$= 2vC_1 - vC_2$$

Hence $(Ug^2)_x = v(2C_1 - C_2)$

Choose $C_1=1$ and let $C_2 = C$

Integrate $Ug^2 = v(2 - C)x$ note $g^2 = vC/U_x$

Combine $\frac{UvC}{U_x} = v(2 - C)x$

Rearrange $\frac{dU}{U} = \frac{C}{2-C} \frac{1}{x}$

Integrate $\ln U = \frac{C}{2-C} \ln x + \ln k$ where $\ln k = \text{constant}$

$$\ln U = \ln x^{\frac{C}{2-C}} + \ln k = \ln kx^{\frac{C}{2-C}}$$

$$U(x) = kx^{C/(2-C)}$$

$$g(x) = \left[\frac{vC}{U_x} \right]^{\frac{1}{2}} \text{ note } U_x = k \frac{C}{2-C} x^{\left(\frac{C}{2-C}-1\right)} = k \frac{C}{2-C} x^{\left(\frac{-2(1-C)}{2-C}\right)}$$

$$g(x) = \left[\frac{vC}{k \frac{C}{2-C} x^{\left(\frac{-2(1-C)}{2-C}\right)}} \right]^{\frac{1}{2}} = \left[\frac{v(2-C)}{k} x^{\left(\frac{-2(1-C)}{2-C}\right)} \right]^{\frac{1}{2}} = \sqrt{\frac{v(2-C)}{k}} x^{\frac{1-C}{2-C}}$$

using $a^{1/2}b^{1/2}=(ab)^{1/2}$ and $(a^m)^n=a^{mn}$

$$\text{Alternatively, } U_x = \frac{C}{2-C} kx^{(C/(2-C))} x^{-1} = \frac{C}{2-C} Ux^{-1}$$

$$\text{Such that } g(x) = \left[\frac{vC}{\frac{C}{2-C} Ux^{-1}} \right]^{\frac{1}{2}} = \left[\frac{v(2-C)x}{U} \right]^{\frac{1}{2}}$$

$$\text{Change constant: } C = \beta = \frac{2m}{m+1} \text{ and } m = \frac{\beta}{2-\beta}$$

$$U(x) = kx^m$$

$$\eta = \frac{y}{g} = y \sqrt{\frac{m+1}{2} \frac{U}{vx}}$$

$$f''' + ff'' + \beta(1 - f'^2) = 0$$

$$f(0) = f'(0) = 0 \text{ and } f'(\infty) = 1$$

Note:

$$2m/(m+1) = \frac{2\beta}{2-\beta} / \left(\frac{\beta}{2-\beta} + 1 \right) = \frac{2\beta}{2-\beta} / \left(\frac{2}{2-\beta} \right) = \beta$$

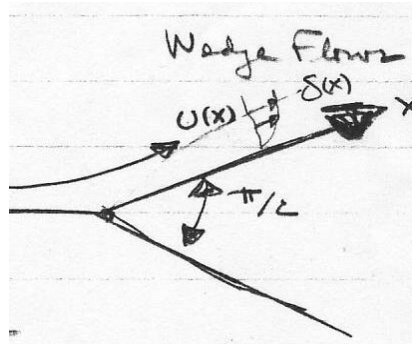
$$2-C = 2 - 2m/(m+1) = 2/(m+1)$$

$$g(x) = \left[\frac{v(2-C)x}{U} \right]^{\frac{1}{2}} = \left[\frac{v2x}{U(m+1)} \right]^{\frac{1}{2}}, \text{ i.e., } g(x)^{-1} = \left[\frac{m+1}{2} \frac{U}{vx} \right]^{\frac{1}{2}}$$

Numerical solutions for $-0.19884 \leq \beta \leq 1.0$.

↑
Separation ($\tau_w = 0$)

$$U(x) = kx^m$$



Solutions show many commonly observed characteristics of BL flow:

- The parameter β is a measure of the pressure gradient, dp/dx . For $\beta > 0$, $dp/dx < 0$ and the pressure gradient is favorable. For $\beta < 0$, the $dp/dx > 0$ and the pressure gradient is adverse.
- Negative β solutions drop away from Blasius profiles as separation approached.
- Positive β solutions squeeze closer to wall due to flow acceleration.
- Accelerated flow: τ_{\max} near wall.
- Decelerated flow: τ_{\max} moves toward $\delta/2$

Inviscid flow past wedges & corners

$\psi(x) = Kx^m$ exact solution potential flow
past wedge or corner shapes

Plane polar coordinates $\nabla^2 \chi = 0$, $\nabla \times \mathbf{v} = 0$

$$\rightarrow \nabla^2 (\chi) + \frac{1}{r^2} \chi_{\theta\theta} = 0 \quad w_r = \frac{1}{r} \chi_\theta \quad w_\theta = -\chi_r$$

$$\chi(r, \theta) = C r^{m+1} \sin[(m+1)\theta] \quad \beta = \frac{2m}{m+1}$$

$$m = \beta / (2 - \beta)$$

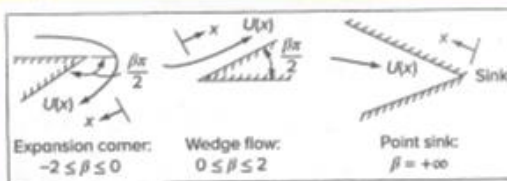


FIGURE 4-15
Some examples of Falkner-Skan potential flows.

