Chapter 3 Solutions of the Newtonian Viscous-Flow Equations

5. Unsteady flows

    a. Stokes 1st problem: sudden acceleration
    b. Diffusion vortex sheet
    c. Decay of a Line Vortex
    d. Stokes 2nd problem: steady oscillations
    e. Starting flow circular pipe
    f. Oscillating pressure gradient pipe flow
    g. Starting flow fixed/moving parallel walls.

a and d are unsteady flows with moving boundaries, of which there are many additional solutions.

Assume parallel flow: $u = u(y, z, t)$, $v = 0$, $w = 0$

$$u_t = -\frac{1}{\rho}\hat{p}_x + \nu(u_{yy} + u_{zz})$$

$\hat{p}_x = f(t)$ only such that

$$u' = u + \int \frac{1}{\rho}\hat{p}_x \, dt$$

$$u'_t = \nu(u'_{yy} + u'_{zz})$$

homogeneous heat-conduction equation, for which many solutions

$$u'_t = u_t + \frac{1}{\rho}\hat{p}_x$$

$u'_{yy} = u_{yy}$ & $u'_{zz} = u_{zz}$

Stokes 1st $\quad u_t = \nu u_{yy}$

Flat plate wall $\quad u(y,0) = 0 \quad$ IC

$y=0 \quad u(0,t) = U \quad$ Impulsive start

$u(\infty,t) = 0 \quad$ stagnant

Diffusion Vortex Sheet

Vortex sheet

$y=0 \quad u_t = \nu u_{yy} \quad$ IC

$u(y,0) = U \operatorname{sgn}(y) \quad$ jump condition

$u(\infty,t) = U$

$u(-\infty,t) = -U \quad \Big\}$ far field $\pm U$

Decay ideal line vortex

$r=0 \quad u_{\theta_t} = \nu \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right]$

$u_\theta(r,0) = \Gamma/2\pi r \quad$ impulsive IC

$u_\theta(0,t) = 0 \quad$ no slip

$u_\theta(\infty,t) = \Gamma/2\pi r \quad$ outer potential vortex

Line vortex impul fluid at rest

$r=0$

Stokes 2nd $\quad u_{\theta_t} = \nu \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right]$

$u_t = \nu u_{yy} \quad u_\theta(r,0^+) = 0$

$u(0,t) = U \cos\omega t \quad u_\theta(r,\infty) = \Gamma/2\pi r$

$u(\infty,t) = 0 \quad u_\theta(\infty,t) = 0$

Unsteady pipe flow: $\rho u_t = -\hat{p}_x + \mu\left(u_{rr} + \frac{1}{r}u_r\right)$

Stokes 1st Problem: Impulsively Started Plate

ρ, μ constant

$u \neq f(x)$

$u = f(y,t)$

$p(x = \pm\infty) = p_\infty$

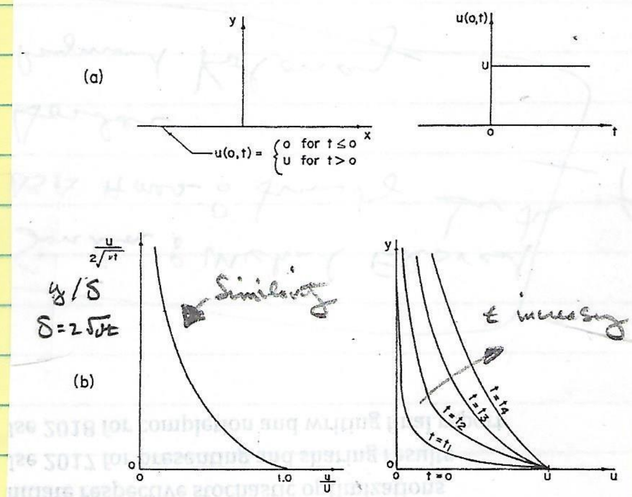


FIGURE 7.4
(*a*) Definition sketch for Stokes' first problem and (*b*) the solution curves in terms of the similarity variable and in terms of the dimensional variables.

Continuity: $u_x + v_y = 0$, $v(0) = 0$ $\Rightarrow v = 0$

Momentum: $\rho u_t = -p_x + \mu u_{yy}$

$0 = -p_y \Rightarrow p(x,t)$ and from x-momentum $u(y,t)$

and $p(x = \pm\infty) = p_\infty$, $p_x = 0$

$\therefore \quad u_t = \nu u_{yy}$

$u(y,0) = 0$ IC

$\left.\begin{array}{l} u(0,t) = U \\ u(\infty,t) = 0 \end{array}\right\}$ BC

Well posed IBVP

$u = f(U, y, t, \nu)$

Dimensional analysis: $u/U = F(\eta)$

η = non dimensional distance $\quad \eta = y/2\sqrt{\nu t}$ (2 for algebraic convenience)

Dimensionally reduced from 2 (y,t) to 1 (η) and similarity solution ie pde → ode

$u_t = U F_t = U F' \eta_t = -U F' \eta / 2t \qquad F' = F_\eta$

$u_y = U F_y = U F' / 2\sqrt{\nu t}$

$u_{yy} = \frac{U}{2\sqrt{\nu t}} F'' \eta_y = \frac{U}{4\nu t} F''$

$\eta = \frac{y}{2\sqrt{\nu}} t^{-\frac{1}{2}}$

$\eta_t = \frac{y}{2\sqrt{\nu}}\left(-\frac{1}{2} t^{-\frac{3}{2}}\right)$

$= \frac{-y}{4\sqrt{\nu}}\left(t^{-1/2}\, t\right)$

$= -\eta/2t$

$-2\eta F' = F'' \qquad F(\infty) = 0 \qquad u(y,0) = 0$

$F(0) = 1 \qquad u(\infty,t) = 0$

$u(0,t) = U$

$\frac{dF'}{F'} = -2\eta\, d\eta$

$-2\eta\, d\eta = \frac{d\left(\frac{dF}{d\eta}\right)}{\frac{dF}{d\eta}}$

$\ln F' = -\eta^2 + \text{constant} \qquad e^{x+y} = e^x e^y$

$F' = A e^{-\eta^2}$

$F(\eta) = A \int_0^\eta e^{-\eta^2} d\eta + B$

$$F(0) = 1 = A\int_0^0 e^{-\eta^2}\,d\eta + B$$

$$B = 1$$

$$F(\infty) = 0 = A\int_0^\infty e^{-\eta^2}\,d\eta + 1$$

$$= \frac{A\sqrt{\pi}}{2} + 1$$

$$A = -2/\sqrt{\pi}$$

$$F = 1 - \underbrace{\frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\eta^2}\,d\eta}_{\mathrm{erf}(\eta)}$$

$$u/U = 1 - \mathrm{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)$$

$\omega(0,0) = \infty$ — delta function vortex sheet
$\omega(y,0) = 0$ — due to impulsive motion

$\int_0^\infty \omega\,dy \neq f(t)$ ie no new ω after $t=0$
initial ω diffuses outward
increasing flow width

$u/U = .05 \Rightarrow \eta = 1.38$

$\delta = 2.76\sqrt{\nu t}$ — width of diffusion layer, which increases as $\sqrt{t}$

$$\omega_z = -\frac{\partial u}{\partial y} = \frac{U}{2\sqrt{\nu t}} F' = \frac{U}{\sqrt{\pi \nu t}} e^{-\eta^2}$$

$$\tau_{yx} = \mu \frac{\partial u}{\partial y} = -\mu \omega_z$$

$$\tau_{yx} \sqrt{\pi \nu t} / \mu U = -e^{-\eta^2}$$ non dimensional shear stress

$= f(\eta)$ only

$$\tau_{yx}(0) = -\mu U / \sqrt{\pi \nu t}$$ $= \infty$ @ $t = 0$

$\propto 1/\sqrt{t}$ $t > 0$

Note: $\mathrm{erf}(\eta) \sim \eta^{-1} e^{-\eta^2}$ $\eta \to \infty$

$\therefore$ influence plate extends to ∞ albeit exponentially small

$$\eta(u/U = .01) = 1.8 = \delta / 2\sqrt{\nu t}$$

$$\delta(u/U = .01) = 3.6 \sqrt{\nu t}$$ diffusion distance ie effects μ/ν within δ and independent U

for $t = 1$ min air $= 10.8$ cm

$\nu_{air} = .15\ cm^2/s$ water $= 2.8$ cm

$\nu_{water} = .01\ cm^2/s$

Stokes (1851)

Rayleigh (1911) $t = x/U_0$ $\tau = \frac{\mu U_0}{\sqrt{\pi \nu}} \sqrt{\frac{U_0}{x}} = \frac{\mu U_0^{3/2}}{\sqrt{\pi \nu}} x^{-1/2}$ $= .56\, \mu U_0^{3/2} / \sqrt{\nu x}$

Blasius

$$D = \int_0^L \tau\, dx = \frac{\mu U_0^{3/2}}{\sqrt{\pi \nu}} 2 L^{1/2}$$ per unit span

$$\tau_N = \frac{\mu U f''(0)}{\sqrt{2 \nu x / U}} = \frac{\mu U^{3/2} .5}{\sqrt{2\nu}} x^{-1/2} = \frac{.35\, U^{3/2}}{\sqrt{\nu x}}$$

$$c_D = \frac{D}{\frac{1}{2}\rho U^2 L} = \frac{2.26}{\sqrt{Re_L}}$$ Blasius $\sim 2.\frac{1.328}{\sqrt{Re_L}}$

$$\omega = u_y = \frac{\Gamma}{\sqrt{\pi \nu t}} e^{-\eta^2} \qquad \eta = y/2\sqrt{\nu t}$$

$$\omega(0,0) = \frac{1}{0} e^0 = \frac{1}{0} = \infty \qquad \eta^2 = \frac{y^2}{4\nu t}$$

initial ω is ∞ $y=0$

$= 0$ $y>0$ $\quad \omega(y,0) = 0 \qquad \eta^2(0,0) = \frac{0}{0} = \left.\frac{2y}{4\nu}\right|_{y=0} = 0$

