1. Similarity solutions

Generally speaking similarity solutions have the following properties:

(1) reduction of variables by one or more through analytical means (coordinate transformation)

(2) reduction of pde to ode's

(3) physically: inertia, pressure, and viscous forces are in some constant proportion (e.g., Re based on local scales is constant)

a. Stagnation point

b. Rotating disc

c. Wedge flows

Also, others with regard to the BL equations and turbulent flow

Stagnation Point Flow

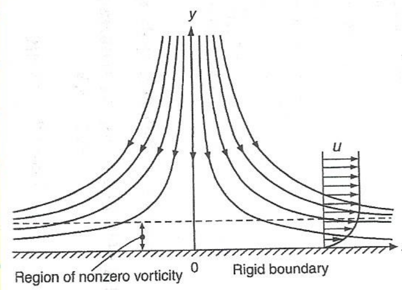


FIGURE 3-24
Stagnation-point flow.

Inviscid Flow

$\psi = Bxy \quad u = Bx = \psi_y \quad v = -By = -\psi_x \quad B = V_0/L$

$p + \frac{1}{2}\rho(u^2 + v^2) = \text{Constant}$ $\quad v = 0$ along wall

$u = 0$ only at Stagnation point

p_0 = Stagnation pressure $\quad p = p_0 - \frac{\rho}{2}B^2(x^2 + y^2)$

Consider $U(y) = U_0(1 - y/L)$ $\quad p + \frac{1}{2}\rho U^2 = C$

$p_y + \rho U U_y = 0$

$p_y = -\rho U U_y$

Adverse pressure gradient $= \underline{\frac{\rho U_0^2}{L}}(1 - y/L)$

Viscous Flow

Analysis extendable curved surface if flow region small compared radius of curvature

$u_x + v_y = 0 \qquad (1)$

$u u_x + v u_y = -p_x/\rho + \nu(u_{xx} + u_{yy}) \qquad (2)$

$u v_x + v v_y = -p_y/\rho + \nu(v_{xx} + v_{yy}) \qquad (3)$

no slip

$\psi = Bx\underbrace{f(y)}_{\neq y \text{ as in } \mu = 0 \text{ solution}} \quad u = Bxf' \quad v = -Bf \quad f(0) = f'(0) = 0$

note: $v_y = -u_x = -Bf'$

Substitution u and v into (2) and (3)

$$B^2 x f'^2 - B^2 x f f'' = -p_x/\rho + \nu B x f''' \quad (4)$$
$$B^2 f f' = -p_y/\rho - \nu B f'' \quad (5)$$

from (5) $p_y = -\rho B(Bff' + \nu f'') = f(y)$ $\therefore p_{yx} = 0$

$$p(x,y) = -\rho B\left(B/2\, f^2 + \nu f'\right) + g(x)$$

Determine $g(x)$ from condition that for large y recover potential flow
ie $f(\infty) = y$, $f'(\infty) = 1$, $f''(\infty) = f'''(\infty) = 0$

$$p(x,\infty) = -\rho B\left(B/2\, y^2 + \nu\right) + g(x)$$
$$= p_0 - \tfrac{1}{2}\rho B^2(x^2+y^2)$$

$$g(x) = p_0 - \tfrac{1}{2}\rho B^2 x^2 + \rho B\nu$$

$$p(x,y) = p_0 - \tfrac{1}{2}\rho B^2 f^2 + \rho B\nu(1-f') - \tfrac{1}{2}\rho B^2 x^2 \quad (6)$$

$p_x = -\rho B^2 x$ $\quad p_y = -\tfrac{1}{2}\rho B^2 2ff' + \rho B\nu f''$
$= -\rho B(Bff' + \nu f'')$

Substitution into (4)

$$B^2 x f'^2 - B^2 x f f'' = B^2 x + \nu B x f'''$$
or $\nu/B\, f''' + ff'' - f'^2 + 1 = 0 \quad (7)$