This expression yields certain radial streamlines that can be interpreted as the "walls" of a wedge or a corner, as shown in Fig. 4-15, depending on the value of $\beta = 2m/(m+1)$. The velocity along these walls has the form $U = Kx^m$, which represents the freestream driving the boundary layer on the wall, with $x = 0$ at the tip of the wedge. The most prominent cases are:

$-2 \leq \beta \leq 0$, $-\frac{1}{2} \leq m \leq 0$: flow around an expansion corner of turning angle $\beta\pi/2$

$\beta = 0$, $m = 0$: the flat plate

$0 \leq \beta \leq +2$, $0 \leq m \leq \infty$: flow against a wedge of half-angle $\beta\pi/2$

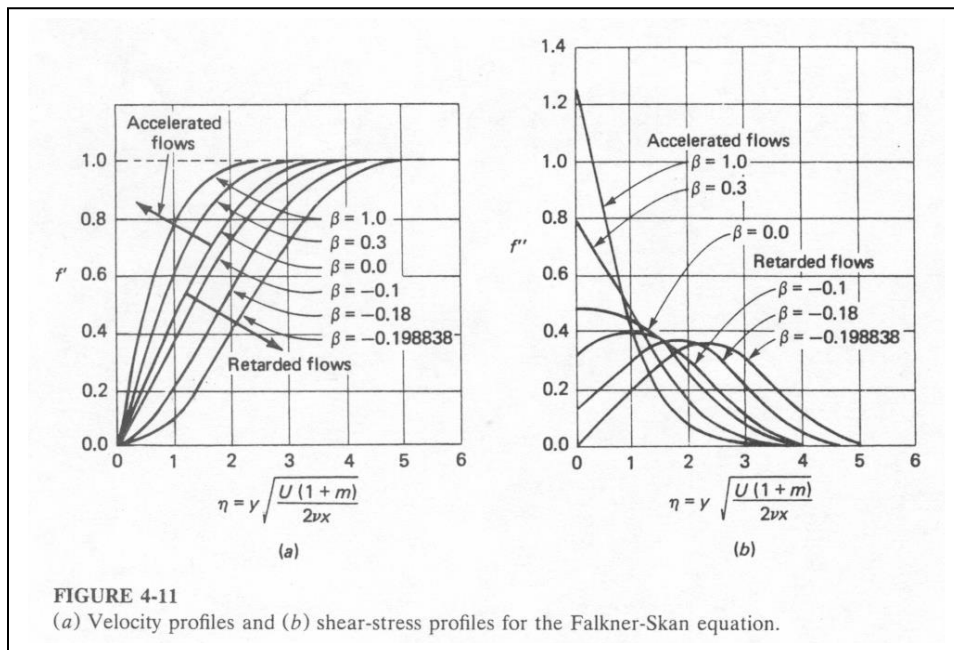
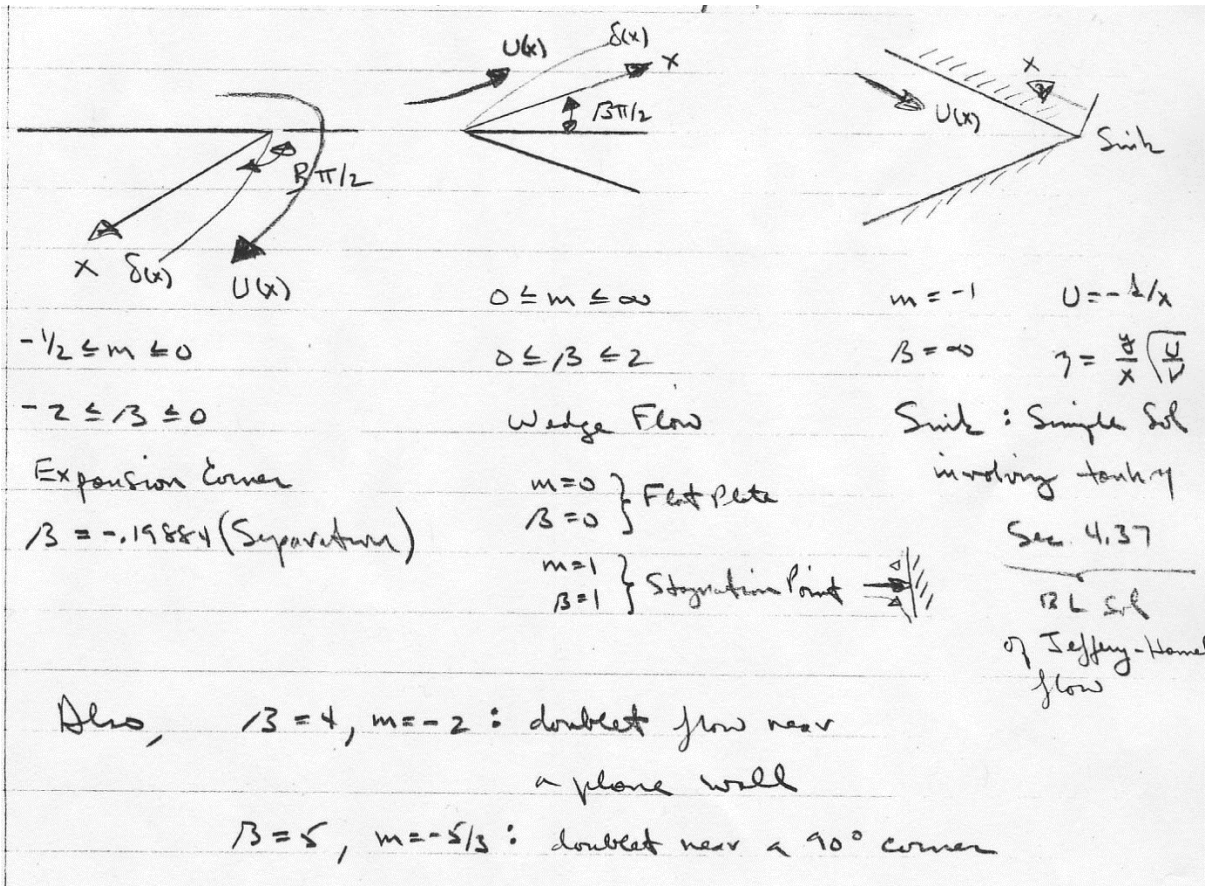
$\beta = 1$, $m = 1$: the plane stagnation point (180° wedge)

