Stokes Flow

Approximate solutions NS for low Re with exact solutions those reduced equations. Similar exact solutions BL equations but counterpoint for very low vs. very high Re

Stokes flow: $\underline{u} \cdot \nabla \underline{u} \sim 0$ i.e. convective inertia negligible

$$\nabla \cdot \underline{u} = 0$$

$$\underline{u}_t = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \underline{u}$$ linear!

more formally using dimensional analysis, which is useful to obtain higher order corrections using matched asymptotic expansions, as follows.

$$\underline{u} = U \underline{u}^*$$

$$p = \frac{\rho \nu U}{l} p^* = \frac{\mu U}{l} p^*$$

$$x_i = l x_i^*$$

$$t = \frac{l^2}{\nu} t$$

U = characteristic velocity

l = characteristic length

$l \propto \sqrt{\nu t}$

$\frac{l^2}{\nu} \propto t$ = time

required viscous diffusion traverse l

Substitution NS

$$\frac{\nu U}{\ell^2} u^*_t + \frac{U^2}{\ell} (u^* \cdot \nabla^* u^*) = -\frac{\nu U}{\ell^2} \nabla^* p^* + \frac{\nu U}{\ell^2} \nabla^{*2} \underline{u}^*$$

$$\times \frac{\ell^2}{\nu U} \qquad u^*_t + Re\,(\underline{u}^* \cdot \nabla^* \underline{u}^*) = -\nabla^* p^* + \nabla^{*2} \underline{u}^* \qquad Re = U\ell/\nu$$

$Re \to 0$ $\qquad \underline{u}_t = -\nabla p + \nabla^2 \underline{u}$ $\qquad$ Stokes equations

drop *

asymptotic limit $Re \to 0$ while space coordinates order unity

Stokes equations:

$$\nabla \cdot \underline{u} = 0 \qquad (1)$$

$$\underline{u}_t = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \underline{u} \qquad (2)$$

$\nabla \cdot (2) \Rightarrow \nabla^2 p = 0 \qquad (3) \qquad p$ harmonic

$\nabla \times (2) \Rightarrow \underline{\omega}_t = \nu \nabla^2 \underline{\omega} \qquad (4) \qquad \underline{\omega}$ harmonic steady flow

Alternative form (2) without p:

$$\nabla \times \nabla \times (2) \Rightarrow \frac{\partial}{\partial t} [\nabla (\nabla \cdot \underline{u}) - \nabla^2 \underline{u}] = \nu \nabla^2 [\nabla (\nabla \cdot \underline{u}) - \nabla^2 \underline{u}]$$

note:

$\nabla \times \nabla \times \underline{u} = \nabla (\nabla \cdot u) - \nabla^2 \underline{u}$

$\nabla \times \nabla p = 0$

$$\therefore \quad \nabla^2 \underline{u}_t = \nu \nabla^4 \underline{u}$$

remove p but 4th order

drop * notation $\Rightarrow$ $$\frac{\partial \underline{V}}{\partial t} + Re(\underline{V} \cdot \nabla \underline{V}) = -\nabla p + \nabla^2 \underline{V} \qquad (7)$$

Let,

$$\underline{V} = \underline{V}^0 + Re\,\underline{V}^1 + \cdots \qquad (8)$$
$$p = p^0 + Re\,p^1$$

By using (8) in (7) and retaining terms of various O(Re) the pde's for $\underline{V}^n$ and p^n can be obtained. For example,

$$\frac{\partial \underline{V}^0}{\partial t} = -\nabla p^0 + \nabla^2 \underline{V}^0 \qquad O(Re=0)$$

Note that the equations for the higher-order $\underline{V}^n$ and p^n will include lower-order terms.

Note that Stokes equations (1) and (2) are linear which is a great advantage since many methods are available for their solution, e.g.

* separation of variables
* linear superposition of elementary solutions (this approach useful in showing which elements produce E and M)

We shall primarily discuss the latter approach

C. Types of low Re problems

1. fully-developed laminar duct flow (inertia terms vanish due to geometry assumed)

2. flow about immersed bodies (microorganisms, etc)

3. flow in narrow passages - lubrication theory

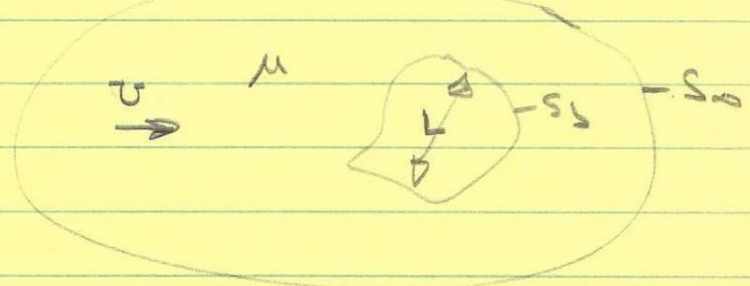
4. flow through porous media ($\nabla^2 \hat{p} = 0$)

In this course, only type 2 flow is considered

D. Creeping flow about immersed bodies

$$\nabla \cdot \underline{V} = 0$$

$$\nabla p = \mu \nabla^2 \underline{V}$$



BC: $\underline{V}(S_b) = 0$

$\underline{V}(\infty) = U_\infty$

Before proceeding with some solutions Stokes paradox is discussed using a dimensional argument

a. Stokes Paradox

2-D body:

$$F' = F'(U, \mu, L)$$ F' = force per unit length

$$\Rightarrow \underbrace{\frac{F'}{\mu U} = \text{constant}}$$

this does not depend on the size of the body (L) & is obviously unrealistic

$\therefore$ ρ can not be neglected

$$F' = F'(U, \mu, L, \rho) \Rightarrow F' = F'(Re)$$

"Stokes paradox"

3-D body:

$$F = F(U, \mu, L)$$

$$\Rightarrow \frac{F}{\mu U L} = \text{constant}$$ Realistic

this can be arranged as

$$\frac{F}{\frac{1}{2}\rho U^2 L} = \frac{\text{Const.}}{Re}$$ but ρ introduced for convenience + in interpretation of turbulent flow

This then demonstrates Stokes Paradox: it is impossible to find a steady two-dimensional solution which satisfies the eqs. of both boundary conditions, this point will be discussed further below where it will be shown that the 3D solution is also not without problem, ie, the inertia terms are not strictly negligible in the far field of the body.

b. Some basic solutions

Since Stokes equations are linear the superposition theorem is valid & solutions can be obtained by combining basic solutions. Recall, that this approach is frequently used in inviscid-flow theory in solving the Laplace equation

1. Uniform flow

$$\underline{V} = U\,\hat{e}_x$$

$$p = \text{constant}$$

Clearly, this velocity & pressure field produces no force or moment. However, as will be shown below this solution can be combined with others to produce the flow pattern for simple geometries

$$\nabla^2 \underline{V} = \nabla(\nabla \cdot \underline{V}) - \nabla \times \underline{\omega}$$

2. Doublet

$= 0$ for incompressible irrotational flow

We have discussed the point that any potential flow is also an exact solution of the full NS equations since the viscous term is then identically zero.

Thus, this is also true for the Stokes equations. However, in this case we also require $\nabla p = 0$, i.e., p = constant. (Keep in mind that for a viscous flow the Bernoulli equation is not valid)

For irrotational flow,

$$\underline{V} = \nabla \varphi$$

$$\nabla^2 \varphi = 0$$

Spherical Coordinates (r, θ, ω)

$$\varphi = A \frac{\cos\theta}{r^2}$$

3-D dipole
Singular at $r = 0$

$$= A x / r^3$$

$$\Rightarrow \quad \underline{V} = A\left(\frac{\hat{e}_x}{r^3} - \frac{3x\hat{e}_r}{r^4}\right)$$

$$p = \text{constant}$$

Again, no force or moment exerted on fluid

3 Rotlet

In this case, a solution is sought for which $\underline{\omega} \neq 0$ and $p = $ constant. Say,

$\nabla\cdot(\underline{a}\times\underline{b}) =$

$\underline{b}\cdot(\nabla\times\underline{a})$

$-\underline{a}\cdot(\nabla\times\underline{b})$

$$\underline{V} = \underline{v} \times \nabla Q \qquad (1)$$

0 by vector identity

$$\nabla\cdot\underline{V} = \underbrace{\nabla Q\cdot(\overset{0}{\nabla\times\underline{v}})}_{\varepsilon_{ijk}\, x_{j,i}\, Q_{,k}} + \underline{v}\cdot(\overset{0}{\nabla\times\nabla Q})$$

$$= 0$$

ε_{ijk} recall alternating tensor; $x_{j,i}$ $= 0$ unless $i = j$ and under this condition $\varepsilon_{ijk} = 0$

$\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1$

$\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1$

$\varepsilon_{ijk} = 0$ otherwise

This shows that the continuity equation is satisfied identically for all forms of Q.