Alternative derivation:

from (3)

$$p_y/\rho = -u\cancel{v_x} - v v_y + \nu(\cancel{v_{xx}} + v_{yy})$$

$$= Bf(-Bf') - \nu B f''$$

$$p_y = -\rho B(Bff' + \nu f'') = f(y) \quad \therefore \quad p_{yx} = 0$$

$$p = -\rho B(\nu f' + B/2\, f^2) + \alpha_1(x) + c \qquad (4)$$

from (2)

$$p_x/\rho = -u u_x - v u_y + \nu(u_{xx} + u_{yy})$$

$$= -Bxf'(Bf') + Bf\,Bxf'' + \nu(Bxf''')$$

$$p_x = \rho Bx(\nu f''' + Bff'' - Bf'^2) \qquad (5)$$

$$p = \tfrac{1}{2}Bx^2(\nu f''' + Bff'' - Bf'^2) + Q_2(y) + c \qquad (6)$$

comparing (4) & (6) or considering $\frac{\partial(5)}{\partial y} = 0$

$$f''' + \frac{B}{\nu}(ff'' - f'^2) = \text{constant} = -B/\nu$$

For large y $u = Bx$ $\therefore$ $f'(\infty) = 1$ & $f'', f''' = 0$

$$f''' + B/\nu(ff'' - f'^2 + 1) = 0 \qquad (7)$$

nondimensionalize using length scale $\sqrt{\nu/B}$
& velocity scale $\sqrt{\nu B}$; recall ν m^2/s &
$B = U_0/L \cdot 1/s$

Alternate derivation using bi-harmonic ψ equation:

$$\psi_y \frac{\partial}{\partial x}(\nabla^2\psi) - \psi_x \frac{\partial}{\partial y}(\nabla^2\psi) = \nu \nabla^4 \psi$$

$$\omega_z = v_x - u_y = -\nabla^2\psi = -(\psi_{xx} + \psi_{yy})$$
$$u = \psi_y \qquad\qquad = v_x - u_y$$
$$v = -\psi_x$$

$$\nabla^4\psi = \psi_{xxxx} + \psi_{yyyy} \qquad \nabla^2\psi = u_y - v_x$$
$$= -v_{xxx} + u_{yyy}$$

$$u\frac{\partial}{\partial x}(u_y - v_x) + v\frac{\partial}{\partial y}(u_y - v_x) = \nu(-v_{xxx} + u_{yyy})$$

$$u(u_{yx} - v_{xx}) + v(u_{yy} - v_{xy}) = \nu(-v_{xxx} + u_{yyy})$$

Stagnation Flow: $\psi = Bxf \quad u = Bxf' \quad v = -Bf$

$$Bxf'(Bf'') - Bf(Bxf''') = \nu(Bxf^{IV})$$

$$\nu Bxf^{IV} + B^2xff''' - B^2xf'f'' = 0$$

$$\frac{\nu}{B}f^{IV} + ff''' - f'f'' = 0$$

integrate: $\frac{\nu}{B}f''' + ff'' - f'^2 = \text{Constant}$

$f''' + B/\nu(ff'' - f'^2) = \frac{-B}{\nu}$ since for large y

$u = Bx$ so $f'(\infty) = 1$

finally: $f''' + B/\nu(ff'' - f'^2 + 1) = 0$ and $f'', f'''\big|_\infty = 0$

$$\frac{\nu}{B} f''' + ff'' - f'^2 + 1 = 0 \qquad (7)$$

Boundary conditions:

$$u(x,0) = 0 \Rightarrow f'(0) = 0$$
$$v(x,0) = 0 \Rightarrow f(0) = 0$$

and for matching with potential flow

$$f'(\infty) = 1$$

non dimensionalize using length scale $\sqrt{\nu/B}$ and velocity scale $\sqrt{\nu B}$; recall ν m^2/s $B = \frac{U_0}{L}$ $\frac{1}{s}$

$$\eta = y\sqrt{B/\nu} \qquad \psi = x F(\eta)\sqrt{\nu B} \qquad \psi = \frac{m^2}{s}$$