$\beta = +4$, $m = -2$: doublet flow near a plane wall

$\beta = +5$, $m = -\frac{5}{3}$: doublet flow near a 90° corner

$\beta = +\infty$, $m = -1$: flow toward a point sink [i.e., the boundary-layer version of the Jeffery-Hamel flow in a convergent wedge (Sec. 3-8)].

These are *similar* flows, i.e., for a given β , the velocity profiles all look alike when scaled by $U(x)$ and $\delta(x)$. They may also be used, with modest success, to predict the behavior of nonsimilar flows.



$$u = U(x) f' \quad \text{where } U(x) = \frac{1}{2} x^m$$

↗ velocity at edge of BL

$$\beta = \frac{2m}{m+1} \quad \gamma = \frac{1}{2} \sqrt{g(x)} = \frac{1}{2} \left[\frac{m+1}{2} \frac{U(x)}{U'(x)} \right]^{1/2}$$

$$\delta(x) = \left[\frac{2\nu}{(m+1)U} \right]^{1/2} x^{(1-m)/2} = \frac{1}{2} \sqrt{\frac{2\nu}{(m+1)U}} x^{(1-m)/2}$$

$$\delta^*(x) = \left(\frac{2\nu}{m+1} \right)^{1/2} \left(\frac{U(x)}{U} \right)^{1/2} \int_0^{\infty} (1-f') dy \quad \delta(x) \quad \left(\frac{2\nu}{(m+1)U} \right)^{1/2} x^{(1-m)/2}$$

$$\tau_w(x) = \mu \left(\frac{m+1}{2} \right)^{1/2} \left(\frac{U^2}{\nu x} \right)^{1/2} f''(0) \quad \mu \left(\frac{m+1}{2\nu} \right)^{1/2} U x^{(3m-1)/2}$$

$$\delta \propto x^{(1-m)/2} \quad \delta^* \propto x^{(1-m)/2} \quad \tau_w \propto x^{(3m-1)/2}$$

Special Cases:

$U(x) = \text{const}$ 1. Blasius: $m=0$ $\delta \propto x^{1/2}$ $\delta^* \propto x^{1/4}$ $\tau_w \propto x^{-1/2}$

$U(x) = 2x$ 2. Stagnation Point: $m=1$ $\delta, \delta^* = \text{const}$ $\tau_w \propto x$

linear increase of velocity U along x of BL exactly balanced by viscous diffusion such that $\delta, \delta^* = \text{const}$

$U = 2x^2$ 3. $m=3$: $\delta \propto x^{-1}$ $\delta^* \propto x^{-1}$ $\tau_w = \text{constant}$

acceleration rate flow tendency increase τ_w

exactly balanced by viscous diffusion

squid momentum away from surface

is reduce τ_w

Alternative derivation Fokker-Planck equation #1

$$u_x + v_y = 0 \quad \text{ie } v_y = -u_x$$

Leibniz rule

$$\int_{a(x)}^{b(x)} \left[\frac{\partial}{\partial x} u(x,y) \right] dy = \frac{d}{dx} \int_{a(x)}^{b(x)} u(x,y) dy + u(x,a) \frac{da}{dx} - u(x,b) \frac{db}{dx}$$

$$\int_0^y dx = - \int_0^y \frac{\partial u}{\partial x} dy = - \frac{\partial}{\partial x} \int_0^y u dy - u(x,y) \frac{dy}{dx} + u(x,0) \frac{dy}{dx}$$

$$v = - \frac{\partial}{\partial x} \int_0^y u dy \quad \uparrow = 0 \text{ since integration for } x = \text{constant}$$

x BL equation: $u u_x + v u_y = v v_x + v u_{yy}$

$$u u_x - \left(\frac{\partial}{\partial x} \int_0^y u dy \right) u_y = \text{RHS}$$

integrate differential equation

Combine x and y single variable $\gamma(x,y)$ to transform pde to ode on $f(\gamma)$ u only
only to achieve similarity.