Since $\frac{\partial}{\partial z} = 0$ and $p = $ constant, Stokes equation reduces to

$$\nabla^2 \underline{V} = 0$$

$$= \frac{\partial^2 V_i}{\partial x_l \partial x_l}$$

$$= \varepsilon_{ijk}\left[\underbrace{\frac{\partial^2 x_j}{\partial x_l \partial x_l}}_{=0}\frac{\partial Q}{\partial x_k} + x_j \frac{\partial^2 Q}{\partial x_l \partial x_l}\right]$$

Rotlet : $\underline{\omega} \neq 0$ & $p =$ constant

$\omega_i = \varepsilon_{ijk} u_{j} v_k$

$\underline{u} = \underline{r} \times \nabla a = \varepsilon_{ijk} x_j \frac{\partial a}{\partial x_k}$

$\underline{\omega} = \underline{u} \times \underline{v}$

$\nabla \cdot \underline{u} = \frac{\partial u_i}{\partial x_i} = \varepsilon_{ijk} \Big(\underbrace{x_{j,i}}_{\text{symmetric}} \frac{\partial a}{\partial x_k} + x_j \underbrace{\frac{\partial^2 a}{\partial x_i \partial x_k}}_{\text{symmetric}} \Big)$

divergence rank r = tensor rank r-1

$\varepsilon_{ijk} x_{j,i} = 0$ since $x_{j,i} = 0$ $j \neq i$ & $\varepsilon_{ijk} = 0$ $j = i$

product antisymmetric & symmetric tensor $= 0$

ie $\nabla \cdot \underline{u} = 0$ alternatively $\nabla \cdot \underline{u} = \nabla a \cdot (\cancel{\nabla \times \underline{v}}) + \underline{v} \cdot (\cancel{\nabla \times \nabla a})$

$= 0$

$\nabla p = \nabla^2 \underline{u}$ but $p =$ constant $\Rightarrow \nabla^2 \underline{u} = 0$

$\nabla^2 u_i = \frac{\partial^2 u_i}{\partial x_e \partial x_e} = \varepsilon_{ijk} \Big[\frac{\partial^2 x_j}{\partial x_e \partial x_e} \frac{\partial a}{\partial x_k} + x_j \frac{\partial}{\partial x_k} \frac{\partial^2 a}{\partial x_e \partial x_e} \Big]$

① $= 0$ ② $= 0$

$\nabla^2 a = 0$

$\frac{\partial}{\partial x_e} \Big(\frac{\partial x_i}{\partial x_e} \Big) = \nabla \cdot (\nabla \cdot \underline{x})$

$= \nabla \cdot (3) = 0$

$\therefore$ $\underline{u} = \underline{r} \times \nabla a$ valid solution Stokes equations for $p =$ constant assuming $\nabla^2 a = 0$. $\therefore$ need axisymmetric solutions $\nabla^2 a = 0$

alternatively $\nabla \times (\nabla \times \underline{u}) = \nabla(\nabla \cdot \underline{u}) - \nabla^2 \underline{u}$

$\nabla^2 \underline{u} = -\nabla \times (\nabla \times \underline{u})$

$(\underline{a} \cdot \nabla)\, \underline{x} = \underline{a}$

$\underline{a} \cdot \nabla = \nabla \cdot \underline{a}$

$\nabla \times \underline{u} = \nabla \times (\underline{r} \times \nabla a) = \underline{r} \cdot (\underbrace{\cancel{\nabla \cdot \nabla a}}_{\nabla^2 a = 0}) + \underbrace{\cancel{(\nabla a \cdot \nabla)\, \underline{r}}}_{\nabla a}$

$- \nabla a \underbrace{(\nabla \cdot \underline{r})}_{3} - \underbrace{(\underline{r} \cdot \nabla) \nabla a}_{\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\nabla a}$

$\nabla \times (\nabla f) = 0$

$= \nabla \chi$

$\chi = \underline{r} \cdot \nabla a - a$

Need to prove that:

$$(\underline{r}\cdot\nabla)\nabla\varphi = \nabla\psi$$

Write $(\underline{r}\cdot\nabla)\nabla\varphi$ by components:

$$(\underline{r}\cdot\nabla)\nabla\varphi = \left(r_1\frac{\partial}{\partial r_1}+r_2\frac{\partial}{\partial r_2}+r_3\frac{\partial}{\partial r_3}\right)\cdot\begin{pmatrix}\frac{\partial\varphi}{\partial r_1}\\ \frac{\partial\varphi}{\partial r_2}\\ \frac{\partial\varphi}{\partial r_3}\end{pmatrix} = \begin{pmatrix} r_1\frac{\partial^2\varphi}{\partial r_1^2}+r_2\frac{\partial^2\varphi}{\partial r_1\partial r_2}+r_3\frac{\partial^2\varphi}{\partial r_1\partial r_3}\\ r_1\frac{\partial^2\varphi}{\partial r_2\partial r_1}+r_2\frac{\partial^2\varphi}{\partial r_2^2}+r_3\frac{\partial^2\varphi}{\partial r_2\partial r_3}\\ r_1\frac{\partial^2\varphi}{\partial r_3\partial r_1}+r_2r_1\frac{\partial^2\varphi}{\partial r_3\partial r_2}+r_3\frac{\partial^2\varphi}{\partial r_3^2}\end{pmatrix}$$

Take the elements in the first row:

$$r_1\frac{\partial^2\varphi}{\partial r_1^2} = \frac{\partial}{\partial r_1}\left(r_1\frac{\partial\varphi}{\partial r_1}\right)-\frac{\partial r_1}{\partial r_1}\frac{\partial\varphi}{\partial r_1} = \frac{\partial}{\partial r_1}\left(r_1\frac{\partial\varphi}{\partial r_1}\right)-\frac{\partial\varphi}{\partial r_1}$$

$$r_2\frac{\partial^2\varphi}{\partial r_1\partial r_2} = \frac{\partial}{\partial r_1}\left(r_2\frac{\partial\varphi}{\partial r_2}\right)-\frac{\partial r_2}{\partial r_1}\frac{\partial\varphi}{\partial r_2} = \frac{\partial}{\partial r_1}\left(r_2\frac{\partial\varphi}{\partial r_2}\right)$$

$$r_3\frac{\partial^2\varphi}{\partial r_1\partial r_3} = \frac{\partial}{\partial r_1}\left(r_3\frac{\partial\varphi}{\partial r_3}\right)-\frac{\partial r_3}{\partial r_1}\frac{\partial\varphi}{\partial r_3} = \frac{\partial}{\partial r_1}\left(r_3\frac{\partial\varphi}{\partial r_3}\right)$$

Where the following rule has been applied:

$$\frac{\partial r_i}{\partial r_j} = \delta_{ij}$$

Doing this process for all the terms gives:

$$(\underline{r}\cdot\nabla)\nabla\varphi = \begin{pmatrix}\frac{\partial}{\partial r_1}\left(r_1\frac{\partial\varphi}{\partial r_1}\right)-\frac{\partial\varphi}{\partial r_1}+\frac{\partial}{\partial r_1}\left(r_2\frac{\partial\varphi}{\partial r_2}\right)+\frac{\partial}{\partial r_1}\left(r_3\frac{\partial\varphi}{\partial r_3}\right)\\ \frac{\partial}{\partial r_2}\left(r_1\frac{\partial\varphi}{\partial r_1}\right)+\frac{\partial}{\partial r_2}\left(r_2\frac{\partial\varphi}{\partial r_2}\right)-\frac{\partial\varphi}{\partial r_2}+\frac{\partial}{\partial r_2}\left(r_3\frac{\partial\varphi}{\partial r_3}\right)\\ \frac{\partial}{\partial r_3}\left(r_1\frac{\partial\varphi}{\partial r_1}\right)+\frac{\partial}{\partial r_3}\left(r_2\frac{\partial\varphi}{\partial r_2}\right)+\frac{\partial}{\partial 3}\left(r_3\frac{\partial\varphi}{\partial r_3}\right)-\frac{\partial\varphi}{\partial r_3}\end{pmatrix}$$