ω = Gaussian with width increase $\sqrt{t}$

& max value decrease $1/\sqrt{t}$

Total amount vorticity

$$\int_{-\infty}^{\infty} \omega\, dy = 2\sqrt{\nu t} \int_{-\infty}^{\infty} \omega\, d\eta = \frac{2\Gamma}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\eta^2} d\eta = 2\Gamma$$

$\therefore$ no new vorticity created after $t=0$ $\qquad \neq f(t)$

$=$ y integral initial delta function ω

initial vorticity simply diffuses outward resulting in increase flow width

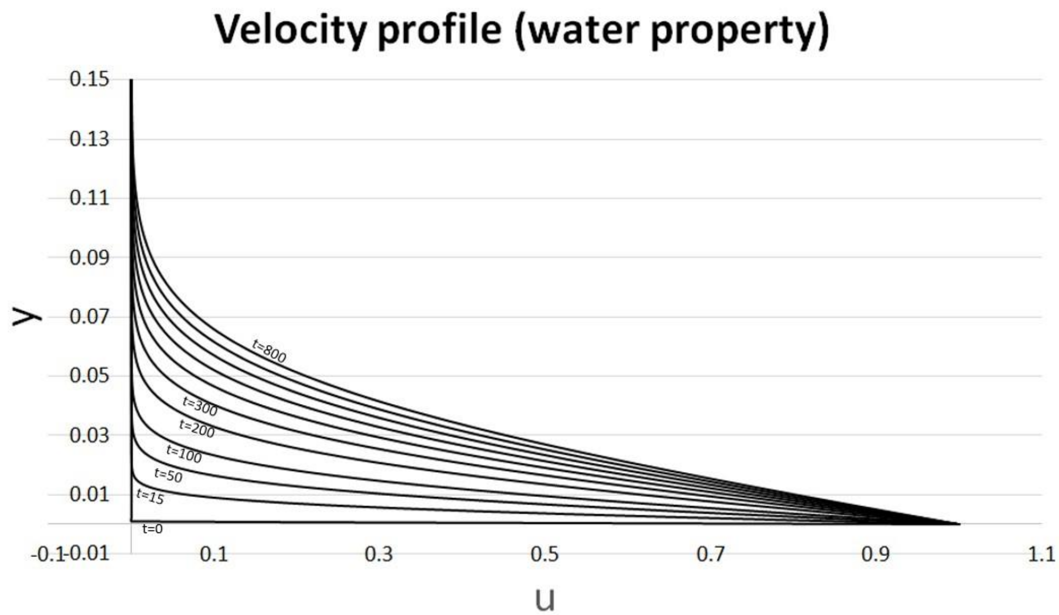
$$\omega(y,0) = \frac{\Gamma}{0} e^{-\frac{y^2}{0}} = \frac{0}{0}$$

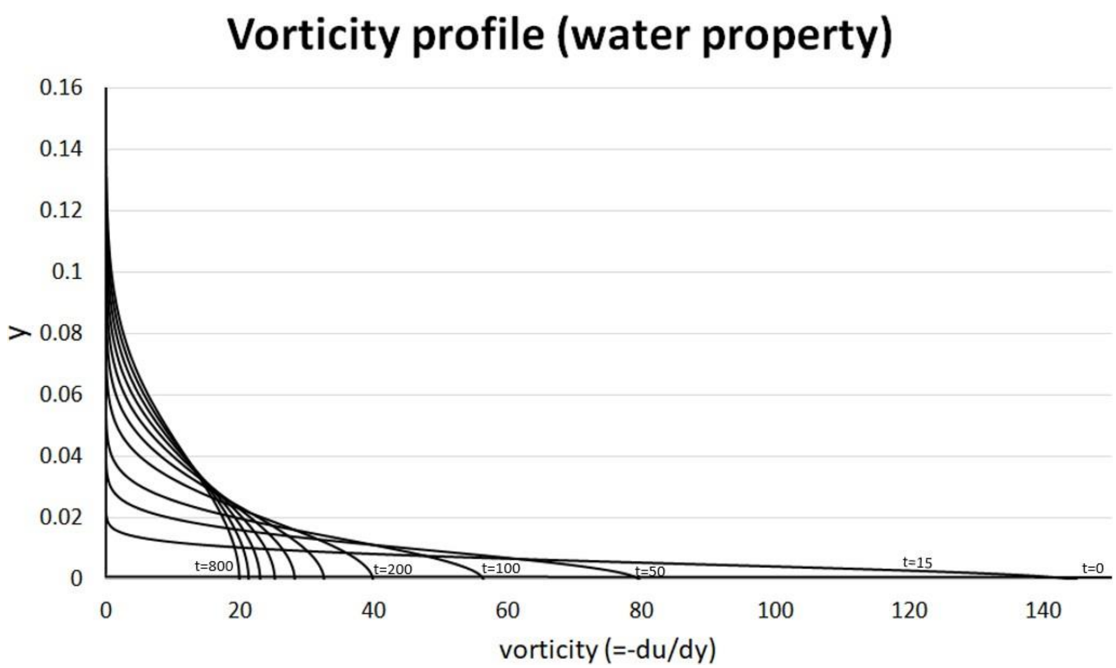
$$= \frac{\Gamma e^{-\eta^2}(-2\eta)}{\sqrt{\pi\nu}\left(\frac{1}{2} t^{-1/2}\right)} \qquad \eta = \frac{y}{0} = \infty \qquad \eta^2 = \frac{y^2}{0} = \infty$$

$$= \frac{\Gamma e^{-\eta^2}(-2\eta)\, 2\sqrt{t}}{\sqrt{\pi\nu}} = \Gamma e^{-\eta^2}\left(-\frac{2y}{2\sqrt{\nu t}}\right) 2\sqrt{t}$$

$$= 0 \,!$$

$$\mu: 0.001 kg/m\ s$$
$$\rho: 1000 \mathrm{kg}/m^3$$
$$\nu: 1 \times 10^{-6} m^2/s$$
$$U = 1 \mathrm{m/s}$$
$$\frac{u}{U} = 1 - erf\left(\frac{y}{2\sqrt{\nu t}}\right)$$



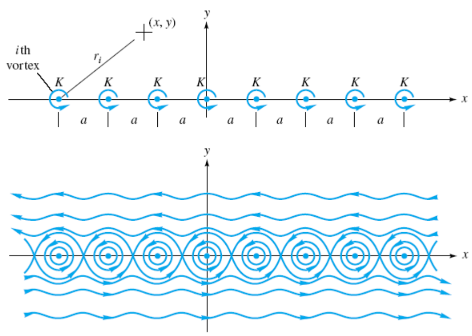


Diffusion vortex sheet

First recall potential flow solution for vortex sheet. The point vortex singularity is important in aerodynamics, since, their distributions can be used to represent airfoils and wings. To see this, consider as an example of an infinite row of vortices:

$$\psi = -K\sum_{i=1}^{\infty} \ln r_i = -\frac{1}{2}K\ln\left[\frac{1}{2}(\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a})\right]$$

Where r_i is radius from origin of ith vortex.



Superposition infinite row equally spaced vortices of equal strength

For $|y| \geq a$ the flow approaches uniform flow with

$$u = \frac{\partial \psi}{\partial y} = \pm\frac{\pi K}{a}$$

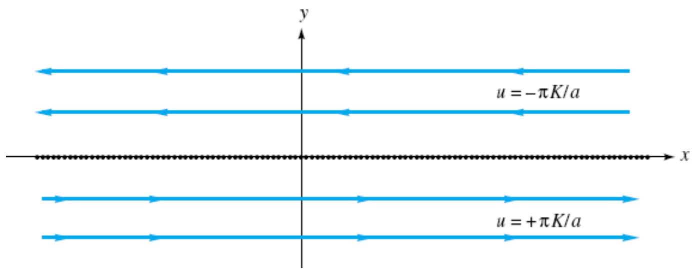
+: below x axis

-: above x axis

Note: this flow is just due to infinite row of vortices and there isn't any pure uniform flow

Potential Flow Vortex sheet:

From afar (i.e. $|y| \geq a$) looks like a thin sheet with velocity discontinuity.



Define $\gamma = \dfrac{2\pi K}{a}$ = strength of vortex sheet

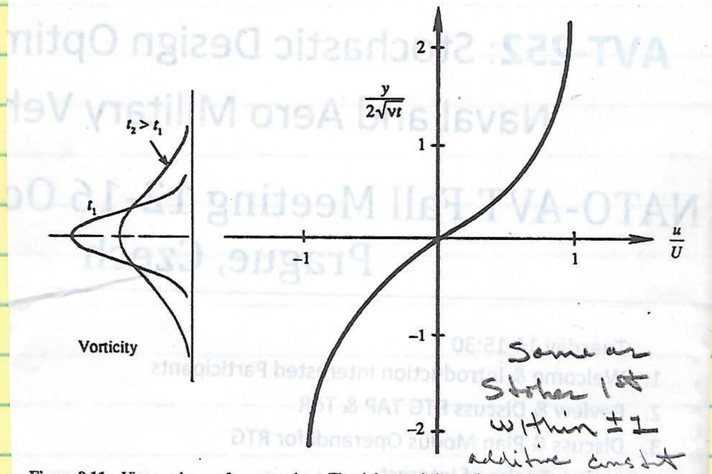
$d\Gamma = \underline{V} \cdot d\,\underline{s}$ (around closed contour)

$$d\Gamma = u_l dx - u_u dx = (u_l - u_u)dx = \frac{2\pi K}{a} dx$$

i.e. $\gamma = \dfrac{d\Gamma}{dx}$ = Circulation per unit span

Note: There is no flow normal to the sheet so that vortex sheet can be used to simulate a body surface. This is the basis of airfoil theory where we let $\gamma = \gamma(x)$ to represent body geometry.