assume: $u(x,y) = U(x) f(\gamma)$ where γ dimensionless and $U(x)$ not arbitrary but part of solution

$$u_x = U_x f + U f' \gamma_x$$

$$u_y = U f'' \gamma_y = U f'' h$$

$$\gamma_{yy} = U f''' \gamma_y^2 + U f'' \gamma_{yy} \quad \text{let } \gamma = y h(x)$$

$$\gamma_x = y h_x = \frac{2 h_x}{h}$$

$$\gamma_y = h(x) \quad \gamma_{yy} = 0$$

$$U f' (U_x f + U f' \frac{2 h_x}{h}) - \left(\frac{\partial}{\partial x} \int_0^y U f dy \right) U f'' h$$

$$= U U_x + U U f''' h^2$$

$$\int f' dy = h^{-1} \int \frac{df}{dy} dy = f/h \quad \frac{\partial}{\partial x} (U f/h) = U_x f/h - \frac{U f h_x}{h^2} + \frac{U}{h} f' \frac{2 h_x}{h}$$

Since $U(x) \wedge h(x) \neq f(\gamma) \wedge f(\gamma)$ for fixed x

$$\sigma f' (\sigma_x f' + \sigma f' \frac{\gamma h_x}{h}) + (-\sigma_x f/h + \frac{\sigma f h_x}{h^2} - \frac{\sigma}{h} f' \gamma h_x) \sigma f'' h$$

$$= \sigma \sigma_x + \sigma \sigma f'' h^2$$

$$\cancel{\sigma \sigma_x f''} + \cancel{\sigma^2 \gamma h_x f' f''} - \cancel{\sigma \sigma_x f f''} + \frac{\sigma^2 h_x}{h} f f'' - \cancel{\sigma^2 \gamma h_x f' f''}$$

$$= \sigma \sigma_x + \sigma \sigma h^2 f''$$

$$\sigma \sigma_x (f'^2 - f f'' - 1) + \frac{\sigma^2 h_x}{h} f f'' = \sigma \sigma h^2 f''$$

$$f''' = \frac{\sigma_x}{\sigma h^2} (f'^2 - f f'' - 1) + \frac{\sigma h_x}{\sigma h^2} f f''$$

$\neq f(x)$ *for similarity* $\neq f(x)$
 $\text{only } f(x)$

assume $\gamma = c_1 x^a$ $\sigma = k x^m$ $m = 2a + 1$

$$h = [k \frac{m+1}{2v}]^{1/2} x^{m/2} \quad h = c x^a \quad \sigma_x = k m x^{m-1} \quad a = \frac{m-1}{2} \quad a-1 = \frac{m-1}{2} - 1$$

$$= [k \frac{m+1}{2v} x^m x^{-1}]^{1/2} \quad h_x = a c x^{a-1} \quad = \frac{m-3}{2}$$

$$= [\frac{m+1}{2v} \frac{\sigma}{k v}]^{1/2}$$

$$= f(x) \quad \frac{k m x^{m-1}}{v c x^{m-1}} = \frac{k m}{v c} \quad \frac{k x^m (\frac{m-1}{2}) c x^{\frac{m-3}{2}}}{v c^2 x^{\frac{2(m-1)}{2}}} = \frac{k (\frac{m-1}{2}) c x^{\frac{2m+m-3}{2}}}{v c^2 x^{\frac{2m-2+2(m-1)}{2}}}$$

$$= \frac{k \frac{m-1}{2}}{v c^2}$$

$$f''' + \frac{\sigma_x}{\sigma h^2} (1 - f'^2 + f f'') - \frac{\sigma_x}{\sigma h^2} f f'' = 0$$

$$k (m - \frac{m-1}{2}) / v c^2$$

$$f''' + \frac{\sigma_x}{\sigma h^2} (1 - f'^2) + \underbrace{\left(\frac{\sigma_x}{\sigma h^2} - \frac{d \sigma_x}{\sigma h^2} \right)}_1 f f'' = 0$$

$$k (\frac{m-1}{2}) / v c^2$$

$$f = 1 \quad c^2 = k (m+1) / 2v$$

$$\beta = \frac{k m}{v c^2} = \frac{k m}{v k (m+1) / 2v}$$

$$\therefore f''' + \beta (1 - f'^2) + f f'' = 0$$

$$= 2m / m + 1$$

Alternative description Falkner-Skan equation #2

what types of external flows
see similarity solutions vbl eq.?