Group the $\frac{\partial}{\partial r_i}$ terms:

$$(\underline{r}\cdot\nabla)\nabla\varphi = \begin{pmatrix}\frac{\partial}{\partial r_1}\left(r_1\frac{\partial\varphi}{\partial r_1}+r_2\frac{\partial\varphi}{\partial r_2}+r_3\frac{\partial\varphi}{\partial r_3}\right)-\frac{\partial\varphi}{\partial r_1}\\ \frac{\partial}{\partial r_2}\left(r_1\frac{\partial\varphi}{\partial r_1}+r_2\frac{\partial\varphi}{\partial r_2}+r_3\frac{\partial\varphi}{\partial r_3}\right)-\frac{\partial\varphi}{\partial r_2}\\ \frac{\partial}{\partial r_3}\left(r_1\frac{\partial\varphi}{\partial r_1}+r_2\frac{\partial\varphi}{\partial r_2}+r_3\frac{\partial\varphi}{\partial r_3}\right)-\frac{\partial\varphi}{\partial r_3}\end{pmatrix}$$

Rewrite as dot product of two vectors:

$$(\underline{r}\cdot\nabla)\nabla\varphi = \begin{pmatrix}\frac{\partial}{\partial r_1}(\underline{r}\cdot\nabla\varphi)-\frac{\partial\varphi}{\partial r_1}\\ \frac{\partial}{\partial r_2}(\underline{r}\cdot\nabla\varphi)-\frac{\partial\varphi}{\partial r_2}\\ \frac{\partial}{\partial r_3}(\underline{r}\cdot\nabla\varphi)-\frac{\partial\varphi}{\partial r_3}\end{pmatrix} = \begin{pmatrix}\frac{\partial}{\partial r_1}(\underline{r}\cdot\nabla\varphi-\varphi)\\ \frac{\partial}{\partial r_2}(\underline{r}\cdot\nabla\varphi-\varphi)\\ \frac{\partial}{\partial r_3}(\underline{r}\cdot\nabla\varphi-\varphi)\end{pmatrix} = \nabla(\underline{r}\cdot\nabla\varphi-\varphi) = \nabla\psi$$

$\therefore$ if $\nabla^2 \alpha = 0$ Stokes equation is satisfied. The problem again reduces to that of obtaining solutions to Laplace equation.

Here we are interested in solutions to $\nabla^2 \alpha$ for axisymmetric flow. Recall from inviscid-flow theory that the solutions are in terms of spherical harmonics. The first harmonic (source) is

$$\alpha \propto 1/r \qquad \underline{V} = \underline{r} \times \nabla \alpha$$

$$\Rightarrow \nabla \alpha \propto \hat{e}_r \quad \text{so that} \quad \underline{V} \propto \underbrace{r\hat{e}_r}_{\underline{r}} \times \hat{e}_r = 0$$

The next solution (doublet) is $\quad \hat{e}_r = \cos\theta\, \hat{e}_x + \sin\theta \cos\omega\, \hat{e}_y + \sin\theta \sin\omega\, \hat{e}_z$

$$\alpha = B \frac{\cos\theta}{r^2}$$

$$= B\, x/r^3$$

$$\underline{V} = B\, \underline{r} \times \nabla \left(x/r^3 \right)$$

$$= B\, \underline{r} \times \left(\frac{\hat{e}_x}{r^3} - 3 \frac{x \hat{e}_r}{r^4} \right)$$

$$= B \frac{\hat{e}_r \times \hat{e}_x}{r^2}$$ Streamlines $\perp$ $\hat{e}_r + \hat{e}_x$, i.e., circles whose centers lie on x-axis

and as stated earlier

$$p = \text{constant}$$

A rotlet exerts a moment but not a force on the fluid

$$F_i = -\int_S \sigma_{ij}\, n_j\, dS$$

S – sphere about rotlet

$$= -\int_S \left[-p\,\delta_{ij} + \mu\,(u_{i,j} + u_{j,i}) \right] n_j\, dS$$

$u_i \sim r^{-2}$ $\quad dS \sim r^2 \quad \Rightarrow \quad F_i \sim \frac{1}{r^3}\, r^2 = \frac{1}{r}\Big|_{r \to \infty} = 0$

$u_{i,j} \sim r^{-3}$

$$\Rightarrow \quad F_i = 0$$

$\underline{M} = \int_S \underline{r} \times \underline{P}\, ds$ $\quad \underline{P} =$ surface force vector

$P_k = \sigma_{kl} n_l$ $\quad M_i = \int_S \varepsilon_{ijk}\, x_j\, \sigma_{kl}\, n_l\, dS \qquad n_l = x_l / r$

which can be evaluated to yield

Many steps, which should be added now!

$$\underline{M} = 8\pi B \mu\, \hat{e}_x$$

$$m_i = \frac{\mu}{V}\int_S \varepsilon_{ijk} x_j x_\ell (u_{k,\ell} + u_{\ell,k})\, dS$$

$x_\ell u_{k,\ell} = -2u_k$ since: $\underline{u} = B\dfrac{\hat{e}_r \times \hat{e}_x}{r^2}$ $\quad$ $\underline{u} \perp \hat{e}_r$ & $\hat{e}_x$

Homogeneous function degree m $f(\frac{x}{\lambda}, \frac{y}{\lambda}, \frac{z}{\lambda}) = \lambda^m f(x,y,z)$ $\quad$ homogeneous function degree 2

& Euler theorem $x f_x + y f_y + z f_z = -m f$

$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\cos\omega\,\hat{j} + \sin\theta\sin\omega\,\hat{k}$

$\hat{e}_r \times \hat{e}_x = \sin\theta\cos\omega\,\hat{k} - \sin\theta\sin\omega\,\hat{j} = \hat{e}_u$

$\underline{u} = \frac{B}{r^2}\hat{e}_u$ $\qquad \frac{\partial u_i}{\partial r} = -2r^{-3} = -2/r^3$ $\qquad \frac{\partial r_i}{\partial x_i} = x_i/r$

$u_{1,1} = -\frac{2B}{r^3} \times \frac{x}{r}$ $\qquad r = [x^2+y^2+z^2]^{1/2}$

$r_x = \frac{1}{2}[\ ]^{-1/2}\, 2x$

$-\frac{2B}{r^4}\underbrace{(x^2+y^2+z^2)}_{r^2} = x_\ell u_{1,\ell} = -\frac{2B}{r^2}u_1 = -2u_1$ $\qquad = x/[\ ]^{1/2} = x/r$

$\therefore x_\ell u_{k,\ell} = -2u_k$

$x_{\ell,k} = \delta_{\ell k}$

$x_\ell u_{\ell,k} = \underbrace{\frac{\partial}{\partial x_k}(x_\ell u_\ell)} - u_\ell \frac{\partial x_\ell}{\partial x_k} = -u_\ell \delta_{\ell k} = -u_k$

$u_\ell x_{\ell,k} + x_\ell u_{\ell,k} = \nabla\cdot(\underline{r}\cdot\underline{u}) = 0$ $\quad \underline{r} \wedge \underline{u} \perp$

$$m_i = \frac{\mu}{V}\int_S \varepsilon_{ijk} x_j (-2u_k - u_k)\, dS$$

$$\underline{M} = -\frac{3\mu}{V}\int_S \underline{r} \times \underline{u}\, dS$$

$\underline{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$\hat{e}_r = a\hat{\imath} + b\hat{\jmath} + c\hat{k} = \cos\theta\,\hat{\imath} + \sin\theta\cos\omega\,\hat{\jmath} + \sin\theta\sin\omega\,\hat{k}$