Diffusion of a Vortex Sheet



Same as Stokes 1st within ±1 additive const

Figure 9.11 Viscous decay of a vortex sheet. The right panel shows the nondimensional solution and the left panel indicates the vorticity distribution at two times.

$$u_t = \nu u_{yy}$$
$$u(y,0) = U\,\mathrm{sgn}(y)$$
$$u(\infty,t) = U$$
$$u(-\infty,t) = -U$$

Same as Stokes 1st

initial ω diffuses away from $y=0$

$$u/U = F(\eta) \qquad \eta = y/2\sqrt{\nu t}$$

$$F'' = -2\eta F'$$
$$F(\infty) = 1$$
$$F(-\infty) = -1$$

note: $\frac{d}{dy}\mathrm{erf}(y) = \frac{2}{\sqrt{\pi}} e^{-y^2}$

$$\omega = -\frac{\partial u}{\partial y} = \frac{-U}{\sqrt{\pi \nu t}} e^{-y^2/4\nu t}$$

Gaussian with width increases $\sqrt{t}$

$$\int_{-\infty}^{\infty} \omega\, dy = 2\sqrt{\nu t}\int_{-\infty}^{\infty} \omega\, d\eta$$
$$= \frac{2U}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\eta^2} d\eta$$
$= 2U \neq f(t)$ as y integral initial ω

$$F(\eta) = \mathrm{erf}(\eta)$$

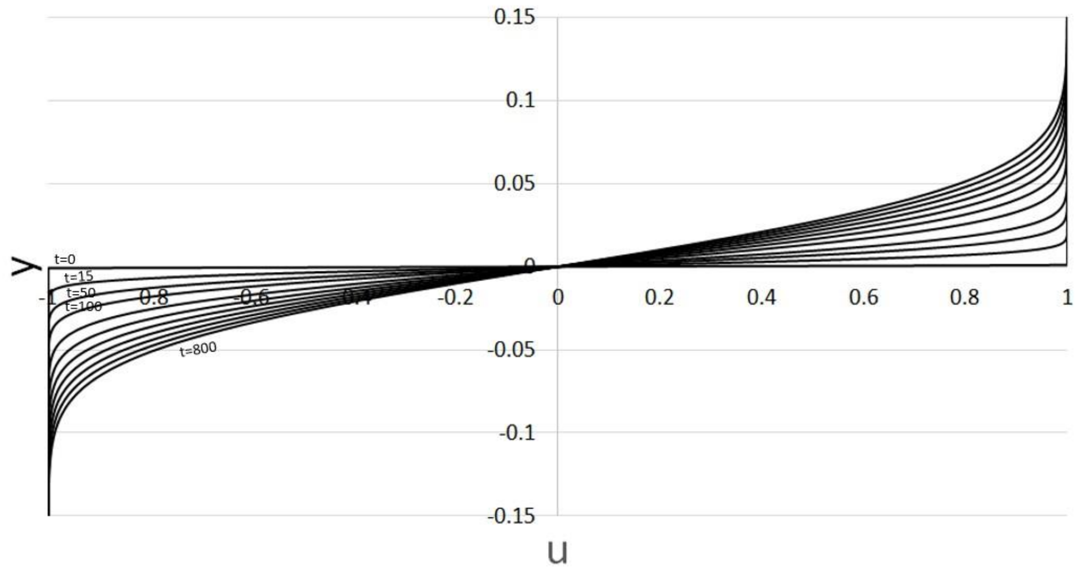
$$u = U\,\mathrm{erf}(\eta)$$

$$u = \pm .95U$$
$$\eta = \pm 1.38$$
$$\delta = \pm 5.52\sqrt{\nu t}$$

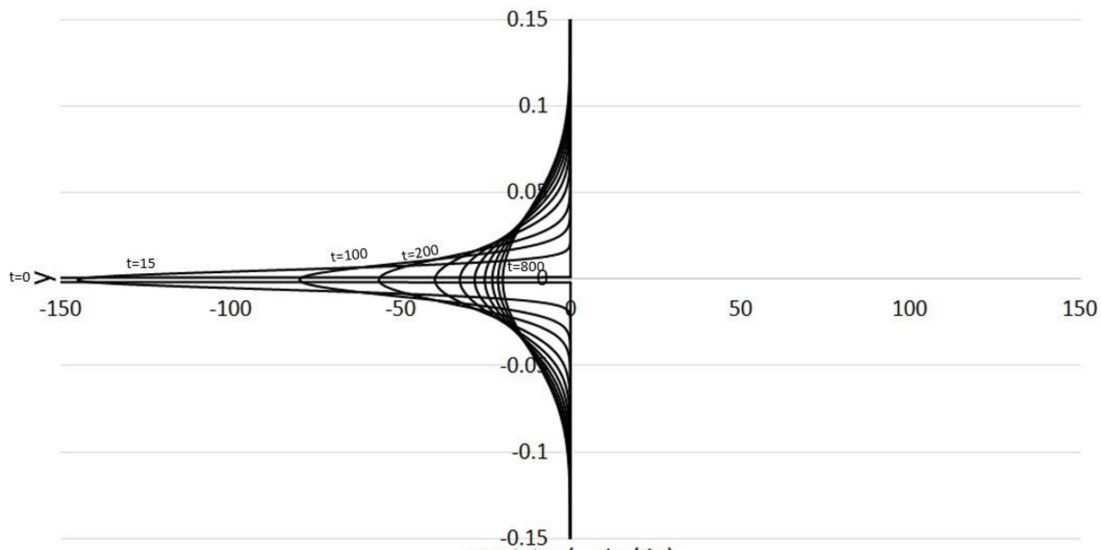
$$\mu : 0.001 kg/m\ s$$
$$\rho : 1000 kg/m^3$$
$$\nu : 1 \times 10^{-6} m^2/s$$
$$U = 1\mathrm{m/s}$$

$$\boldsymbol{u = Uerf\left(\frac{y}{2\sqrt{\nu t}}\right)}$$

Velocity profile (water property)



Vorticity profile (water property)



Decay of an ideal line vortex : Oseen Vortex

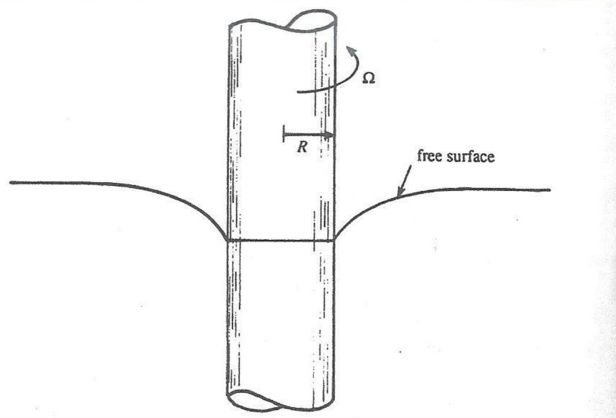


Figure 9.7 Rotation of a solid cylinder of radius R in an infinite body of viscous fluid. The shape of the free surface is also indicated. The flow field is viscous but irrotational.

$$u_\theta = \Omega R^2/r = \frac{\Gamma}{2\pi r} \qquad \Gamma = 2\pi \Omega R^2$$

Irrotational vortex. $\tau_{r\theta} = \mu \left[r \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta} \right]$

$$= -\frac{2\mu \Omega R^2}{r^2}$$

Work done per unit height

$$= 2\pi R \, \tau_{r\theta} u_\theta \Big|_{r=R}$$

$$= 2\pi R \, (2\mu\Omega)(\Omega R) = 4\pi \mu R^2 \Omega^2$$

= viscous dissipation such that there is no net force at a point

Suppose $r \to 0$ while $\Omega \uparrow$ such that $\Gamma = 2\pi \Omega R^2$ is unchanged. In the limit we have a potential line vortex with singularity at the origin

Exercise 9.15. Consider a solid cylinder of radius a, steadily rotating at angular speed Ω in an infinite viscous fluid. The steady solution is irrotational: $u_\theta = \Omega a^2/R$. Show that the work done by the external agent in maintaining the flow (namely, the value of $2\pi R u_\theta \tau_{r\theta}$ at $R = a$) equals the viscous dissipation rate of fluid kinetic energy in the flow field.

Solution 9.15. Using the given velocity field, the shear stress is:

$$\tau_{R\varphi} = \mu R \frac{\partial}{\partial R}\left(\frac{u_\varphi}{R}\right) = \mu\Omega a^2 R \frac{\partial}{\partial R}\left(\frac{1}{R^2}\right) = -2\mu\Omega a^2 \frac{1}{R^2}.$$

The work done per unit height $= \left\{2\pi a \tau_{R\varphi} u_\varphi\right\}_{R=a} = 2\pi a \cdot 2\mu\Omega \cdot \Omega a = 4\pi\mu a^2\Omega^2$.