Assume self similar velocity profile: $u(x,y) = u_c(x) f(\eta)$
 $\eta = y / \delta(x)$

u_c, δ determined as part of solution.

$$dx = x_x dx + x_y dy$$

for $x = \text{constant}$ $x(x,y) - x(x,0) = \int_0^y x_y dy$

$$x=0 \text{ at } y=0 \Rightarrow f(0)=0 = \delta u_c \int_0^{\frac{y}{\delta}} \frac{y}{\delta} \eta d(\frac{y}{\delta})$$

at $x/\delta u_c = f(\eta) = \delta u_c \int_0^{\eta} f' d\eta = \delta u_c f(\eta)$
ie $x = \delta u_c f(\eta)$

$$\eta = y \delta^{-1}$$

$$\eta_x = -y \delta^{-2} \delta_x$$

$$= -\eta \delta^{-1} \delta_x$$

$$u = x_y v = -x_x = -f(\delta u_c)_x - (\delta u_c) f' \eta_x$$

$$= -f(\delta_x u_c + \delta u_{cx}) + (\delta u_c) f' \eta \delta^{-1} \delta_x$$

$$= -f(\delta_x u_c + \delta u_{cx}) + \eta u_c f' \delta_x$$

continuity automatically satisfied

$$u_y = u_c f'' \delta^{-1}$$

$$u_{yy} = u_c f''' \delta^{-2}$$

$$u u_x + v u_y = u u_{cx} + v u_{yy}$$

$$u_c f' (u_c \eta f' - u_c f'' \eta \delta^{-1} \delta_x) + [\dots] u_c f'' \delta^{-1} = u_c u_{cx} + v u_c f''' \delta^{-2}$$

$$u_c u_{cx} f' - u_c^2 \eta \delta^{-1} f'' f' - (\delta_x u_c^2 + \delta u_c u_{cx}) \frac{f f'}{\delta} + u_c^2 \eta \delta^{-1} \frac{f'' f'}{\delta} = \text{RHS}$$

$$u_c u_{cx} f'^2 - (\delta_x u_c^2 + \delta u_c u_{cx}) \frac{f f'}{\delta} - u_c u_{cx} = \frac{v u_c f'''}{\delta^2}$$

$$f''' + \frac{u_c u_{cx} \delta^2}{v u_c} (1 - f'^2) + \frac{\delta}{v u_c} (\delta_x u_c^2 + \delta u_c u_{cx}) f f' = 0$$

$$f''' + \frac{2\alpha x \delta^2}{\sqrt{}} (1-f^2) + \frac{\delta}{\sqrt{}} (\delta x u_c + \delta u_{cx}) f f'' = 0$$

$$\frac{d}{dx} (\delta u_c)$$

$$f''' + \alpha f f'' + \beta (1-f^2) = 0 \quad \beta = \frac{u_c \delta^2}{\sqrt{}}$$

ODE for f assuming $\alpha, \beta \neq f(x)$ $\alpha = \frac{\delta}{\sqrt{}} (\delta u_c) x$
 = constant

$$\text{note: } 2\alpha - \beta = \frac{1}{\sqrt{}} \frac{d}{dx} (\delta^2 u_c)$$

$$(2\alpha - \beta)(x - x_0) = \frac{1}{\sqrt{}} \delta^2 u_c$$

Assume $x_0 = 0$ and $\delta(0) = 0 \Rightarrow \delta = \left[\frac{\sqrt{(2\alpha - \beta)x}}{u_c} \right]^{1/2}$

usually $u_c, x > 0$; however, may have opposite signs

$$2\alpha - \beta = 1 \quad 2\alpha - \beta = -1 \quad 2\alpha - \beta = -1$$

any value acceptable

since δ/δ similarity

variable δ/δ also OK

$$\delta = \left[\pm \frac{\sqrt{x}}{u_c} \right]^{1/2}$$

original equation, whereas $\alpha = \text{constant}$

often $\alpha = 1$ and such that $\delta = \left[\frac{\sqrt{(2-\beta)x}}{u_c} \right]^{1/2}$

$u_c(x)$ from $\beta = \frac{\pm \sqrt{x}}{u_c}$ $u_{cx} = \pm \frac{x}{u_c} u_{cx} = \pm \frac{x}{u_c} \frac{1}{x} u_c$

$$\beta = \pm \frac{x}{u_c} \frac{du_c}{dx}$$

$$\beta \frac{dx}{x} = \pm \frac{du_c}{u_c}$$

$$\ln |x| + \beta \ln x = \pm \ln u_c + \ln u_0^{-1}$$

$$\ln \left(\frac{x}{u_c} \right)^\beta = \pm \ln \frac{u_c}{u_0}$$

L, u_0 same sign u_c, x

Self similar BL u_c
 simple power law

$$u_c = u_0 \left(\frac{x}{u_c} \right)^m$$

$m = \beta$ u_c, x same sign
 $= -\beta$ u_c, x opposite sign

$$\gamma = \frac{y}{\delta} = \frac{y}{[\pm vx/u_c]^{1/2}} = \frac{y/L}{\sqrt{Re} \left(\frac{x}{L}\right)^{1/2}} \quad Re = \frac{u_c L}{\nu}$$

For u_c, x same sign $m = \beta \quad 2\alpha - m = 1$

$$\alpha = m + 1/2$$

$$f''' + \frac{m+1}{2} f f'' + m(1-f'^2) = 0$$

where for BL flow $u=0, v=0 \quad y=0$

$$u = u_c \quad y \rightarrow \infty$$

$$u(x, y) = u_c(x) f'(\gamma) \quad \gamma = y/\delta$$

$$v(x, y) = -f(u_c \delta)_x + \gamma u_c f' \delta_x$$

$$\infty \quad f(0) = 0 \quad f'(0) = 0 \quad \wedge \quad f'(\infty) = 1$$

I_c not needed; however, $x_1 = 0 \quad \gamma$ singular

FS equation nonlinear ODE so a priori not known which we have solutions or whether or not unique.