$\hat{e}_r \times \hat{e}_x = +c\hat{\jmath} - b\hat{k}$ $\qquad (a\hat{\imath} + b\hat{\jmath} + c\hat{k}) \times \hat{\imath} = -b\hat{k} + c\hat{\jmath}$

$\underline{r} \times (\hat{e}_r \times \hat{e}_x) = (x\hat{\imath} + y\hat{\jmath} + z\hat{k}) \times (+c\hat{\jmath} - b\hat{k})$

$\hat{\imath}\hat{\jmath}\hat{k}\hat{\imath}\hat{\jmath}\hat{k}$ $\qquad = +xc\hat{k} + xb\hat{\jmath} - yb\hat{\imath} - cz\hat{\imath}$

$= -(yb + cz)\hat{\imath} + xb\hat{\jmath} + xc\hat{k}$

$(\underline{r} \cdot \hat{e}_x)\hat{e}_r - (\underline{r} \cdot \hat{e}_r)\hat{e}_x$

$x(a\hat{\imath} + b\hat{\jmath} + c\hat{k}) - (xa + yb + zc)\hat{\imath}$

$(\cancel{xa} - \cancel{xa} - yb - zc)\hat{\imath} + xb\hat{\jmath} + xc\hat{k} = -(yb + cz)\hat{\imath} + xb\hat{\jmath} + xc\hat{k}$

$-(yb + cz)\hat{\imath} + \underbrace{xa\hat{\imath} + xb\hat{\jmath} + xc\hat{k}}_{x\hat{e}_r} - xa\hat{\imath}$

$= x\hat{e}_r + (-xa - yb - cz)\hat{\imath} = x\hat{e}_r - (xa + yb + cz)\hat{\imath} = x\hat{e}_r - r\hat{\imath}$

$-(yb + cz)\hat{\imath} + \underbrace{xb\hat{\jmath} + xc\hat{k} + xa\hat{\imath}}_{+x\hat{e}_r} - xa\hat{\imath}$

$+x\hat{e}_r - (xa + yb + cz)\hat{\imath}$ $\qquad \underline{r} \cdot \hat{e}_r = \underline{r} \cdot \frac{\underline{r}}{|r|} = r$

$+x\hat{e}_r - r\hat{\imath}$

$\underline{r} \times \underline{u} = \frac{B}{r^2} \underbrace{\underline{r} \times (\hat{e}_r \times \hat{r}_x)}_{(\underline{r} \cdot \hat{e}_x)\hat{e}_r - (\underline{r} \cdot \hat{e}_r)\hat{e}_x} = \frac{Bx}{r^2}\hat{e}_r - \frac{B}{r}\hat{e}_x$ $\qquad -\frac{3\mu}{r}\left(\frac{Bx}{r^2}\hat{e}_r - \frac{B}{r}\hat{e}_x\right)$

$-\frac{3B\mu}{r^2}\left(\frac{x}{r}\hat{e}_r - \hat{e}_x\right)$

$\underline{M} = -3B\mu \int_s \left(\frac{x}{r}\hat{e}_r - \hat{e}_x\right)\frac{ds}{r^2}$

$x = r\cos\theta \qquad dS = r^2 \sin\theta \, d\theta \, d\omega$

$$\underline{M} = -3B\mu \int_0^{2\pi} d\omega \int_0^{\pi} \left[\cos\theta \, (a\hat{i} + b\hat{j} + c\hat{k}) - \hat{i} \right] \sin\theta \, d\theta$$

$$\cos\theta \, (\cos\theta \, \hat{i} + \sin\theta \cos\omega \, \hat{j} + \sin\theta \sin\omega \, \hat{k}) - \hat{i}$$

$$\underbrace{(\cos^2\theta - 1)\,\hat{i} + \cos\theta \sin\theta \cos\omega \, \hat{j} + \cos\theta \sin\theta \sin\omega \, \hat{k}}$$

$$\underline{M} = -3B\mu \int_0^{2\pi} d\omega \int_0^{\pi} [(\quad)] \sin\theta \, d\theta$$

$M_i = 8\pi B\mu \, \hat{e}_x \qquad$ rotlet $\quad F_i = 0 \quad M_i \neq 0$

$\propto B$ = velocity magnitude $\qquad \underline{u} = B \dfrac{\hat{e}_r \times \hat{e}_x}{r^2}$

right hand rule in x direction

r = constant

Euler's Homogeneous Function Theorem

Let $f(x, y)$ be a homogeneous function of order n so that

$$f(t x, t y) = t^n f(x, y). \tag{1}$$

Then define $x' \equiv x t$ and $y' \equiv y t$. Then

$$n t^{n-1} f(x, y) = \frac{\partial f}{\partial x'} \frac{\partial x'}{\partial t} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial t} \tag{2}$$

$$= x \frac{\partial f}{\partial x'} + y \frac{\partial f}{\partial y'} \tag{3}$$

$$= x \frac{\partial f}{\partial (x t)} + y \frac{\partial f}{\partial (y t)}. \tag{4}$$

Let $t = 1$, then

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n f(x, y). \tag{5}$$

This can be generalized to an arbitrary number of variables

$$x_i \frac{\partial f}{\partial x_i} = n f(\mathbf{x}), \tag{6}$$

where Einstein summation has been used.

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Weisstein, Eric W. "Euler's Homogeneous Function Theorem." From *MathWorld*--A Wolfram Web Resource. https://mathworld.wolfram.com/EulersHomogeneousFunctionTheorem.html

SUBJECT CLASSIFICATIONS

Calculus and Analysis › Functions ›

4 Stokeslet

Now we seek a solution for which $p \neq$ constant. Since the gde for p is the Laplace equation we can again immediately write down solutions

$p \propto 1/r$ source

$p \propto \cos\theta / r^2$ dipole

etc.

The source solution turns out not to be of interest, but the dipole one leads to the Stokeslet

$$p = 2c\mu\, x / r^3 \qquad \cos\theta = x/r$$

Then, from Stokes equation

$$\nabla^2 \underline{v} = \frac{1}{\mu} \nabla p$$

or

$$\nabla^2 u = c\left(\frac{2}{r^3} - 6\frac{x^2}{r^5}\right)$$

$$\nabla^2 v = -6c\,\frac{xy}{r^5}$$

$$\nabla^2 w = -6c\,\frac{xz}{r^5}$$

$$p = 2c\mu\, x/r^3$$

$$u_{xx}+u_{yy}+u_{zz} = \frac{1}{\mu} p_x = \frac{1}{\mu}\left[2c\mu \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)\right] \qquad \frac{\partial r_i}{\partial x_i} = \frac{x_i}{r}$$

$$= 2c\left[r^{-3} + x\left(-3r^{-4}\, x/r\right)\right.$$

$$= 2c\left[r^{-3} - 3x^2/r^5\right]$$

$$\nabla^2 u = c\left(2/r^3 - 6x^2/r^5\right.$$

$$\nabla^2 v = \frac{1}{\mu} p_y = \frac{1}{\mu}\left[2c\mu x\left(-3r^{-4}\, y/r\right)\right]$$

$$= 2cx\left(-3y/r^5\right)$$

$$= -6c\, xy/r^5$$

$$\nabla^2 w = \frac{1}{\mu} p_z = -6c\, xz/r^5$$

$$\nabla^2 v = -6c\, xy/r^5$$

$$\phi = r^{-3} \qquad \chi = xy$$

$$\nabla^2 \phi = 6/r^5$$

$$\nabla \phi = -3/r^4\, \hat{e}_r$$

$$\nabla^2 \chi = 0$$

$$\nabla \chi = y\,\hat{e}_x + x\,\hat{e}_y$$

$$\nabla\phi \cdot \nabla\chi = -6\, xy/r^5$$

$$\nabla^2(\phi\chi) = \nabla^2\left(xy/r^3\right) = -6\, xy/r^5$$

$$\Rightarrow v = c\, xy/r^3$$

the particular integrals to these nonhomogeneous pde's can be obtained using the properties of harmonic functions and the following identities

$$\nabla^2\left(\frac{1}{r^n}\right) = \frac{n(n-1)}{r^{n+2}} \qquad r^2 = x^2+y^2+z^2$$

$$\nabla^2(Q\chi) = \chi\nabla^2 Q + Q\nabla^2\chi + 2\nabla Q\cdot\nabla\chi$$

$$\Rightarrow \qquad u = c\frac{x^2}{r^3} \qquad Q = r^{-3} \quad \chi = x^2$$

$$v = c\frac{xy}{r^3} \qquad Q = r^{-3} \quad \chi = xy$$

$x = r\cos\theta$

$$w = c\frac{xz}{r^3} \qquad Q = r^{-3} \quad \chi = xz$$

$y = r\sin\theta\cos\omega$

$z = r\sin\theta\sin\omega$

$$\underline{V} = (u, v, w) + \underline{V}' \qquad \text{where } \underline{V}' = \text{solution to the homogeneous equation } \nabla^2\underline{V} = 0$$

$$= c\frac{x}{r^2}\hat{e}_r + \underline{V}'$$

$$\hat{e}_r = \cos\theta\,\hat{e}_x + \sin\theta\cos\omega\,\hat{e}_y + \sin\theta\sin\omega\,\hat{e}_z \qquad \cos\theta = \frac{x}{r}$$