From (4.58) the viscous dissipation rate of kinetic energy per unit volume for an incompressible flow is $\rho\varepsilon = 2\mu S_{ij}S_{ij}$, where ε is the viscous dissipation of kinetic energy per unit mass. For the given flow field there is only one non-zero independent strain component:

$$S_{R\varphi} = S_{\varphi R} = \frac{R}{2}\frac{\partial}{\partial R}\left(\frac{u_\varphi}{R}\right) = \frac{\Omega a^2}{2} R \frac{\partial}{\partial R}\left(\frac{1}{R^2}\right) = -\Omega a^2 \frac{1}{R^2}.$$

Therefore:

$$\rho\varepsilon = 2\mu S_{ij}S_{ij} = 2\mu\left(S_{R\varphi}^2 + S_{\varphi R}^2\right) = 4\mu\Omega^2\frac{a^4}{R^4},$$

so the kinetic energy dissipation rate per unit height is:

$$\int_a^\infty \rho\varepsilon 2\pi R dR = 8\pi\mu\Omega^2 a^4 \int_a^\infty \frac{1}{R^3} dR = 4\pi\mu\Omega^2 a^2,$$

which equals the work done turning the cylinder.

$$\frac{\partial}{\partial x_j}(u_i \sigma_{ij}) = \sigma_{ij}\frac{\partial u_i}{\partial x_j} + u_i \frac{\partial \sigma_{ij}}{\partial x_j}$$

total work of surface force = deformation work w/o a and lost to internal energy + increase of KE since contributes fluid a

$$\sigma_{ij}\frac{\partial u_i}{\partial x_j} = \sigma_{ij}(\underbrace{\varepsilon_{ij} + \omega_{ij}}_{e_{ij}}) = \sigma_{ij}\varepsilon_{ij} \quad \text{since } \sigma_{ij}\omega_{ij} = 0$$

$$= \left[-(p + \tfrac{2}{3}\mu\nabla\cdot\underline{v})\delta_{ij} + 2\mu\varepsilon_{ij}\right]\varepsilon_{ij} \quad \text{since } \varepsilon_{ij}\delta_{ij} = \varepsilon_{ii} = \nabla\cdot\underline{v}$$

$$= -p\nabla\cdot\underline{v} + \underbrace{2\mu\varepsilon_{ij}\varepsilon_{ij} - \tfrac{2}{3}\mu(\nabla\cdot\underline{v})^2}_{\Phi}$$

$$= -p\nabla\cdot\underline{v} + \Phi$$

Next, suppose (infinitely small R of fast or cylinder) suddenly stops rotating at $t=0$; thereby, reducing the velocity at $r=0$ to zero impulsively. Then the fluid would slow down due to viscous diffusion ie viscous decay of a line vortex. Circular analog of the diffusion of a vortex sheet

$$u_{\theta_t} = \nu \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right]$$

$$u_\theta(r,0) = \Gamma/2\pi r$$

$$u_\theta(0,t) = 0$$

$$u_\theta(\infty,t) = \Gamma/2\pi r$$

Assume $u' = \dfrac{u_\theta}{\Gamma/2\pi r} = \underbrace{f(r,t,\nu)}_{\text{must be nondimensional}} = F(\eta) \quad \eta = \dfrac{r^2}{4\nu t}$

expect similarity since
r & t have no natural scales
from BC; & eliminate $\Gamma/2\pi r$ via nondimensional u_θ
Note η is square of form used for Stokes 1st problem. Substitute u' into GDE

$$F'' + F' = 0$$

$$F(\infty) = 1$$

$$F(0) = 0$$

$$u_\theta = \frac{\Gamma}{2\pi r} u' = \frac{\Gamma}{2\pi r} F(\eta) \qquad \eta = \frac{r^2}{4\nu t} = \frac{r^2}{4\nu} t^{-1}$$

$$\eta_t = \frac{-r^2}{4\nu} t^{-2}$$

$$\eta_r = \frac{2r}{4\nu t} = \frac{r}{2\nu t}$$

$$u_{\theta t} = \frac{\Gamma}{2\pi r} F' \left(\frac{-r^2}{4\nu t^2}\right)$$

$$= \frac{-\Gamma r}{8\pi \nu t^2} F'$$

$$r u_\theta = \frac{\Gamma}{2\pi} F \qquad \frac{\partial}{\partial r}(r u_\theta) = \frac{\Gamma}{2\pi} F' \frac{r}{2\nu t} = \frac{\Gamma r}{4\pi \nu t} F' \quad \div r \quad = \frac{\Gamma}{4\pi \nu t} F'$$

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right] = \frac{\Gamma}{4\pi \nu t} F'' \frac{r}{2\nu t} = \frac{\Gamma r}{8\nu^2 t^2} F''$$

$$\frac{-\Gamma r}{8\pi \nu t^2} F' = \frac{\Gamma r}{8\pi \nu t^2} F'' \qquad F'' + F' = 0$$

$$\frac{F''}{F'} = -1 \qquad \frac{\frac{d}{d\eta}(F')}{F'} = -1$$

$$\frac{dF'}{F'} = -d\eta$$

$$\ln F' = -\eta + c$$

$$F(\infty) = 1 \qquad D = 1 \qquad F' = c e^{-\eta}$$

$$F(0) = 0 \qquad c + D = 0 \qquad F = c e^{-\eta} + D$$

$$c = -D \qquad F = 1 - e^{-\eta}$$

$$u_\theta = \frac{\Gamma}{2\pi r}\left[1 - e^{-\frac{r^2}{4\nu t}}\right]$$

$$F = 1 - e^{-\eta}$$

$$u_\theta = \frac{\Gamma}{2\pi\nu}\left[1 - e^{-r^2/4\nu t}\right]$$

in this case $\eta = \frac{r}{\sqrt{\nu t}}$ $= \frac{\Gamma}{2\pi\sqrt{\nu t}}\left(\frac{1}{\eta}\right)\left[1 - \exp\left(-\frac{\eta^2}{4}\right)\right]$

One of family of vortices that satisfy NS equations. Another is the Taylor vortex

$$u_\theta = \frac{H}{8\pi}\frac{r}{\nu t^2}\exp\left(\frac{-r^2}{4\nu t}\right)$$

H = amount of angular momentum in the vortex, which is infinite for Oseen vortex

Oseen
$V^* \propto u_\theta/\sqrt{t}$

Taylor
$V^* \propto u_\theta/t^{-3/2}$

$\eta = r/\sqrt{\nu t}$

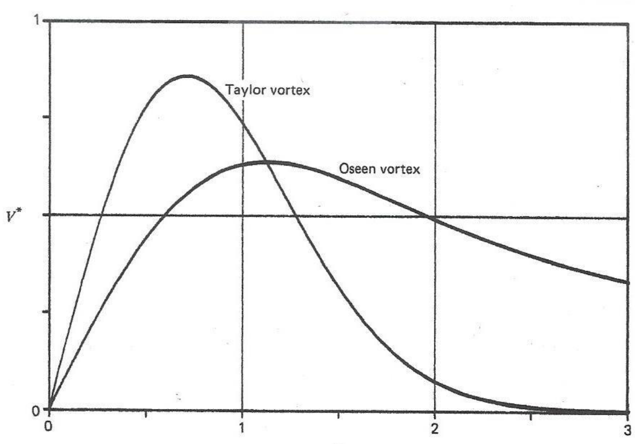


Figure 11.8 Profiles for Oseen and Taylor vortices in similarity variables. For the Oseen vortex, $V^* \propto v_\theta/t^{-1/2}$, while for the Taylor vortex, $V^* \propto v_\theta/t^{-3/2}$. In each case $\eta = r/\sqrt{\nu t}$.

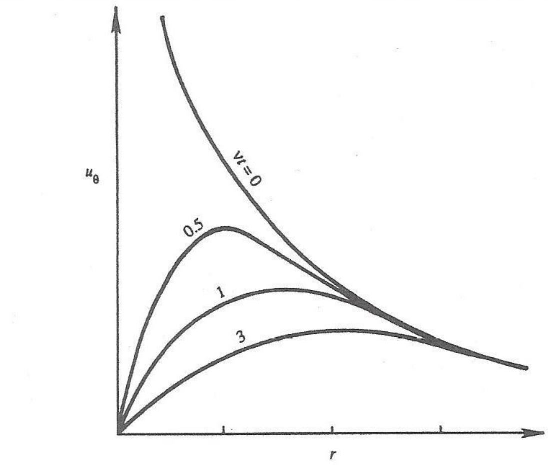


Figure 9.12 Viscous decay of a line vortex showing the tangential velocity at different times.

for $r << 2\sqrt{\nu t}$ rigid body rotation
$r >> 2\sqrt{\nu t}$ irrotational vortex

Alternatively, a corresponding solution can be obtained for a line vortex suddenly imposed on a fluid at rest. Impulsive start of infinitely small R fast or cylinder. In this case

$$u_\theta = \frac{\Gamma}{2\pi r} e^{-r^2/4\nu t}$$

For the Oseen vortex

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) = \frac{\Gamma}{4\pi\nu t} \exp\left(-\frac{\eta^2}{4}\right) \quad \eta = \frac{r}{\sqrt{\nu t}}$$

ie Gaussian bell curve profile at each instant.

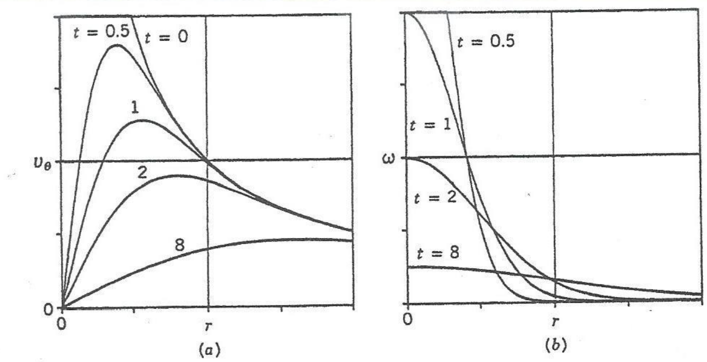


Figure 11.7 Viscous decay of an ideal vortex: (*a*) velocity profiles and (*b*) vorticity profiles at corresponding times. Scales are arbitrary.