$y = \sqrt{\nu/B}\,\eta$

$\frac{\partial}{\partial y} = \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} = \sqrt{B/\nu}\frac{\partial}{\partial \eta}$

$$u = BxF' \qquad v = -F\sqrt{\nu B}$$

(7) becomes: $F''' + FF'' + 1 - F'^2 = 0$ nonlinear 3rd order ODE

$F(0) = F'(0) = 0$ $\quad F'(\infty) = 1$ Solved numerically

$F = \sqrt{B/\nu}\, f$

$f = \sqrt{\nu/B}\, F$

Note: $f' = \sqrt{\nu/B}\, F' \sqrt{B/\nu} = F'$

$f'' = \sqrt{B/\nu}\, F''$

$f''' = B/\nu\, F'''$

The following would be a typical asymptotic analysis: in $F''' + FF'' + 1 - F'^2 = 0$, as η becomes large, $F \to a + \eta$ and $(1 - F'^2) \to 0$. Therefore, at large η, a conservative view of this equation is:

$$\frac{F'''}{F''} \sim -\eta \quad or \;\; F'' \approx e^{-\eta^2/2}$$

Viscous Flow, F. White, 4th Edition

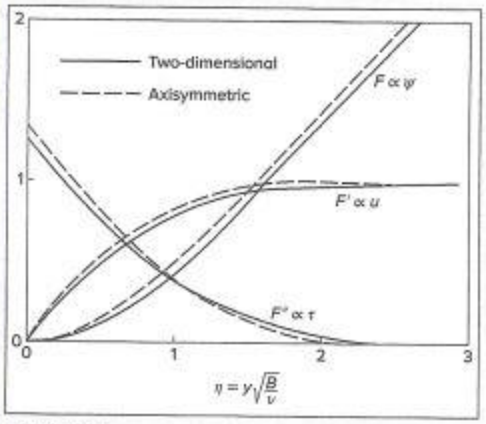


FIGURE 3-32
Numerical solutions of viscous stagnation flow for planar [Eq. (3-206)] and axisymmetric [Eq. (3-219)] conditions.

TABLE 3-4
Numerical solutions for stagnation flow

	$F' = u/U$	
η	Planar $F''(0) = 1.23259$ $\eta^* = 0.6479$	Axisymmetric $F''(0) = 1.31194$ $\eta^* = 0.5689$
0.1	0.11826	0.12619
0.2	0.22661	0.24239
0.3	0.32524	0.34863
0.4	0.41446	0.44499
0.5	0.49465	0.53160
0.6	0.56628	0.60871
0.7	0.62986	0.67663
0.8	0.68954	0.73577
0.9	0.73508	0.78666
1.0	0.77787	0.82987
1.1	0.81487	0.86608
1.2	0.84667	0.89598
1.3	0.87381	0.92032
1.4	0.89681	0.93983
1.5	0.91617	0.95522
1.6	0.93235	0.96718
1.7	0.94578	0.97631
1.8	0.95684	0.98316
1.9	0.96588	0.98822
2.0	0.97322	0.99190
2.2	0.98386	0.99635
2.4	0.99055	0.99847
2.6	0.99464	0.99940
2.8	0.99705	0.99979
3.0	0.99843	0.99993

δ = thickness stagnation layer

$\eta = 2.4 = y/\sqrt{B/\nu} = \delta/\sqrt{B/\nu} \Rightarrow \delta = 2.4\sqrt{\nu/B}$

$\neq f(x)$

$p_x = -\rho B^2 x$
$= -\rho U U_x$

Thinning due to favorable p_x = viscous diffusion

δ small: air approaching 10 cm D cylinder
at $V_0 = 10$ m/s $\Rightarrow B = 4V_0/D = 400\ s^{-1} \Rightarrow \delta = .46$mm
White Sec. 7.3   .5%D
$= U_0/L$

Also exact solution BL equations, i.e., Falkner-Skan similarity solution $\alpha = \beta = 1$.