Next, $\underline{V}'$ is determined from the continuity equation

$$\nabla\cdot\underline{V} = 0$$

$$= c\nabla\cdot\left(\frac{x}{r^2}\hat{e}_r\right) + \nabla\cdot\underline{V}'$$

$$\nabla \cdot \left(\frac{x}{r^2} \hat{e}_r \right) = \nabla \cdot \left(\frac{x}{r^3} \underline{r} \right)$$

$$= \underline{r} \cdot \nabla \left(\frac{x}{r^3} \right) + \frac{x}{r^3} \nabla \cdot \underline{r} \qquad \nabla \cdot \underline{r} = 3$$

$$= \left[x \left(\frac{1}{r^3} - \frac{3x^2}{r^5} \right) - y \frac{3xy}{r^5} - z \frac{3xz}{r^5} \right] + 3 \frac{x}{r^3}$$

$$= \frac{x}{r^3}$$

$$\nabla \cdot \underline{v} = c \frac{x}{r^3} + \nabla \cdot \underline{v}' = 0$$

$$\Rightarrow \underline{v}' = c \frac{\hat{e}_x}{r}$$

also, note that $\nabla^2 \underline{v}' = 0$

So that the final solution is

$$\left. \begin{aligned} p &= 2c\mu\, x / r^3 \\ \underline{v} &= c \left(\frac{x}{r^2} \hat{e}_r + \frac{1}{r} \hat{e}_x \right) \end{aligned} \right\} \text{Stokeslet}$$

Again, singular at origin

In this case, the force is nonzero but there is no moment

$$\underline{F} = 8\pi c \mu \hat{e}_x$$

$$\underline{M} = 0$$

Again many complex steps need to be added notes!

$$f_i = -\int_S \sigma_{ij} n_j \, ds$$

$$\sigma_{ij} = -p\delta_{ij} + \mu (u_{i,j} + u_{j,i})$$

$$F_i = -\int_S \sigma_{ij} n_j \, dS \qquad \sigma_{ij} = -p\delta_{ij} + \mu(u_{i,j} + u_{j,i})$$

$$F_i = -\int_S \left[-\frac{2c\mu x}{r^3}\delta_{ij} + \mu(u_{i,j}+u_{j,i}) \right] n_j \, dS \qquad n_j = x_j / r$$

$$= -\int_S \left[-\frac{2c\mu x}{r^3}\frac{x_i}{r} + \frac{\mu}{r} x_j (u_{i,j} + u_{j,i}) \right] dS$$

$$= 6c\mu \int_S \frac{x x_i}{r^4} \, dS \qquad x(x\hat{i} + y\hat{j} + z\hat{k})$$

$$x(r\cos\theta\hat{i} + r\sin\theta\cos\omega\hat{j} + r\sin\theta\sin\omega\hat{k})$$

Substituting $x, \hat{e}_r, dS$

$$\underline{F} = 6c\mu \int_S \frac{x}{r^3}\hat{e}_r \, dS \qquad x r \hat{e}_r$$

$x = r\cos\theta$

$\hat{e}_r = \cos\theta\hat{i} + \sin\theta\cos\omega\hat{j} + \sin\theta\sin\omega\hat{k}$

$dS = r^2\sin\theta \, d\theta \, d\omega$

$$\underline{F} = 6c\mu \int_0^{2\pi} d\omega \int_0^{\pi} \cos\theta(\cos\theta\hat{e}_x + \sin\theta\cos\omega\hat{e}_y + \sin\theta\sin\omega\hat{e}_z) \sin\theta \, d\theta$$

$$\underline{F} = 8\pi c\mu \hat{e}_x \qquad \propto c$$

direction $+x$ $c > 0$

Note: $x_j u_{i,j} = -u_i = -c\left[\frac{x}{r^3}x_i + \frac{\delta_{i1}}{r}\right]$

$$x_j u_{j,i} = c\left[\frac{\delta_{i1}}{r} - 3\frac{x x_i}{r^3}\right]$$

$$\underline{u} = c\left[\frac{x}{r^2}\left(\cos\theta\,\hat{e}_x + \sin\theta\cos\omega\,\hat{e}_y + \sin\theta\sin\omega\,\hat{e}_z\right) + \frac{\hat{e}_x}{r}\right]$$

$$= c\left[\left(\frac{x\cos\theta}{r^2} + \frac{1}{r}\right)\hat{e}_x + \frac{x}{r^2}\left(\sin\theta\cos\omega\,\hat{e}_y + \sin\theta\sin\omega\,\hat{e}_z\right)\right]$$

$x = r\cos\theta$

$\cos\theta = \frac{x}{r}$

$$-\underline{u} = -c\left[\left(\frac{x^2}{r^3} + \frac{1}{r}\right)\hat{e}_x + \frac{xy}{r^3}\hat{e}_y + \frac{xz}{r^3}\hat{e}_z\right]$$

$$-u_i = -c\left[\frac{x}{r^3}x_i + \frac{\delta_{i1}}{r}\right] = x_j\frac{\partial u_i}{\partial x_j}$$

$y = r\sin\theta\cos\omega$

$\frac{y}{r} = \sin\theta\cos\omega$

$$x_j u_{j,i} = \underbrace{\frac{\partial}{\partial x_i}(x_j u_j)}_{\frac{\partial x_j}{\partial x_i}u_j + x_j\frac{\partial u_j}{\partial x_i}} - u_j\frac{\partial x_j}{\partial x_i} = \frac{\partial}{\partial x_i}(\underline{r}\cdot\underline{u}) - u_i$$

$z = r\sin\theta\sin\omega$

$\frac{z}{r} = \sin\theta\sin\omega$

$\underline{u} = c\left(\frac{x}{r^2}\hat{e}_r + \frac{\hat{e}_x}{r}\right)$ $\quad \underline{r}\cdot\underline{u} = r\hat{e}_r\cdot\underline{u} = \frac{cx}{r} + c\cos\theta = \frac{2cx}{r}$ $\quad \hat{e}_r = \frac{\underline{r}}{|r|} = \frac{\underline{r}}{r}$

$\hat{e}_r = \cos\theta\,\hat{e}_x + \sin\theta\cos\omega\,\hat{e}_y + \sin\theta\sin\omega\,\hat{e}_z$ $\quad r\hat{e}_r = \underline{r}$

$$\frac{\partial}{\partial x_i}\left(\frac{2cx}{r}\right) - c\left[\frac{x}{r^3}x_i + \frac{\delta_{i1}}{r}\right] \qquad \frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) = -\frac{x_i}{r^3}$$

$$= c\left(\frac{2}{r}\delta_{i1} - \frac{2xx_i}{r^3}\right) - c\left[\frac{x}{r^3}x_i + \frac{\delta_{i1}}{r}\right]$$

$$= c\left(\frac{\delta_{i1}}{r} - 3\frac{xx_i}{r^3}\right)$$

$$F_i = -\int_S\left[-2c\mu\frac{x}{r^3}\frac{x_i}{r} - \frac{\mu}{r}c\left(\frac{x}{r^3}x_i + \cancel{\frac{\delta_{i1}}{r}}\right) + \frac{\mu}{r}c\left(-3\frac{xx_i}{r^3} + \cancel{\frac{\delta_{i1}}{r}}\right)\right]dS$$

$$= 6c\mu\int_S\frac{xx_i}{r^4}\,dS$$

c. Rotating Sphere

One might anticipate that the solution for a sphere rotating with constant angular velocity Ω about the x-axis can be obtained from a rotlet with its strength adjusted to satisfy the no-slip condition.