Where $\omega_z \neq 0$ flow is viscous, whereas where $\omega_z = 0$ it remains the original potential flow vortex. Height falls off as t^{-1} & width increases by viscous diffusion $\sqrt{\nu t}$

Note: $\int_0^\infty \omega_z \, 2\pi r \, dr = \Gamma$

Solution useful estimate decay of wing/propeller tip vortices, especially if ν_t used

Squire (1965)

$\nu_t = f\left(Re_{vortex} = \Gamma/\nu = \frac{2\pi r v_\theta}{\nu}\right)$

decay time $t = \frac{z - z_0}{U}$

$\Delta z = z - z_0$ distance behind wing

U = aircraft speed

$u_{\theta max}$ decay $\propto z^{-1/2}$

core growth $\propto z^{1/2}$

z_0 = effective origin

Alternative derivation viscous decay line vortex

Similarity solutions:

$$\gamma = A t^{-n} F(\xi/\delta(t)) = A t^{-n} F(\eta)$$

or

$$\gamma = A \xi^{-n} F(\xi/\delta(t)) = A \xi^{-n} F(\eta)$$

γ = dependent field variable, eg, velocity component
A = constant $\gamma \times t^n$ or $\gamma \times \xi^n$
ξ = independent spatial variable
t = time
$\eta = \xi/\delta$ = Similarity variable
$\delta(t)$ = time dependent length scale
At^{-n} or $A\xi^{-n} \times F$ needed when solutions are infinite or zero at t or $\xi = 0$

Thin rapidly spinning cylinder $u_\theta = \Gamma/2\pi r$
ie ideal vortex strength Γ located $r=0$. At $t=0$ cylinder stops spinning.

$$u_\theta(r,t) = A r^{-n} F(r/\delta(t)) = A r^{-n} F(\eta)$$

$$u_\theta(r,0) = \Gamma/2\pi r = u_\theta(r\to\infty, t)$$
$$u_\theta(0,t) = 0 \quad t>0$$

ie $F(\eta\to\infty) = 1$ and $F(0)=0$ $\quad Ar^{-n} = \Gamma/2\pi r$

$\frac{\partial u_\theta}{\partial t} = \nu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right)$ $\quad u_\theta = \left(\frac{\Gamma}{2\pi r}\right) F(\eta)$ $\quad \eta = r/\delta(t)$

$\eta_t = -r/\delta^2\, \delta_t$

$\frac{\partial u_\theta}{\partial t} = \frac{\Gamma}{2\pi r} F'\left(-\frac{r}{\delta^2}\right)\delta_t = -\frac{\Gamma}{2\pi r}(\delta^{-1}\delta_t)\,\eta F'$ $\quad \frac{\partial u_\theta}{\partial r} = -\frac{\Gamma}{2\pi r^2}F$

$r u_\theta = \frac{\Gamma}{2\pi}F$ $\quad \frac{\partial}{\partial r}(r u_\theta) = u_\theta + \frac{\Gamma}{2\pi}(F'/\delta - F/r)$ $\quad + \frac{\Gamma}{2\pi r}F'\delta^{-1}$

$\frac{\partial}{\partial r}\left(u_\theta/r + \frac{\Gamma}{2\pi r}(F'/\delta - F/r)\right)$ $\quad = \frac{\Gamma}{2\pi r}(F'/\delta - F/r)$

$\frac{\partial}{\partial r}\left(\frac{\Gamma}{2\pi r^2}\cancel{F} + \frac{\Gamma}{2\pi r}(F'\delta^{-1} - \cancel{F/r})\right)$

$\frac{\Gamma}{2\pi}\frac{\partial}{\partial r}(F'/r\delta) = \frac{\Gamma}{2\pi}\left[\frac{1}{r\delta^2}F'' - F'/r^2\delta\right]$

$-\frac{\Gamma}{\cancel{2\pi} r}\left(\frac{1}{\delta}\delta_t\right)\eta F' = \frac{\nu\Gamma}{\cancel{2\pi}}\left[\frac{1}{r\delta^2}F'' - F'/r^2\delta\right]$

$-\boxed{\frac{r^2}{\nu\delta}\frac{d\delta}{dt}}\,\eta F' = \frac{r^2}{\delta^2}F'' - \frac{r}{\delta}F'$ $\quad$ For similarity $[\;] = f(\eta)$

$-\frac{\eta^3}{2}F' = \eta^2 F'' - \eta F'$ $\quad \neq f(r,t)$

$-\frac{\eta}{2}F' + \frac{1}{\eta}F' = F''$ $\quad$ ∴ assume $\delta = \sqrt{\nu t}$

$\left(\frac{1}{\eta} - \frac{\eta}{2}\right)F' = \frac{d}{d\eta}F'$ $\quad \delta_t = \frac{\nu^{1/2}}{2}t^{-1/2}$

$\left(\frac{1}{\eta} - \frac{\eta}{2}\right)d\eta = \frac{dF'}{F'}$ $\quad \frac{r^2}{\nu\sqrt{\nu}\,t^{1/2}}\frac{\nu^{1/2}}{2}t^{-1/2} = \frac{r^2}{2\nu t} = \frac{\eta^2}{2}$

$\ln\eta - \frac{\eta^2}{4} + c = \ln F'$

$\exp\left(\ln\eta - \frac{\eta^2}{4} + c\right) = F' = \eta e^{-\frac{\eta^2}{4}} c$

$c\int \eta \exp\left(-\frac{\eta^2}{4}\right)d\eta + D = F(\eta)$ $\quad x^2 = \frac{\eta^2}{4}$

$F(\infty) = 1$ $\quad -2c\, e^{-\eta^2/4} + D = F(\eta)$ $\quad 2x\,dx = \frac{\eta}{2}d\eta$

$F(0) = 0$ $\quad -2c + D = 0$ $\quad c = D/2$ $\quad 4x\,dx = \eta\,d\eta$

$D = 1$

$F = 1 - e^{-\eta^2/4}$ $\quad \int \eta \exp\left(-\frac{\eta^2}{4}\right)d\eta$

$u_\theta = \left(\frac{\Gamma}{2\pi r}\right)\left(1 - e^{-\eta^2/4}\right)$ = Gaussian Vortex $\quad 4\int x\exp(-x^2)\,dx$

$r \ll \delta$ rigid body rotation $\sigma^2 = 4\nu t$ $\quad 4\left(-\frac{1}{2}e^{-x^2}\right)$

$r \gg \delta$ ideal vortex $\quad -2e^{-\eta^2/4}$

Decay line vortex

Exercise 3.28. Starting from (3.29), show that the maximum u_θ in a Gaussian vortex occurs when $1+2(r^2/\sigma^2)=\exp(r^2/\sigma^2)$. Verify that this implies $r \approx 1.12091\sigma$.

Solution 3.28. Differentiate the u_θ equation from (3.29) with respect to r and set this derivative equal to zero.

$$\frac{d}{dr}(u_\theta(r)) = \frac{\Gamma}{2\pi}\frac{d}{dr}\left(\frac{1-\exp(-r^2/\sigma^2)}{r}\right) = \frac{\Gamma}{2\pi}\left(-\frac{1-\exp(-r^2/\sigma^2)}{r^2} - \frac{\exp(-r^2/\sigma^2)}{r}\left(-\frac{2r}{\sigma^2}\right)\right) = 0.$$

Eliminate common factors assuming $r \neq 0$.

$$0 = -1 + \exp(-r^2/\sigma^2) + (2r^2/\sigma^2)\exp(-r^2/\sigma^2) = -1 + (1 + 2r^2/\sigma^2)\exp(-r^2/\sigma^2).$$

This can be rearranged to:

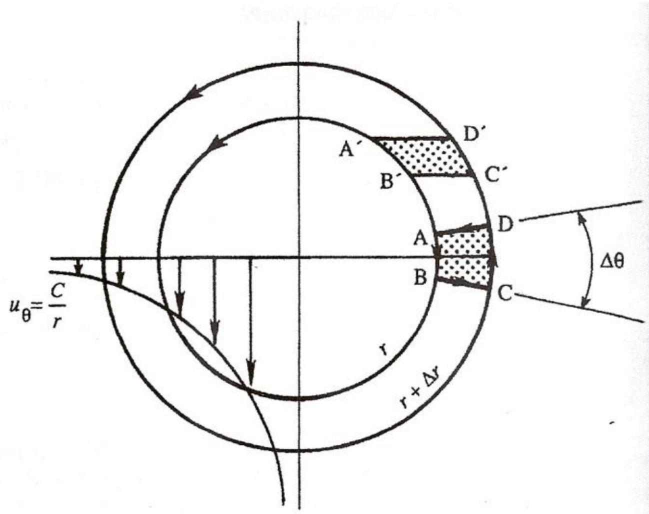
$$\exp(r^2/\sigma^2) = 1 + 2r^2/\sigma^2,$$

which is the desired result. When $r/\sigma \approx 1.12091$, then

$$\exp(r^2/\sigma^2) = 3.51289 \quad \text{and} \quad 1 + 2r^2/\sigma^2 = 3.51288,$$

which is suitable numerical agreement.

FIGURE 3.17 Irrotational vortex. The streamlines are circular, as for solid body rotation, but the fluid velocity varies with distance from the origin so that fluid elements only deform; they do not spin. The vorticity of fluid elements is zero everywhere, except at the origin where it is infinite.