Note:

$$p_x = -\rho B^2 x = -\rho U U_x$$ inviscid flow pressure gradient

$$p_y = -\rho B (B f f' + \nu f'')$$
$$= -\rho B (B \sqrt{\nu/B} F F' + \nu \sqrt{B/\nu} F'')$$
$$= -\rho B \sqrt{\nu B} (F F' + F'') = O(\sqrt{\nu})$$

Similar as boundary layer theory ie $p_y \sim 0$ across δ

$$\tau_w = \mu (u_y + v_x)\Big|_{y=0}$$

$$\tau_w = \mu B x f'' = \mu B x \sqrt{B/\nu}\, F''\Big|_{y=0}$$

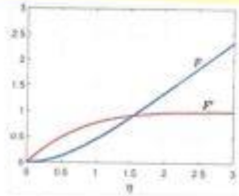
$$\tau_w = U \sqrt{\mu \rho B}\, F_0'' \;\propto U$$

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{2F_0''}{\sqrt{Re_x}} \qquad Re_x = \frac{Ux}{\nu}$$

$$\propto Re_x^{-1/2}$$ Again similar laminar boundary layer theory

Many extensions of theory such as: oblique, two fluid, axisymmetric, circular cylinder, and unsteady flows

The flow due to the presence of a solid surface at $z=0$ in planar stagnation-point flow was described first by Karl Hiemenz in 1911,[6] whose numerical computations for the solutions were improved later by Leslie Howarth.[7] A familiar example where Hiemenz flow is applicable is the forward stagnation line that occurs in the flow over a circular cylinder.[8][9]



Two-dimensional stagnation point flow

The solid surface lies on the xy. According to potential flow theory, the fluid motion described in terms of the stream function ψ and the velocity components $(v_x, 0, v_z)$ are given by

$$\psi = kxz, \quad v_x = kx, \quad v_z = -kz.$$

The stagnation line for this flow is $(x, y, z) = (0, y, 0)$. The velocity component v_x is non-zero on the solid surface indicating that the above velocity field do not satisfy no-slip boundary condition on the wall. To find the velocity components that satisfy the no-slip boundary condition, one assumes the following form

$$\psi = \sqrt{\nu k}xF(\eta), \quad \eta = \frac{z}{\sqrt{\nu/k}}$$

where ν is the Kinematic viscosity and $\sqrt{\nu/k}$ is the characteristic thickness where viscous effects are significant. The existence of constant value for the viscous effects thickness is due to the competing balance between the fluid convection that is directed towards the solid surface and viscous diffusion that is directed away from the surface. Thus the vorticity produced at the solid surface is able to diffuse only to distances of order $\sqrt{\nu/k}$; analogous situations that resembles this behavior occurs in asymptotic suction profile and von Kármán swirling flow. The velocity components, pressure and Navier–Stokes equations then become

$$v_x = kxF', \quad v_z = -\sqrt{\nu k}F, \quad \frac{p_o - p}{\rho} = \frac{1}{2}k^2x^2 + k\nu F' + \frac{1}{2}k\nu F^2$$
$$F''' + FF'' - F'^2 + 1 = 0$$

The requirements that $(v_x, v_z) = (0,0)$ at $z=0$ and that $v_x \to kx$ as $z \to \infty$ translate to

$$F(0) = 0, \; F'(0) = 0, F'(\infty) = 1.$$

The condition for v_z as $z \to \infty$ cannot be prescribed and is obtained as a part of the solution. The problem formulated here is a special case of Falkner-Skan boundary layer. The solution can be obtained from numerical integrations and is shown in the figure. The asymptotic behaviors for large $\eta \to \infty$ are

$$F \sim \eta - 0.6479, \quad v_x \sim kx, \quad v_z \sim -k(z-\delta^*), \quad \delta^* = 0.6479\delta$$

where δ^* is the displacement thickness.