$$\underline{v} = B \frac{\hat{e}_r \times \hat{e}_x}{r^2}$$

BC: $\underline{v}(\infty) = 0$, $\underline{v}(r=a) = \Omega a \, \hat{e}_r \times \hat{e}_x$

$$\Rightarrow \quad B = \Omega a^3$$

$$\Rightarrow \quad \underline{v} = \frac{\Omega a^3}{r^2} \, \hat{e}_r \times \hat{e}_x$$

$$\underline{M} = -8\pi\mu\Omega a^3 \hat{e}_x$$

moment on sphere (acts in a direction which opposes the motion of the sphere)

d. Uniform flow past an immersed sphere

The solution corresponding to uniform flow past a sphere can be obtained by superimposing the solutions for uniform flow, a doublet, and a Stokeslet. Alternatively, the solution can be obtained through the use of the stream function in spherical coordinates and the technique of separation of variables (See 3(9.2)-3(9.3))

$$\underline{V} = U\hat{e}_x + A\left(\frac{\hat{e}_x}{r^3} - 3\frac{x\hat{e}_r}{r^4}\right) + c\left(\frac{x}{r^2}\hat{e}_r + \frac{\hat{e}_x}{r}\right)$$

$$p = 2c\mu\, x/r^3$$

BC: $\underline{V}(\infty) = U\hat{e}_x$, $\underline{V}(a) = 0$

The condition at ∞ is automatically satisfied. The two constants A and c are now determined to enforce the no-slip condition

$$\underline{V}(x=r=a) = 0$$

$$= U\hat{e}_x + A\left(\frac{\hat{e}_x}{a^3} - 3\frac{\hat{e}_r}{a^3}\right) + c\left(\frac{\hat{e}_r}{a} + \frac{\hat{e}_x}{a}\right)$$

$$0 = U + A/a^3 + c/a$$

$\Rightarrow$

$$0 = -3\frac{A}{a^3} + c/a$$

$$A = -U a^3/4$$

$$C = -3/4\, U a$$

$$\underline{V} = U\left[\hat{e}_x - \frac{1}{4}\frac{a}{r}\left(\frac{a^2}{r^2}+3\right)\hat{e}_x + \frac{3}{4}\frac{ax}{r^2}\left(\frac{a^2}{r^2}-1\right)\hat{e}_r\right]$$

$$p = p_\infty - \frac{3}{2}\mu U \frac{ax}{r^3} = p_\infty - \frac{3\mu a U}{2r^2}\cos\theta \qquad \boxed{\begin{array}{c}\propto \mu \\ \text{antisymmetric} \\ + \text{front} \; - \text{rear}\end{array}}$$

Also, see 3(9.2)-3(9.3) t for ψ and $\underline{V} = (u_r, u_\theta)$. The solution is shown in Fig 3-38. Also shown for comparison is the potential flow solution.

- velocity field independent of μ
- velocity field symmetric (since convection neglected)
- near sphere $|\underline{V}| < U$ (ie, no high velocity region near shoulder as indicated by inviscid solution)
- effect of sphere spreads to large distances: at $r = 10a$ the velocities are still 10% below their free-stream values

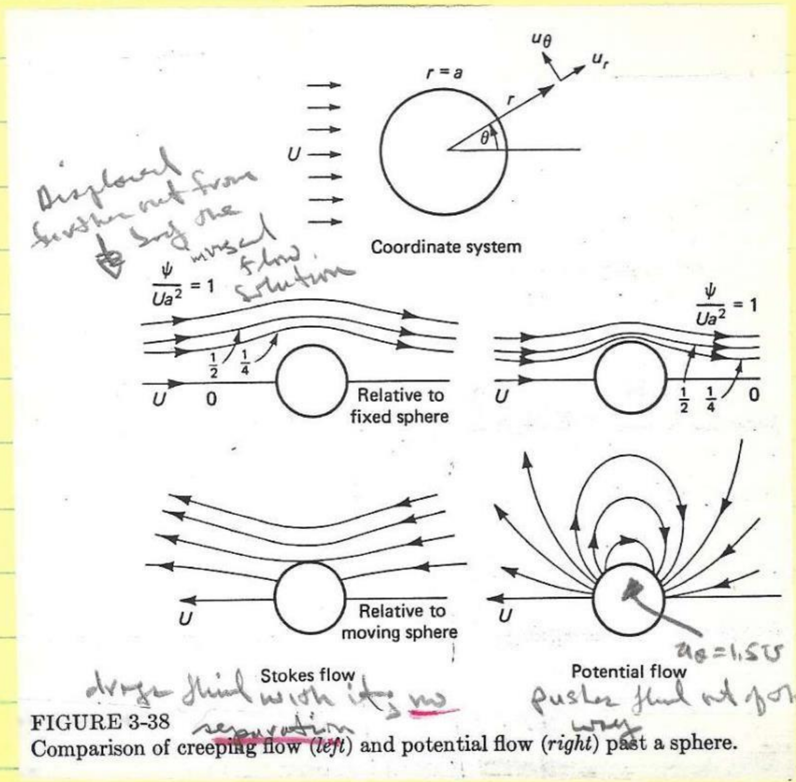

FIGURE 3-38
Comparison of creeping flow (*left*) and potential flow (*right*) past a sphere.

The force on the sphere is

$$\underline{F} = 6\pi\mu U a\, \hat{e}_x \qquad \text{Stokes drag law}$$

$F \propto \mu U L$ as anticipated from dimensional analysis

$F = \frac{2}{3}$ friction $+ \frac{1}{3}$ pressure

See 3(9.2)-3(9.3)

$$\Rightarrow \quad C_D = \frac{2F}{\rho U^2 \pi a^2} = \frac{24}{Re} \qquad Re = \frac{2a\rho U}{\mu} \qquad Re \ll 1$$

This formula is one of the few analytic drag formulas available and the only one for a sphere out of the entire Re range! Also, actually valid for $Re \lesssim 1$.

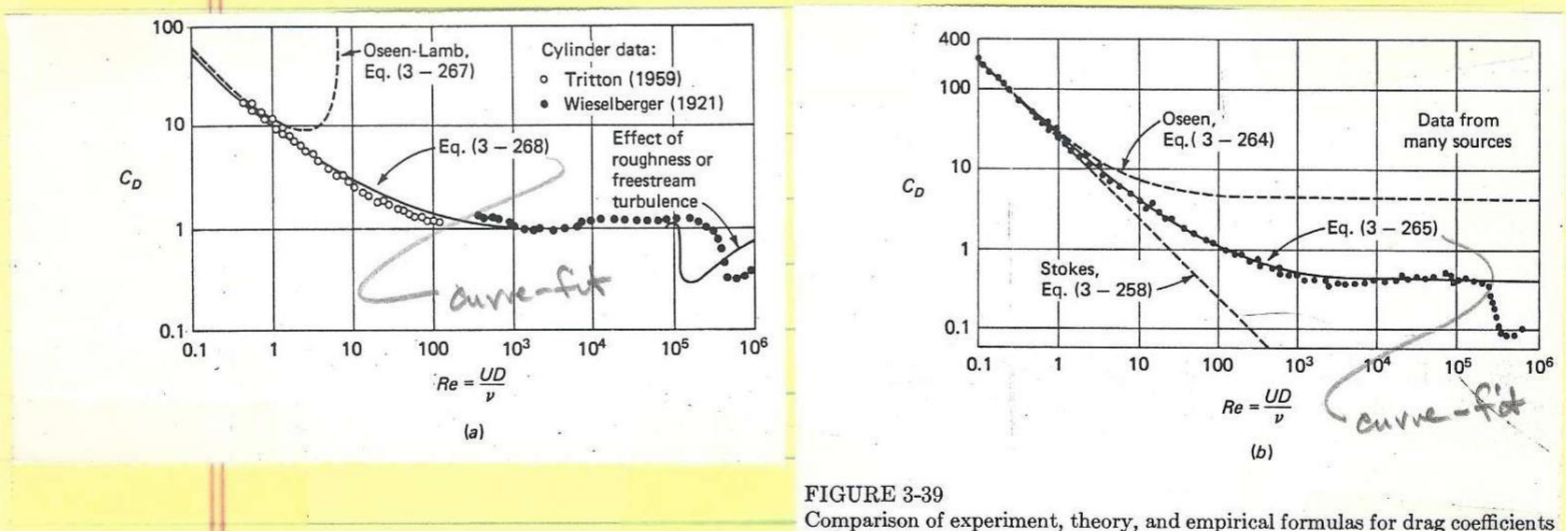

FIGURE 3-39
Comparison of experiment, theory, and empirical formulas for drag coefficients of a cylinder and a sphere (smooth walls): (a) cylinder; (b) sphere.