Instead, real vortices combine elements of the ideal vortex flows described by (3.22) and (3.25). Near the center of rotation, a real vortex's core flow is nearly solid-body rotation, but far from this core, real-vortex-induced flow is nearly irrotational. Two common idealizations of the this behavior are the Rankine vortex defined by:

$$\omega_z(r) = \begin{Bmatrix} \Gamma/\pi\sigma^2 = \text{const. for } r \le \sigma \\ 0 \qquad\qquad \text{for } r > \sigma \end{Bmatrix} \quad \text{and} \quad u_\theta(r) = \begin{Bmatrix} (\Gamma/2\pi\sigma^2)r & \text{for} \quad r \le \sigma \\ \Gamma/2\pi r & \text{for} \quad r > \sigma \end{Bmatrix}, \qquad (3.28)$$

and the Gaussian vortex defined by:

$$\omega_z(r) = \frac{\Gamma}{\pi\sigma^2}\exp(-r^2/\sigma^2), \quad \text{and} \quad u_\theta(r) = \frac{\Gamma}{2\pi r}(1 - \exp(-r^2/\sigma^2)) \qquad (3.29)$$

In both cases, σ is a core-size parameter that determines the radial distance where real vortex behavior transitions from solid-body rotation to irrotational-vortex flow. For the Rankine vortex, this transition is abrupt and occurs at $r = \sigma$ where u_θ reaches its maximum. For the Gaussian vortex, this transition is gradual and the maximum value of u_θ is reached at $r/\sigma \approx 1.12091$ (see Exercise 3.28).

Line vortex suddenly introduced into fluid at rest

 = impulsive rotational start

infinitely thin and fast rotating cylinder at $r=0$

Exercise 9.34. Suppose a line vortex of circulation Γ is suddenly introduced into a fluid at rest at $t = 0$. Show that the solution is $u_\theta(r,t) = (\Gamma/2\pi r)\exp\{-r^2/4\nu t\}$. Sketch the velocity distribution at different times. Calculate and plot the vorticity, and observe how it diffuses outward.

Solution 9.34. The solution to this problem is very similar to the decay of a line vortex (see Example 9.8). In two-dimensional (r,θ)-polar coordinates, the governing equation is:

$$\frac{\partial u_\theta}{\partial t} = \nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right)\right].$$

The boundary conditions on the velocity $u_\theta(r, t)$ are

$$u_\theta(r,0^+) = 0,\ u_\theta(r,\infty) = \Gamma/2\pi r, \text{ and } u_\theta(\infty, t) = 0.$$

In this case the second boundary condition suggests a similarity solution of the form:

$$u_\theta = \frac{\Gamma}{2\pi r}f(\eta) = \frac{\Gamma}{2\pi r}f\left(\frac{r}{\sqrt{\nu t}}\right).$$

For this solution form the time and radial derivatives are:

$$\frac{\partial u_\theta}{\partial t} = \frac{\Gamma}{2\pi r}\frac{df}{d\eta}\frac{\partial\eta}{\partial t} = \frac{\Gamma}{2\pi r}\frac{df}{d\eta}\left(-\frac{\eta}{2t}\right) = -\frac{\Gamma}{2\pi r}\left(\frac{\eta}{2t}\right)\frac{df}{d\eta},\quad \frac{\partial(ru_\theta)}{\partial r} = \frac{\Gamma}{2\pi}\frac{df}{d\eta}\frac{\partial\eta}{\partial r} = \frac{\Gamma}{2\pi}\frac{df}{d\eta}\left(\frac{1}{\sqrt{\nu t}}\right), \text{ and}$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right) = \frac{\partial}{\partial r}\left(\frac{\Gamma}{2\pi r}\frac{1}{\sqrt{\nu t}}\frac{df}{d\eta}\right) = -\frac{\Gamma}{2\pi r^2}\frac{1}{\sqrt{\nu t}}\frac{df}{d\eta} + \frac{\Gamma}{2\pi r}\frac{1}{\nu t}\frac{d^2 f}{d\eta^2}.$$

Reassemble the governing equation and divide out the common factor of $\Gamma/2\pi r$:

$$-\left(\frac{\eta}{2t}\right)\frac{df}{d\eta} = -\frac{\nu}{r\sqrt{\nu t}}\frac{df}{d\eta} + \frac{\nu}{\nu t}\frac{d^2 f}{d\eta^2} = -\frac{1}{t}\left(\frac{1}{\eta}\frac{df}{d\eta} - \frac{d^2 f}{d\eta^2}\right).$$

Multiply by t and put the second derivative on the left: $\dfrac{d^2 f}{d\eta^2} = \left(-\dfrac{\eta}{2} + \dfrac{1}{\eta}\right)\dfrac{df}{d\eta}$.

Integrate to find: $\ln\dfrac{df}{d\eta} = -\dfrac{\eta^2}{4} + \ln\eta + const$. Exponentiate $\dfrac{df}{d\eta} = e^{const.}\eta\exp\{-\eta^2/4\}$

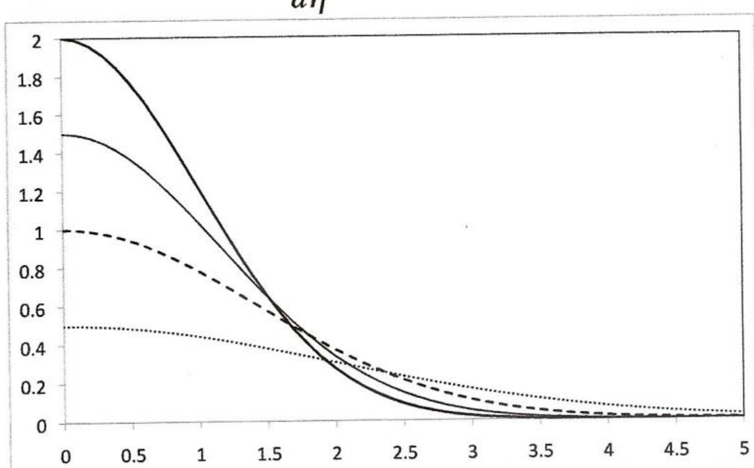
and integrate again:

$$f = A + B\exp\{-\eta^2/4\}.$$

The constants A and B can be determined from the boundary conditions: $f(0) = 1$, and $f(\infty) = 1$; $A = 0$, and $B = 1$. Thus, the velocity field is:

$$u_\theta(r,t) = \frac{\Gamma}{2\pi r}\exp\left\{-\frac{r^2}{4\nu t}\right\}.$$

In this flow the z-component of the vorticity is the only non-zero component.



$$-\omega_z(r,t) = \frac{1}{r}\frac{\partial}{\partial r}(ru_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial\theta} = \frac{\Gamma}{4\pi\nu t}\exp\left\{-\frac{r^2}{4\nu t}\right\} = \frac{\Gamma}{4\pi\nu t}\exp\left\{-\frac{\eta^2}{4}\right\}.$$

The plot above shows ω_z (vertical axis) vs. r (horizontal axis) at four different times. With increasing time, the vorticity ar $r = 0$ decreases but it spreads outward in the radial direction.

Stokes 1st: impulsive plate motion

$u = U(1 - \mathrm{erf}(\eta)) \qquad \eta = y/2\sqrt{\nu t} \qquad \eta_y = \frac{1}{2\sqrt{\nu t}}$

$\omega_z = -u_y = \frac{U}{\sqrt{\pi \nu t}} e^{-\eta^2} = \omega_z(y,t) \qquad \frac{d}{dy}(\mathrm{erf}\, y) = \frac{2}{\sqrt{\pi}} e^{-y^2}$

$\omega_z = f(y,t)$: $\omega_z(0,0) = \infty \quad \omega_z(y,0) = 0$

$u/U = .05 \Rightarrow \eta = 1.38 \qquad \omega_z(0, t>0) = U/\sqrt{\pi \nu t}$

$\delta = 2.76\sqrt{\nu t} \qquad \omega_z(y>0, t>0) = \frac{U}{\sqrt{\pi \nu t}} e^{-\eta^2}$

Diffusion Vortex Sheet

Similar Stokes 1st and laminar BL.

1. $u_s = U - U_0$
2. $y>0$ = temporally developing BL with irrotational uniform stream far from wall

$u = U\,\mathrm{erf}(\eta) \qquad \eta = y/2\sqrt{\nu t}$

Same conclusions ω as Stokes 1st: $\omega = -u_y = -\frac{U}{\sqrt{\pi \nu t}} e^{-\eta^2}$

$\tau_w = \mu u_y\big|_{y=0} = \frac{\mu U}{\sqrt{\pi \nu t}}$

$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{2}{\sqrt{\pi}}\sqrt{\frac{\nu}{U^2 t}}$

$Ut = x$ in spatially developing BL

Such that $\sqrt{\frac{\nu}{U^2 t}} = \left(\frac{\nu}{Ux}\right)^{1/2} = Re_x^{-1/2}$

$u = \pm .95U \quad \eta = \pm 1.38 \quad \delta = 5.52\sqrt{\nu t}$

$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{2}{\sqrt{\pi}}\sqrt{\frac{\nu}{U^2 t}} \qquad t = \frac{x}{U}$