Another exact solution to the boundary-layer equations that may be obtained from the Falkner-Skan similarity solution is that corresponding to a stagnation-point flow. The values of the constants α and β that yield this solution are $\alpha = \beta = 1$. But this is equivalent to letting β be unity in the solution for the flow over a wedge. Then the angle of the wedge becomes π, which means the flow impinges on a flat surface yielding a plane stagnation point.

The solution may be obtained by setting $\beta = 1$ in Eqs. (9.7). This gives

$$U(x) = cx \tag{9.8a}$$

$$\xi(x) = \sqrt{\frac{\nu}{c}} \tag{9.8b}$$

$$f''' + ff'' + 1 - (f')^2 = 0 \tag{9.8c}$$

$$f(0) = f'(0) = 0 \tag{9.8d}$$

$$f'(\eta) \to 1 \quad \text{as } \eta \to \infty \tag{9.8e}$$

$$\psi(x,y) = \sqrt{c\nu}\, xf\left(\frac{y}{\sqrt{\nu/c}}\right) \tag{9.8f}$$

It will be noticed that this is precisely the exact solution to the full Navier-Stokes equations that was obtained by Hiemenz for a stagnation point. This solution is given by Eqs. (7.7a), (7.7b), and (7.7c). Thus the exact solution to the boundary-layer equations is also an exact solution to the full Navier-Stokes equations in this instance.

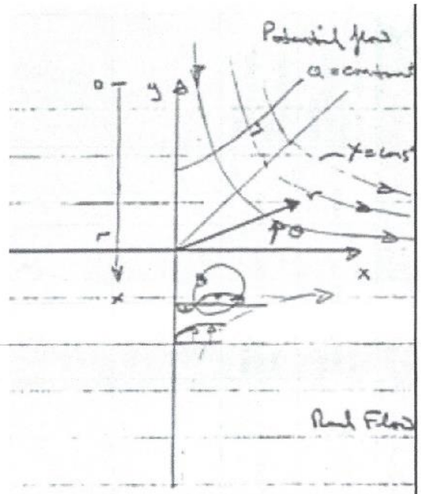
Consider the complex potential

$$F(z)=\frac{a}{2}z^2=\frac{a}{2}r^2e^{2i\theta}$$

$$\varphi=\mathrm{Re}[F(z)]=\frac{a}{2}r^2\cos 2\theta$$

$$\psi=\mathrm{Im}[F(z)]=\frac{a}{2}r^2\sin 2\theta$$

Orthogonal rectangular hyperbolas



φ: asymptotes y = ± x

ψ : asymptotes x=0, y=0

$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

$$\left\{\begin{array}{l}\underline{V}=\nabla\varphi=\varphi_r\hat{e}_r+\dfrac{1}{r}\varphi_\theta\hat{e}_\theta\\ v_r=ar\cos 2\theta\\ v_\theta=-ar\sin 2\theta\end{array}\right.$$

$\frac{\pi}{2}\le\theta\le 0$ (flow direction as shown)

$$\underline{V}=v_r\left(\cos\theta\hat{i}+\sin\theta\hat{j}\right)+v_\theta\left(-\sin\theta\hat{i}+\cos\theta\hat{j}\right)=$$

$$\left(v_r\cos\theta-v_\theta\sin\theta\right)\hat{i}+\left(v_r\sin\theta+v_\theta\cos\theta\right)\hat{j}$$

Potential flow slips along surface: (consider $\theta=90^\circ$)

$\psi = Bxy \quad B = U/L$

$u = Bx = \psi_y$

$v = -By = -\psi_x$

$p + \frac{1}{2}\rho(u^2+v^2) = C$

$p + \frac{1}{2}\rho B^2(x^2+y^2) = C$

$p(0,0) = C = p_0$

$p = p_0 - \frac{1}{2}\rho\frac{U^2}{L^2}(x^2+y^2)$

$p_x = -\rho\frac{U^2}{L^2}x$

$p_y = -\rho\frac{U^2}{L^2}y$

1) determine a such that $v_r=U_0$ at r=L, $\theta=90^\circ$

$v_r=aL\cos(2\times 90)=U_0\Rightarrow aL=-U_0$, i.e. $a=-\frac{U_0}{L}$

2) let $U(x)=v_r$ at x=L-r:

$\Rightarrow v_r=a(L-x)\cos(2\times 90)=U(x)$

$$Or:\ U(x)=-a(L-x)=\frac{U_0}{L}(L-x)=U_0(1-\frac{x}{L})$$

$U_x = -\frac{U_0}{L}$

$p + \frac{1}{2}\rho U^2 = C$

$p_x + \rho U U_x = 0$

$p_x = -\rho U U_x = -\rho U_0(1-\frac{x}{L})(-\frac{U_0}{L}) = \rho\frac{U_0^2}{L}(1-\frac{x}{L})$

3.81 Flow near an infinite rotating disk

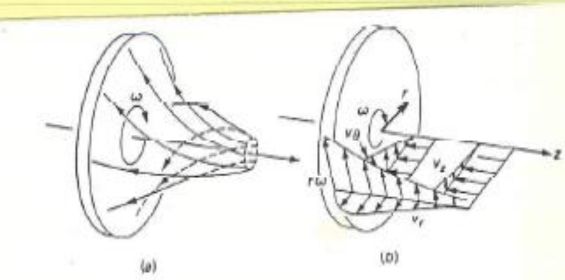


FIGURE 3-21
Laminar flow near a rotating disk: (a) streamlines; (b) velocity components.

Flow is rotationally symmetric : $\frac{\partial}{\partial\theta} = 0$

(1) $$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{\partial}{\partial z}(v_z) = 0$$

(2) $$v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2}\right)$$

(3) $$v_r\frac{\partial v_\theta}{\partial r} + v_z\frac{\partial v_\theta}{\partial z} - \frac{1}{r}v_r v_\theta = \nu\left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} + \frac{\partial^2 v_\theta}{\partial z^2} - \frac{v_\theta}{r^2}\right)$$

(4) $$v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2}\right)$$

The boundary conditions are :

$z = 0$: $\quad v_r = v_z = 0 \quad v_\theta = r\omega \quad p = 0$ ← convenient constant

$z \to \infty$: $\quad v_r = v_\theta = 0$

note : $v_z(\infty)$ not prescribed (centrifugal pumping)
also $\frac{\partial p}{\partial r} = 0$ so that p is uncoupled
from $\underline{v}$ & can be solved from (1) – (3) & then (4) for p

There are no obvious velocity and length scales for this problem.

Karman (1921): $\frac{v_r}{r}, \frac{v_\theta}{r}, v_z, p = f(z)$ only

$\Rightarrow$ define $z^* = z/\sqrt{\nu/\omega}$ = similarity variable
(similar to $y/\sqrt{\nu t}$ + others)

$v_r = r\omega F(z^*)$ $\quad v_\theta = r\omega G(z^*)$ $\quad v_z = \sqrt{\omega\nu}\, H(z^*)$

$p = \rho\omega\nu P(z^*)$

Substitution into (1)–(4) results in

$$\underbrace{H' = -2F \quad F'' = -G + F^2 + F'H \quad G'' = 2FG + HG'}_{\text{solve for } F, G, H}$$

$$\underbrace{P' = 2FH - 2F'}_{\text{then solve for } p}$$

$'$ denotes $\frac{\partial}{\partial z^*}$

$F(0) = H(0) = P(0) = 0 \quad G(0) = 1$
$F(\infty) = G(\infty) = 0$

Solution is obtained by using standard techniques for system of ODE (see discussion in text)

Reality: laminar flow unstable at $Re = 3.2\times10^5$

$Re = \dfrac{\omega r^2}{\nu}$

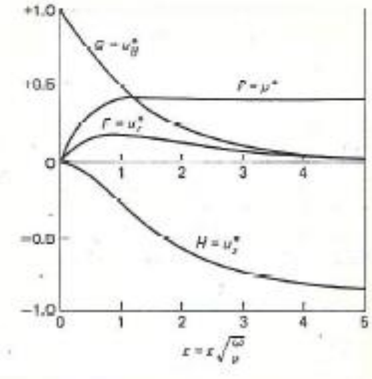


FIGURE 3-23
Numerical solutions of Eqs. (3-155) for the infinite rotating disk.