$m \geq 0$ unique solutions for:

$m = 0$:	Blasius flow over a flat plate with a sharp leading edge; also the local flow at any cusp leading edge
$0 < m < 1$:	Flow over a wedge with half-angle $\theta_{1/2} = m/(m+1)$ with $0 < \theta_{1/2} < \pi/2$
$m = 1$:	Hiemenz flow toward a plane stagnation point: Section 11.9
$1 < m < 2$:	Flow into a corner with $\theta_{1/2} > \pi/2$; a flow of this type may be difficult to produce experimentally
$m > 2$:	No corresponding simple ideal flow

Many people have contributed to the classification and computation of Falkner-Skan flows. Rosenhead (1963) contains a good summary by C. W. Jones and E. J. Watson. From these works some of the complicated behavior at negative values of m can be pieced together. When $-0.0904 < m < 0.0$, there are an infinite number of solutions for each value of m (Fig. 20.4). However, not all of these solutions are physically acceptable. One of the main arguments in establishing boundary layer theory is that the viscous effects are confined to a thin region near the wall. In light of this fact, people have proposed that a boundary layer should approach the free stream exponentially:

$$1 - u^* = 1 - f' \sim Ae^{-B\eta} \quad \text{as } \eta \rightarrow \infty$$

If this condition is applied, there are only two known acceptable solutions for each m in the range $-0.0904 < m < 0$. One of the solutions has $u > 0$ for all η , while the other has the interesting characteristic that there is backflow for a small region near the wall [Stewartson's (1954) reverse-flow profiles]. When m is exactly equal to -0.0904 , only one solution exists. This profile has zero shear stress at the wall and therefore is on the verge of separating for all x .

For $-1 < m < -0.0904$, all solutions for a given m tend to oscillate about $f' = 1$ as η becomes infinite. At each value of m , one of these solutions has just one region where the velocity $f' > 1$, and then $f' \rightarrow 1$ exponentially. Because a laminar boundary layer with these super velocities may be difficult to produce experimentally, some workers reject these solutions as physically impossible.

The case $m = -1$ with u_x and x having opposite signs, $u = -a_0/x$ ($u_0 = -a_0$), represents a solid wall in the flow field of an ideal line sink. When two walls are present, the problem represents the flow into a wedge. The differential equation in this case has an exact closed-form solution (Problem 20.9). On the other hand, the equivalent problem with the sign changed so the flow comes from a source ($m = -1$ and $u = +a_0/x$) has no solution. This means that boundary layer theory does not produce a similarity solution for flows in a flat-wall diffuser. These flows require a nonsimilar solution.

Most of the complicated behavior in Falkner-Skan solutions happens when m is between -1 and 0 . When $m < -1$ we again find a unique solution. All solutions in the range $m < -1$ have the flow going from large x toward $x = 0$. Hence, the flows are strongly accelerated with $u = -a_0(x/L)^m$, $m < -1$.

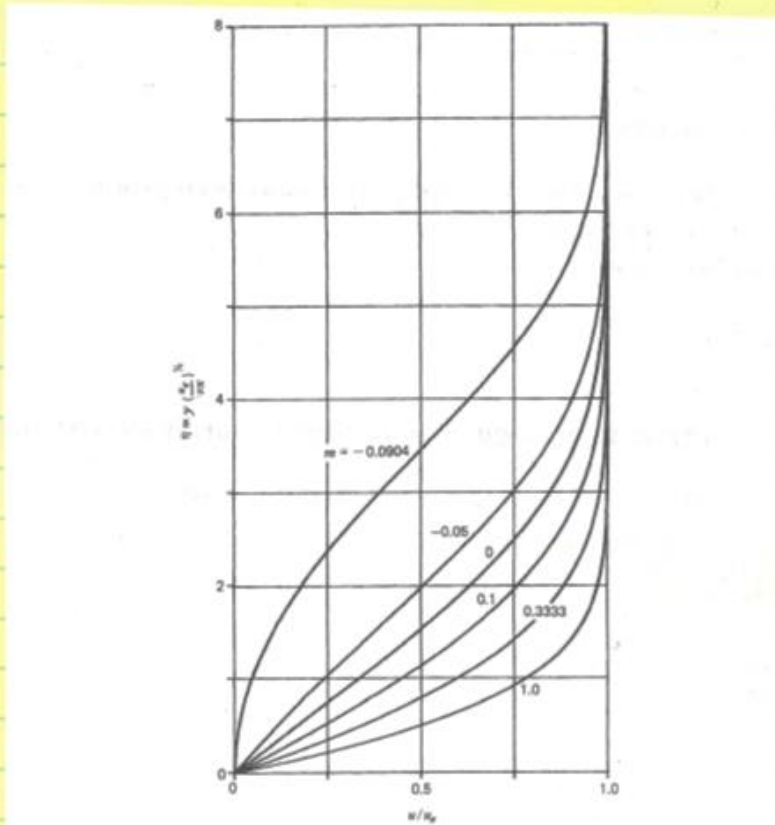


Figure 20.4 Falkner-Skan profiles. The parameter m indicates the external velocity variation through $u_e = u_0 x^m$.