$= \frac{2}{\sqrt{\pi}} Re_x^{-1/2}$

$= 1.13\, Re_x^{-1/2}$

Blasius: $C_f = .664\, Re_x^{-1/2}$

Decay ideal line vortex : Oseen vortex

$$u_\theta = \frac{\Gamma}{2\pi r}\left[1 - e^{-\frac{r^2}{4\nu t}}\right] = \frac{\Gamma}{2\pi\sqrt{\nu t}}\left(\frac{1}{\eta}\right)\left[1 - e^{-\frac{\eta^2}{4}}\right]$$

$$\eta = r/\sqrt{\nu t} \qquad \eta_r = 1/\sqrt{\nu t}$$

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) \qquad r u_\theta = \frac{\Gamma}{2\pi}\left(1 - e^{-\eta^2/4}\right)$$

$$= \frac{\Gamma}{4\pi\nu t} e^{-\eta^2/4} \qquad \frac{\partial}{\partial r}(r u_\theta) = \frac{-\Gamma}{2\pi}\left(e^{-\eta^2/4}\right)\left(-\frac{2\eta}{4} \times \frac{1}{\sqrt{\nu t}}\right)$$

$$= \frac{\Gamma}{2\pi}\frac{\eta}{2\sqrt{\nu t}} e^{-\eta^2/4}$$

$$\omega_z = f(r,t)$$

$$\omega_z(0,0) = \frac{1}{0} = \infty$$

Same behavior
Stokes 1st

$$\omega_z(r,0) = 0 \qquad \omega_z(0, t>0) = \frac{\Gamma}{4\pi\nu t}$$

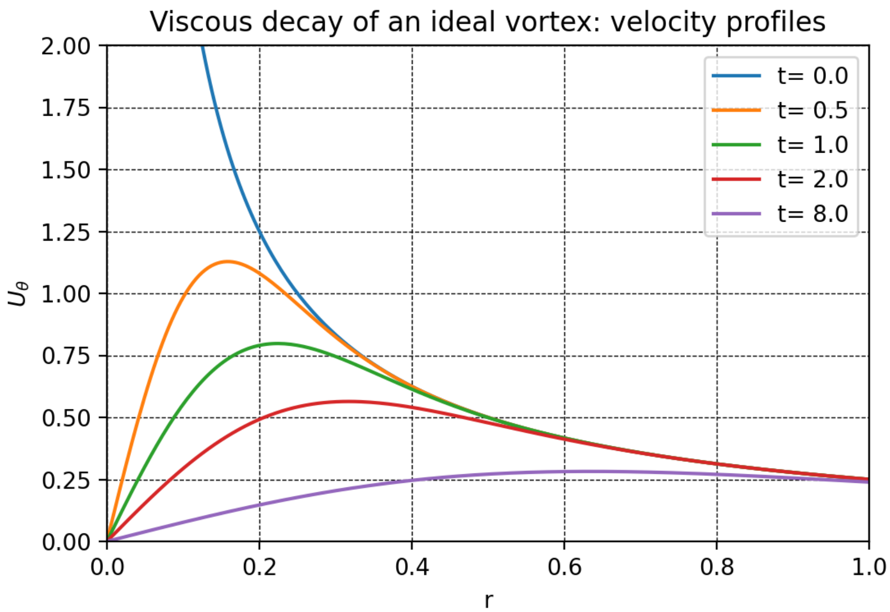
except $\propto t^{-1}$
vs $t^{-1/2}$

Impulsive line vortex

$$u_\theta = \frac{\Gamma}{2\pi r} e^{-\frac{r^2}{4\nu t}}$$

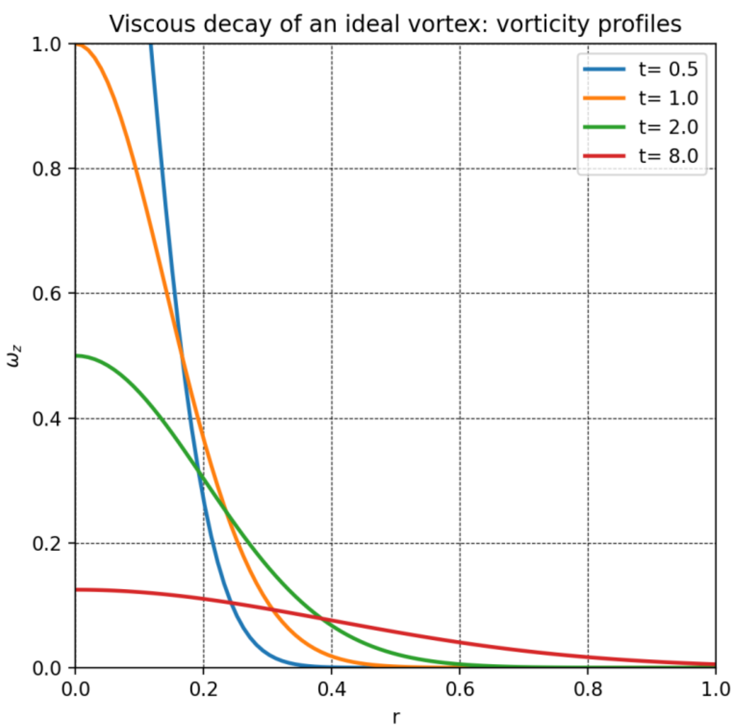
$$\omega_z = \frac{-\Gamma}{4\pi\nu t} e^{-\eta^2/4}$$

Same conclusion decay
vs impulsive or Stokes 1st
vs diffusion vortex sheet



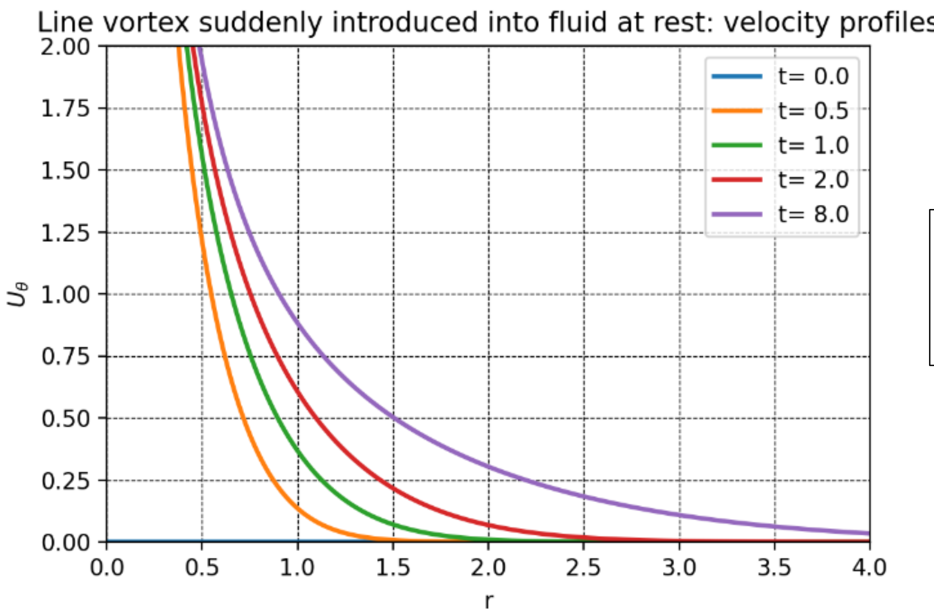
$$u_\theta(0,0) = \infty$$
$$u_\theta(0,t) = 0$$
$$u_\theta(\infty,t) = \frac{\Gamma}{2\pi r} \to 0$$

$$u_\theta = \frac{\Gamma}{2\pi r}\left(1 - e^{-\frac{r^2}{4\nu t}}\right)$$



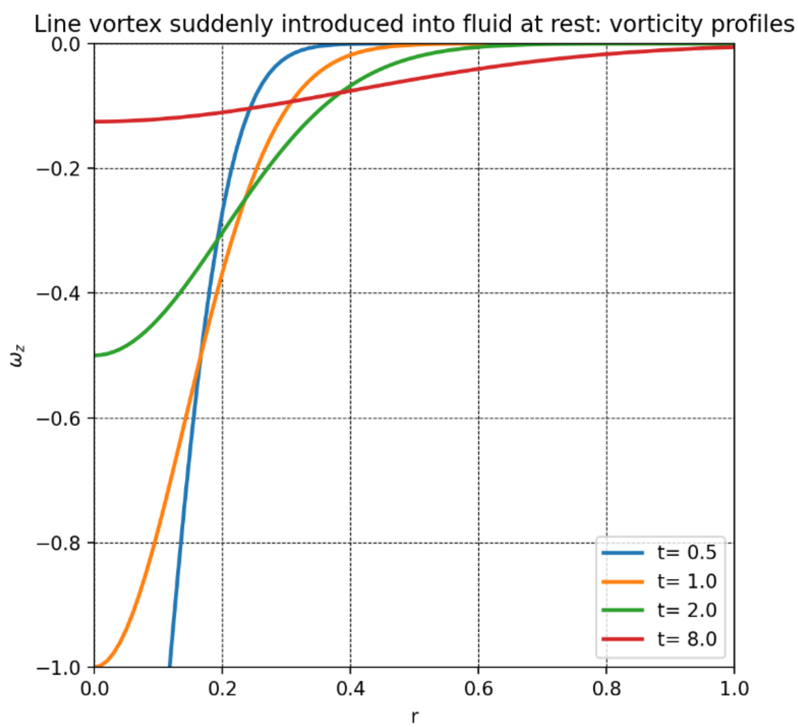
$$\omega_z(0,0) = \infty$$
$$\omega_z(0,t) = \frac{\Gamma}{4\pi\nu t}$$
$$\omega_z(\infty,t) = 0$$

$$\omega_z = \frac{\Gamma}{4\pi\nu t} e^{-\frac{r^2}{4\nu t}}$$



$$u_\theta(0,0) = \infty$$
$$u_\theta(0,t) = \infty$$
$$u_\theta(\infty,t) = 0$$

$$u_\theta = \frac{\Gamma}{2\pi r} e^{-\frac{r^2}{4\nu t}}$$



$$\omega_z(0,0) = -\infty$$
$$\omega_z(0,t) = -\frac{\Gamma}{4\pi\nu t}$$
$$\omega_z(\infty,t) = 0$$

$$\omega_z = -\frac{\Gamma}{4\pi\nu t} e^{-\frac{r^2}{4\nu t}}$$

Width of diffusion layer as a function of time: viscous decay of an ideal vortex

$$u_\theta = \frac{\Gamma}{2\pi r}\left(1 - e^{-\frac{r^2}{4\nu t}}\right)$$

$$\frac{u_\theta}{\frac{\Gamma}{2\pi r}} = 0.95 = 1 - e^{-\frac{r^2}{4\nu t}}$$

$$0.05 = e^{-\frac{r^2}{4\nu t}}$$

$$\log(0.05) \sim -3.0 = -\frac{r^2}{4\nu t}$$

$$r \sim \sqrt{12\nu t} = 3.46\sqrt{\nu t}$$

Width of diffusion layer as a function of time: Line vortex suddenly introduced into fluid at rest