Two other nonspherical solutions of interest are for a circular disk:

normal to free stream: $F = 16 \mu V a$ — 15% sphere

parallel to free stream: $F = \frac{32}{3} \mu V a$ — 43% sphere

$\Rightarrow$ Stokes sphere law is also approximately valid for nonspherical bodies and is often used to estimate drag of roughly spherical bodies (sand grains, dust, etc.)

e. Two-dimensional creeping flow and Oseen's Improvement

Previously Stokes paradox was mentioned. We now show this more formally and outline the procedure for its remedy.

We now attempt to solve the problem of 2-D flow around a circular cylinder. In this case, the technique of separation of variables is used since we have not established any basic solutions to Stokes equations for 2-D flow. Beginning with the vorticity-transport eq. in terms of χ:

$$\nabla^4 \chi = 0$$

where for polar coordinates (r,θ)

$$\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)$$

Try $\quad \chi(r,\theta) = f(r)\sin\theta$

(since $\chi(\infty) = Ur\sin\theta$)
ie unif. flow

$$\Rightarrow \quad f(r) = Ar^3 + Br\log r + cr + D/r$$

from $\chi(\infty,\theta) = Ur\sin\theta \quad \Rightarrow \quad A = B = 0$
$\qquad c = U$

$$\chi(r,\theta) = (Ur + D/r)\sin\theta$$

on $r=a \quad \chi_\theta = \chi_r = 0 \qquad$ no-slip condition

from $\chi_\theta = 0 \Rightarrow \chi(a,\theta) = \text{constant} = 0$
$\qquad \chi_r(a,\theta) = 0$

no choice of D will satisfy both these conditions. Alternatively, had we satisfied the near field boundary conditions first it would have been found that it is impossible to satisfy the far field conditions. Thus, we conclude that there is no solution to Stokes equations in 2D which can satisfy both the near- & far-field conditions: Stokes Paradox recall that solutions to Stokes equations can be expressed in terms of asymptotic or perturbation expansions

$$\chi = \chi_0 + Re\,\chi_1 + Re^2\,\chi_2 + \cdots$$

we see that for 2D the $O(0)$ expansion is singular (ie, has no solution). For 3D, it can be shown that $O(Re)$ expansion is singular (Whitehead's paradox). Such expansions are called singular perturbation expansions.
(or nonuniform expansions)

The reason for the singular behavior (which also leads to the remedy) is the fact that inertia is not negligible in the far field. In the far field, the inertia force is actually larger than the viscous force since the convection velocity is $\sim U$ & velocity gradients are small. (& not 0 as in Stokes approximation)

An alternative low Re approximation put forth by Oseen in order to resolve this difficulty is

$$(\underline{v} \cdot \nabla)\underline{v} \simeq U \frac{\partial \underline{v}}{\partial x} \quad \text{linearized inertia term}$$

Oseen equations:

$$\nabla \cdot \underline{v} = 0$$

$$\rho\left[\frac{\partial \underline{v}}{\partial t} + U\frac{\partial \underline{v}}{\partial x}\right] = -\nabla p + \mu \nabla^2 \underline{v}$$

} Still linear !

Solutions to Oseen equations can be obtained in a similar manner as for Stokes equations; however, these can be shown to not be accurate in the near field. The remedy is to use the technique of matched asymptotic expansions whereby the near- and far-field solutions are matched to provide a uniformly valid approximation (see Van Dyke: Perturbation Methods in Fluid Mechanics)

Some solutions to the Oseen equations are given in the text for a sphere, flat plate normal to the flow, and a circular cylinder. Note that the velocity field predicted for a sphere no longer shows fore-aft symmetry, but now includes a broader aft wake. However, there is no improvement in the prediction of C_D, Fig 3-39

We shall return to this topic later in the course

* Heat Transfer in Creeping Motion: read text, especially discussion of King's law for calibrating hot wires

3.11 Digital-Computer Solutions: read text (although somewhat outdated)

Appendix 3D Potential Flow

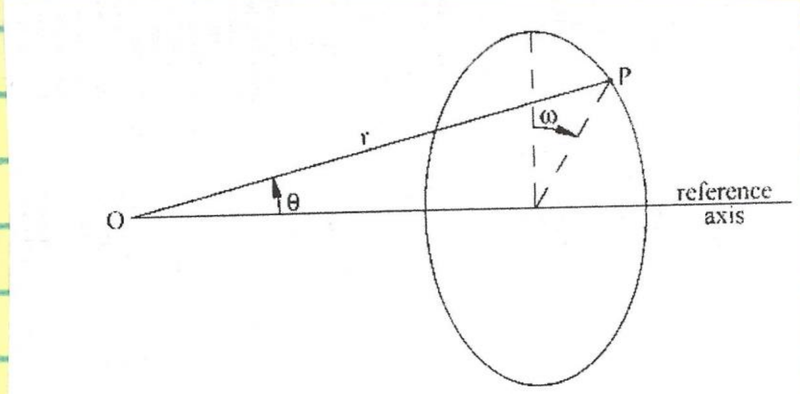

FIGURE 5.1 Definition sketch of spherical coordinates.

3D axisymmetric bodies of interest

Spherical coordinates $P = P(r, \theta, \omega)$

but u_ω & $\frac{\partial}{\partial \omega} = 0$

$\underline{u} = \nabla \phi \quad \nabla^2 \phi = 0$

$u_r = \phi_r \qquad = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \phi_r) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, \phi_\theta)$

$u_\theta = \frac{1}{r}\phi_\theta$

$\nabla \cdot \underline{u} = 0 \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(u_\theta \sin\theta) = 0$

Stokes stream function:

$u_r = \frac{1}{r^2 \sin\theta}\psi_\theta$

$u_\theta = -\frac{1}{r\sin\theta}\psi_r$

Continuity identically satisfied

General solution $\nabla^2\phi = 0$ by separation of variables provides fundamental solutions that can be combined to obtain solutions about simplified geometries: uniform stream, source/sink, doublet, blunt nose, sphere, etc.

Spherical harmonics

$$\phi(r,\theta) = R(r)\,T(\theta) \quad \text{since} \neq f(\omega)$$

Substitution $\nabla^2\phi = 0$: $\frac{T}{r^2}\frac{d}{dr}\left(r^2 R_r\right) + \frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, T_\theta\right) = 0$

$$\times \frac{r^2}{RT} \Rightarrow \underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2 R_r\right)}_{f(r)} = \underbrace{-\frac{1}{T\sin\theta}\frac{d}{d\theta}\left(\sin\theta\, T_\theta\right)}_{f(\theta)}$$

$$\therefore \quad LHS = RHS = \text{constant} = \underbrace{\ell(\ell+1)}_{\text{for some } T(\theta) \text{ solution}}$$

$\frac{1}{R}\frac{d}{dr}\left(r^2 R_r\right) = \ell(\ell+1)$ equidimensional equation: $R(r) = K r^\alpha$

$\alpha(\alpha+1)K r^\alpha - \ell(\ell+1) K r^\alpha = 0$ ie $\alpha = \ell$ or $\alpha = -(\ell+1)$

$$\therefore \quad R_\ell(r) = A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}$$

R_ℓ valid any ℓ for arbitrary constants A_ℓ and B_ℓ

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta \, T_\theta) + l(l+1)T = 0$$