Vertical Velocity

From outside BL, as $Re \uparrow \infty$ $u_e/u_0 \rightarrow 0$ since $\delta \rightarrow 0$

From inside BL, u_e some other remaining term.

X-momentum equation since

$u_y \rightarrow \infty$ as $\delta \rightarrow 0$ as $Re \rightarrow \infty$

is $0 \times \infty$ finite.

$$u_e \sim \frac{u_0}{L} \left(\frac{L}{\delta}\right) = \frac{u_0}{\delta} \sqrt{Re}$$

$$\frac{\delta}{L} \sim \frac{1}{\sqrt{Re}}$$

$$u_x \sim \frac{u_0^2}{L} \times \delta \sim \frac{u_0^2}{\delta}$$

$$\delta^2 \sim \frac{u_0 L}{u_0^2} \text{ i.e. } \frac{\delta}{L} \sim \frac{1}{\sqrt{Re}}$$

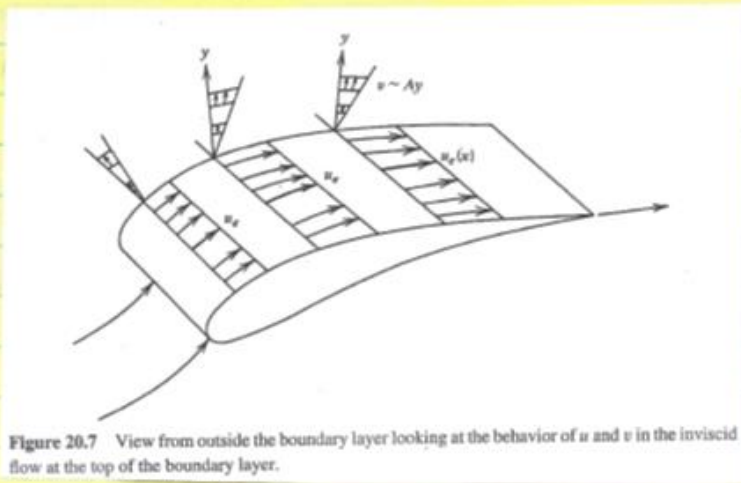


Figure 20.7 View from outside the boundary layer looking at the behavior of u and v in the inviscid flow at the top of the boundary layer.

① Inviscid flow near wall continuity equation

$$\frac{du_c}{dx} + \frac{dv}{dy} = 0 \Rightarrow v|_{\text{inviscid}} = \int_0^y \frac{\partial u}{\partial y} dy = -\frac{du_c}{dx} \int_0^y dy$$

Inviscid flow near wall

1. $u_c = u_c(x)$ assuming $\frac{du_c}{dx}$ constant $\sim -\frac{du_c}{dx} y$ as $y \rightarrow 0$
2. $\frac{\partial u}{\partial x} = 0$ at $\delta = 0$
3. $v \sim Ay$ i.e. grows linearly

② Viscous BL flow near wall continuity equation

$$v = \int_0^y \frac{\partial u}{\partial y} dy = - \int_0^y \frac{du}{dx} dy = \frac{1}{dx} \int_0^y (u_c - u) dy = -\frac{du_c}{dx} y$$

$$v(y \rightarrow \infty) = -\frac{du_c}{dx} y + \frac{1}{dx} (u_c \delta^*) \quad \text{where } \delta^* = \int_0^{\infty} (1 - \frac{u}{u_c}) dy$$

↑
inviscid v

influence BL profile
on v i.e.

v BL
correction
inviscid flow

2nd order solution
inviscid flow instead
of $v(\text{wall}) = 0$ use
 $v(\text{wall}) = \frac{1}{dx} (u_c \delta^*)$