$$\delta \simeq 5\sqrt{\nu/\omega} \qquad \underbrace{\delta/r_0 \approx 5/\sqrt{Re}}_{\text{actually independent of } r} \qquad Re = \frac{\omega r_0^2}{\nu}$$

$$Q = 2\pi r\int_0^\infty (\ldots)\,dz = .885\,\pi r^2\sqrt{\nu\omega} = \text{amount of fluid pumped outward on one side of the disk}$$

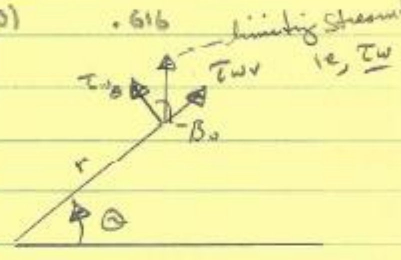
$$\Rightarrow v_z(\infty) = .885\sqrt{\nu\omega} \quad \text{(volume flow per unit area)}$$

$$\underline{\tau_w} = \tau_{wr}\,\hat{e}_r + \tau_{w\theta}\,\hat{e}_\theta$$

$$\tau_{w_r} = \mu\frac{\partial v_r}{\partial z}\Big|_w \qquad \tau_{w_\theta} = \mu\frac{\partial v_\theta}{\partial z}\Big|_w$$

$$\tan\beta = -\frac{\tau_{wr}}{\tau_{w\theta}} = -\frac{F'(0)}{G'(0)} = \frac{.51}{.616} = .828$$

$$\beta = 39.6^\circ$$



3.82 Moment Coefficient

Above solution is strictly only applicable to an infinite disk. However, it should also be valid for a finite disk if

$r_0 \gg \delta \qquad Re \gg 1$

then departures are limited to thin edge effects. It is of interest to calculate the torque on the disk

$$M = \int \tau_{w_\theta}\, r\, dA = \int_0^{r_0} \tau_{w_\theta}\, 2\pi r^2 dr$$

$$\tau_{w_\theta} = \rho \nu \sqrt{\nu \omega^3}\, G_0'$$

$$= \frac{\pi}{2} \rho r_0^4 G_0' (\nu\omega^3)^{1/2}$$

$$G_0' = -.616$$

$$C_m = \frac{-2M}{\frac{1}{2}\rho\omega^2 r_0^5} = \frac{3.87}{\sqrt{Re}} \qquad Re = \frac{\omega r_0^2}{\nu}$$

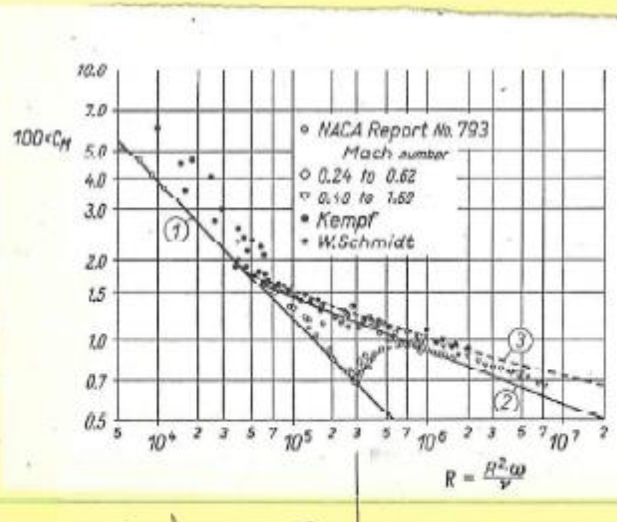


Fig. 5.14. Turning moment on a rotating disk; curve (1) from eqn. (5.56), *laminar*; curves (2) and (3) from eqns. (21.30) and (21.33), *turbulent*

good agreement ← → flow becomes turbulent