Legendre equation can be reduced
Standard form $x = \cos\theta$

$$\frac{d}{dx}\left[(1-x^2)T_x\right] + l(l+1)T = 0$$

Solution Legendre functions 1st kind $P_l(x)$
& 2nd kind $Q_l(x)$ such that

$$T_l(\theta) = C_l P_l(\cos\theta) + D_l Q_l(\cos\theta)$$

$Q_l(\cos\theta)$ diverges $\cos\theta = \pm 1$ ∴ $D_l = 0$
$P_l(\cos\theta)$ diverges $\cos\theta = \pm 1$ for no singularities
unless l = integer ∴ in flow field
l = integer

$$\phi_l(r,\theta) = \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$ C_l absorbed into A_l & B_l

Since $\nabla^2\phi$ linear $\phi(r,\theta) = \sum_{l=0}^{\infty} \phi_l(r,\theta)$

Fundamental solutions superimposed for additional solutions

where $P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$ Legendre polynomial order l

$P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = \frac{1}{2}(3x^2 - 1)$
etc.

uniform flow: $B_\ell = 0$, $A_\ell = 0$ $\quad \ell \neq 1$

$U \quad \ell = 1$

$P_\ell(\cos\theta) = \cos\theta$

$\phi(r,\theta) = Ur\cos\theta \qquad x = r\cos\theta$

$= Ux$

$u_r = \phi_r = U\cos\theta = \frac{1}{r^2\sin\theta}\psi_\theta$

$\psi = \frac{1}{2}Ur^2\sin^2\theta + \cancel{f(r)}$

$u_\theta = \frac{1}{r}\phi_\theta = -U\sin\theta = -\frac{1}{r\sin\theta}\psi_r$

$\psi = \frac{1}{2}Ur^2\sin^2\theta \pm \cancel{g(\theta)} \qquad f(r) = g(\theta) = \text{constant}$

$= 0$

An alternative way of evaluating $\psi(r, \theta)$ is simply to invoke its definition. Then, considering an arbitrary point P in the fluid as shown in Fig. 5.3, the amount of fluid crossing the surface generated by OP due to the uniform flow will be $2\pi\psi$. But the flow area perpendicular to the velocity vector is $\pi(r\sin\theta)^2$. Hence it follows from the definition of ψ that

$$2\pi\psi = U\pi(r\sin\theta)^2$$

or

$$\psi(r,\theta) = \tfrac{1}{2}Ur^2\sin^2\theta$$

This agrees with the result obtained by the other method.

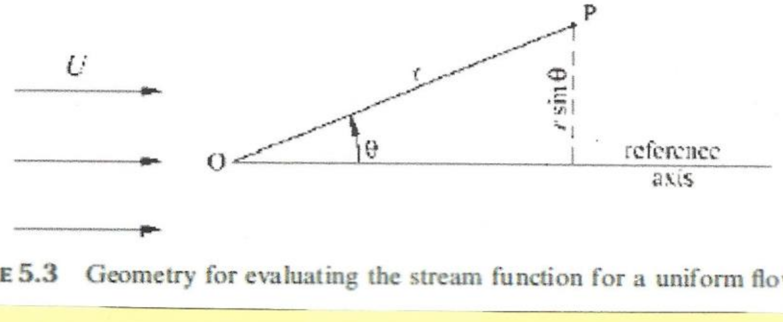

FIGURE 5.3 Geometry for evaluating the stream function for a uniform flow.

Source / sink: $A_l = 0$ all l

$B_l = 0 \quad l \neq 0$

$\quad = B_0 \neq 0 \quad l = 0$

$P_0(\cos\theta) = 1$

$$\phi(r,\theta) = B_0/r \qquad Q = \int_S \underline{u}\cdot\underline{n}\, dS = -4\pi B_0$$

$$u_r = -B_0/r^2 \qquad \phi = -Q/4\pi r$$

$$u_\theta = 0$$

$$\psi(r,\theta) = -Q/4\pi\,(1+\cos\theta)$$

doublet: $\phi(r,\theta) = -Q/4\pi r + Q/4\pi(r-\delta r)$

$$\phi(r,\theta) = \frac{\mu}{4\pi r^2}\cos\theta$$

$$u_r = -\frac{\mu}{2\pi r^3}\cos\theta = \frac{1}{r^2\sin\theta}\psi_\theta$$

$$u_\theta = -\frac{\mu}{4\pi r^3}\sin\theta = \frac{-1}{r\sin\theta}\psi_r$$

$$\psi = -\frac{\mu}{4\pi r}\sin^2\theta$$

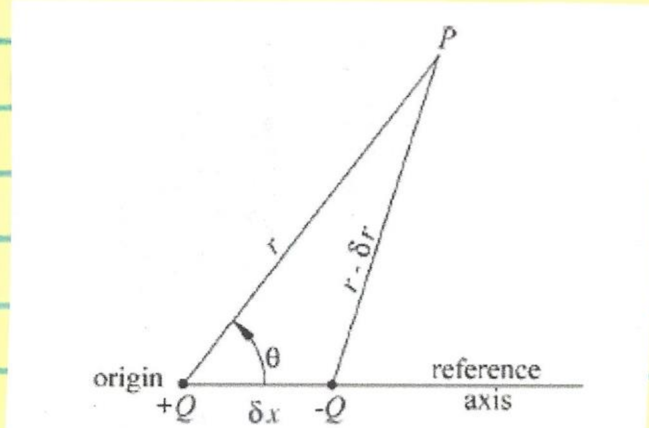

FIGURE 5.5 Superposition of a source and a sink that become a doublet as $\delta x \to 0$.

Sphere: $\psi(r,\theta) = \frac{1}{2}Ur^2\sin^2\theta - \frac{\mu}{4\pi r}\sin^2\theta$

uniform stream + doublet

$$\psi = 0 \quad r = r_0 \Rightarrow r_0 = \left(\frac{\mu}{2\pi U}\right)^{1/3} \text{ ie } \mu = 2\pi U a^3$$

$$\psi(r,\theta) = \frac{1}{2}U\left(r^2 - \frac{a^3}{r}\right)\sin^2\theta$$

$$u_r = U\cos\theta\left(1 - \left(\frac{r_0}{r}\right)^3\right)$$

$$\phi(r,\theta) = U\left(r + \frac{1}{2}\frac{a^3}{r^2}\right)\cos\theta \qquad u_\theta = -U\sin\theta\left(1 + \frac{1}{2}\left(\frac{r_0}{r}\right)^3\right)$$

$u_r(r=r_0) = 0$

$u_\theta(r=r_0) = -U \sin\theta \left(\frac{3}{2}\right)$

stagnation point $0, \pi$

max u_θ $\pm\pi/2$

$u_{\theta\,max} = \pm 1.5\,U$

recall cylinder $u_\theta = 2U\sin\theta$

$C_p = 1 - \frac{9}{4}\sin^2\theta$

recall cylinder $C_p = 1 - 4\sin^2\theta$

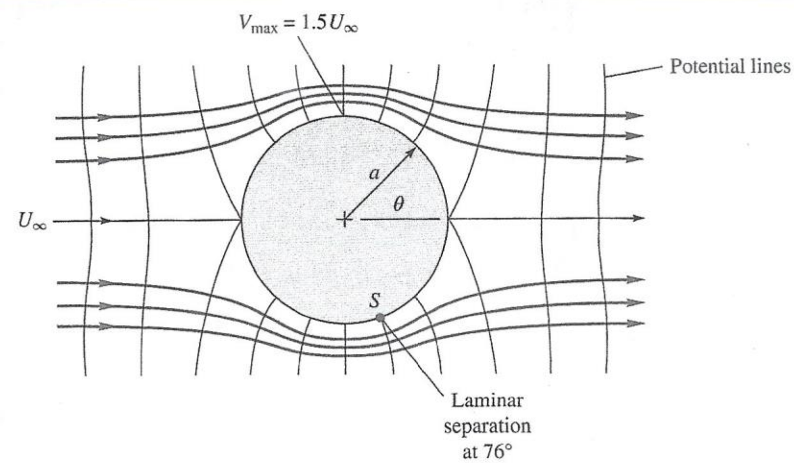

Fig. 8.31 Streamlines and potential lines for inviscid flow past a sphere.