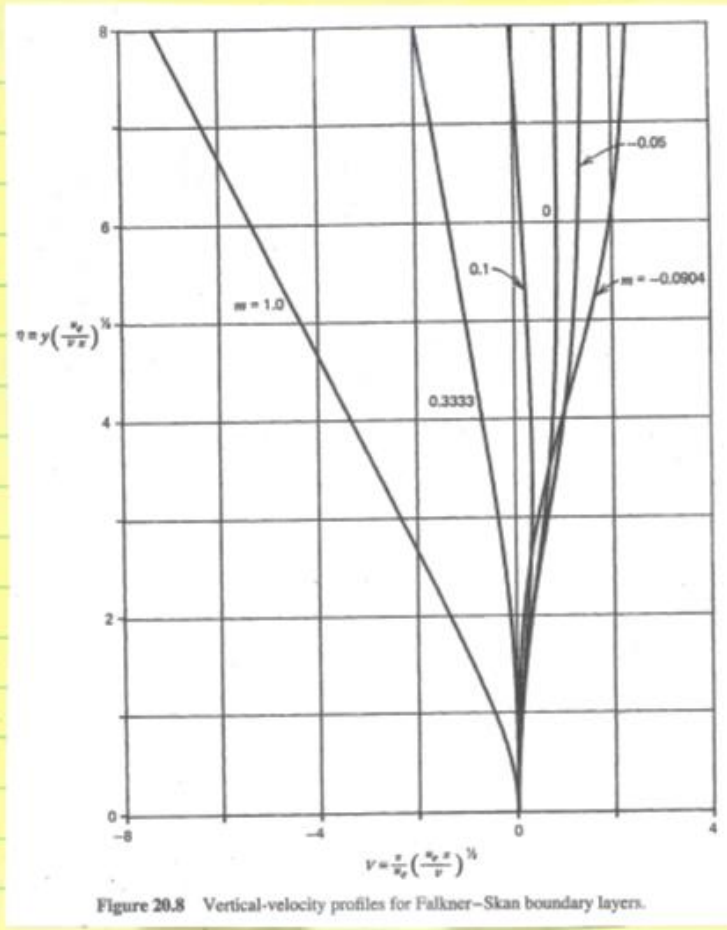


Figure 20.8 Vertical-velocity profiles for Falkner-Skan boundary layers.

$v(\eta)$ for Falkner-Skan BL solutions various in β
 $\beta = -\gamma x = -\tau \frac{d}{dx}(u_e \delta) + \gamma u_e \delta' \frac{dx}{dx}$ For $\gamma \rightarrow 0$ no approach ^{undamental} form prevails ^{form previous} question
 $v(\infty) = \frac{u}{u_e} \left(\frac{u_e \delta}{\nu} \right)^{1/2} = -\beta \eta + \frac{x}{u_e \delta} \frac{d}{dx}(u_e \delta')$
 $\beta \neq 0$ η intercept $\delta = \left[\pm \frac{\nu x}{u_e} \right]^{1/2}$
 $v(\infty) \rightarrow \infty$

BL separates no slip & match inviscid flow as $y \rightarrow \infty$
 $u \rightarrow u_e$
 $v \rightarrow$ curve with slope $-\frac{du_e}{dx}$ some inviscid flow v near wall

Flat Plate with wall Suction or Blowing

Blasius solution with $w_w \ll U \neq 0$ & $w_w = 0$
 Similarly requires

$$w_w = -\gamma_x = \sqrt{\frac{\nu U}{2x}} (-f' - f) \Big|_{\eta=0} \quad \eta = y \sqrt{\frac{U}{2\nu x}}$$

$$= \sqrt{\frac{\nu U}{2x}} (-f(0)) \quad \text{as } f(0) \neq 0$$

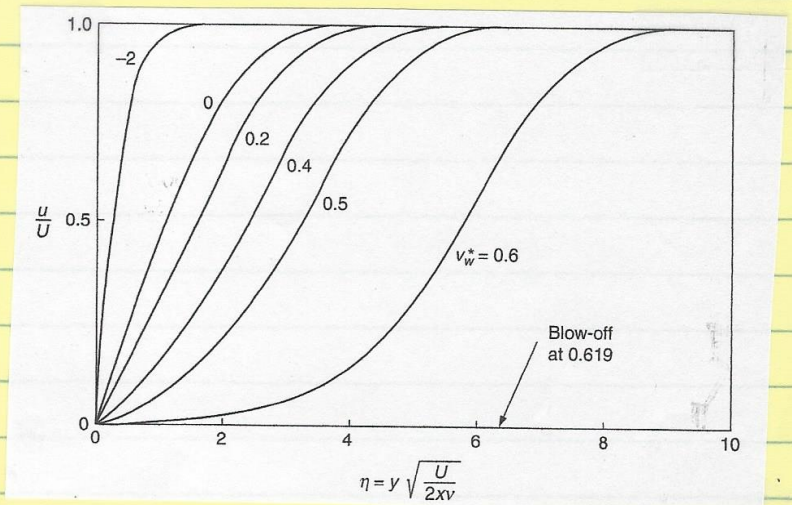
with $w_w \propto x^{-1/2}$

Solve Blasius equation with: $f'(0) = 0, f''(0) = 1$
 & $f(0) \neq 0$

Suction-blowing parameter $w_w^* = \frac{w_w}{U} \sqrt{Re_x} = -f(0) / \sqrt{2}$

$w_w < 0$ suction
 thin BL, increases τ_w
 similar favorable P_x
 & stable transition

$w_w > 0$ blowing
 thickens BL, reduces τ_w
 similar adverse P_x
 S-shaped with inflection point
 i.e. unstable to turbulent transition



$w_w^* = 0.619 \Rightarrow \frac{\partial u}{\partial y} = 0$ at $y=0$ BL blown off i.e. $\tau_w = 0$
 $u=0 \quad y \geq 0$ & BL approximation not valid!