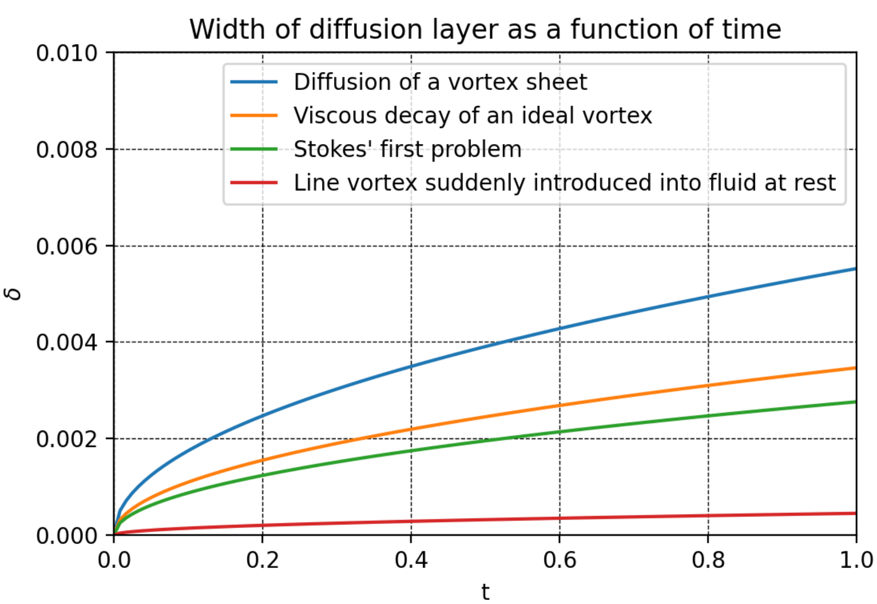
$$u_\theta = \frac{\Gamma}{2\pi r} e^{-\frac{r^2}{4\nu t}}$$

$$\frac{u_\theta}{\frac{\Gamma}{2\pi r}} = 0.95 = e^{-\frac{r^2}{4\nu t}}$$

$$0.95 = e^{-\frac{r^2}{4\nu t}}$$

$$\log(0.95) \sim -0.05 = -\frac{r^2}{4\nu t}$$

$$r \sim \sqrt{0.2\nu t} = 0.45\sqrt{\nu t}$$



	δ	$\dot{\delta}$
Diffusion of a vortex sheet	$5.52\sqrt{\nu t}$	$2.76\sqrt{\nu/t}$
Viscous decay of an ideal vortex	$3.46\sqrt{\nu t}$	$1.73\sqrt{\nu/t}$
Stokes' first problem	$2.76\sqrt{\nu t}$	$1.38\sqrt{\nu/t}$
Sudden line vortex	$0.45\sqrt{\nu t}$	$0.225\sqrt{\nu/t}$

$\dot{\delta} \to 0$ as $t \to \infty$, i.e., rate of diffusion decreases over time

Burgers Vortex

The line vortex viscous spreading can be canceled by superposing a radial inflow towards the core and a steady flow obtained

Consider a $u_\theta(r)$ with axis along z axis and to this flow a symmetric radial inflow is added

$$u_r = -ar \qquad a = \text{strength of radial flow}$$

u_r is unbounded at ∞; thus, current solution is local flow near small r

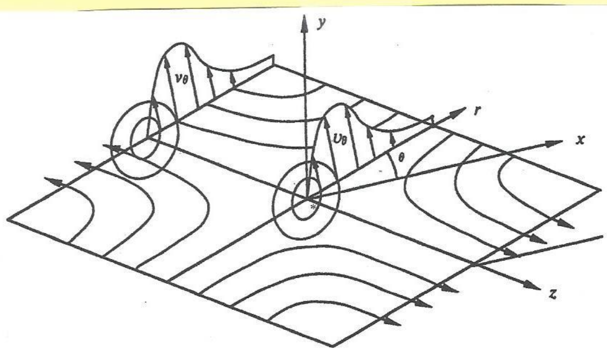


Figure 11.11 Burgers vortex.

Continuity $\quad \frac{\partial u_z}{\partial z} = -\frac{1}{r}\frac{\partial}{\partial r}(r u_r) = 2a$

$$u_z = 2az$$

(u_r, u_z) = axisymmetric inviscid flow toward stagnation point at origin

strain rates $\varepsilon_{rr} = \varepsilon_{\theta\theta} = -a$ & $\varepsilon_{zz} = 2a$
vortex is stretched along its axis at rate a

Assume $u_\theta = u_\theta(r)$ only

$$\nu \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r u_\theta)\right] = -ar\frac{du_\theta}{dr} \qquad (1)$$

$$u_\theta(0) = 0 \qquad u_\theta(\infty) = \frac{\Gamma}{2\pi r}$$

change variables using reduced circulation

$$\gamma = f = \frac{2\pi r u_\theta}{\Gamma}$$

(1) becomes $\quad -a\frac{df}{dr} = \nu\frac{d}{dr}\left(\frac{1}{r}\frac{df}{dr}\right)$

next, transform using similarity variable

$$\eta = \frac{r}{\sqrt{\nu/2a}}$$

$$f'' + \left(\frac{1}{2}\eta - \frac{1}{\eta}\right) f' = 0$$

$$f = 1 - \exp\left(-\frac{\eta^2}{4}\right)$$

$$u_\theta = \frac{\Gamma}{2\pi r}\left[1 - \exp\left(-\frac{r^2}{2\nu/a}\right)\right]$$

vortex core diffusion is cancelled by radial inflow such that flow is steady state

Stokes 2nd Problem: Oscillating Plate

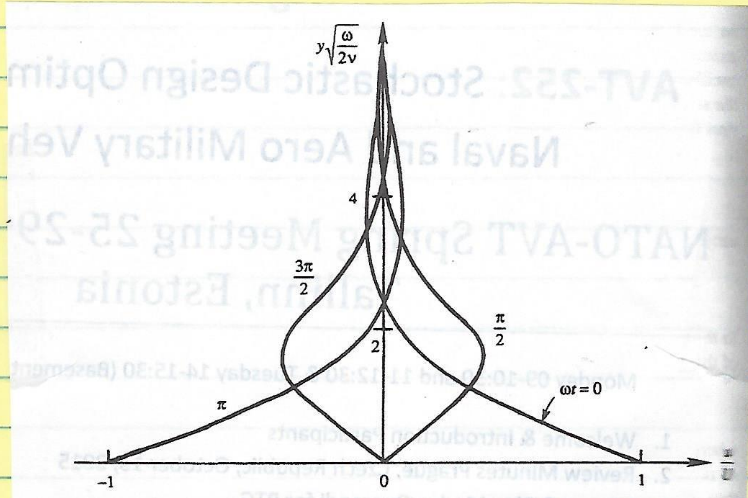


Figure 9.13 Velocity distribution in laminar flow near an oscillating plate. The distributions at $\omega t = 0$, $\pi/2$, π, and $3\pi/2$ are shown. The diffusive distance is of order $\delta = 4\sqrt{\nu/\omega}$.

$$u_t = \nu u_{yy}$$
$$u(0,t) = U \cos \omega t$$
$$u(\infty,t) = 0$$

steady state periodic solution

$$u = e^{i\omega t} f(y)$$

f(y) complex
u = real RHS
which has phase
difference u(0,t)

equation with
constant coefficients

$$i\omega f = \nu f_{yy}$$

has exponential solution of form $f = e^{\lambda y}$ $\quad \lambda = \sqrt{i\omega/\nu} = \pm(i+1)\sqrt{\omega/2\nu}$ $\quad i = \sqrt{-1}$

so $\quad f = A e^{-(1+i)y\sqrt{\omega/2\nu}} + B e^{(1+i)y\sqrt{\omega/2\nu}}$

$$u = A e^{i\omega t} e^{-(1+i)y\sqrt{\omega/2\nu}}$$

$u(\infty,t) = 0$
$\Rightarrow B = 0$

$u(0,t) = U \cos \omega t$

$\Rightarrow A = U$

real part: $u = U e^{-y\sqrt{\omega/2\nu}} \cos\left(\omega t - \underbrace{y\sqrt{\frac{\omega}{2\nu}}}_{\text{phase lag } u \text{ vs. plate}}\right)$

Damped oscillatory motion y direction

at $y = 4\sqrt{\nu/\omega}$ $u = U \exp(-4/\sqrt{2}) = .06\,U$ amplitude

ie wall influence confined $\underbrace{\delta = 4\sqrt{\nu/\omega}}_{\text{diffusion distance}} \neq f(t)$
which decreases with ω

note: $u/U = f(y, t, \omega, \nu)$

Dimensional analysis $u/U = F(\omega t, y\sqrt{\omega/\nu})$
3 Π's and similarity not possible it can not be
y separated t, whereas similarity y and t must represented
combined along with single curve in terms of nondimensional variables

In this case diffusion $\delta \neq f(t)$
vs Stokes 1st δ increases $\sqrt{t}$

Previous solution similarity possible since no imposed/special length or time scales. Stokes 2nd and unsteady fully developed pipe flow have imposed time scale $1/\omega$.

Stokes 1st $u_{yy} > 0$ all y $\Rightarrow u_t > 0$
and momentum constantly diffused outward
and δ increases $\sqrt{t}$

Stokes 2nd u_{yy} and u_t oscillating sign
and momentum confined constant δ

phase lag relative y = 0

$\cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}\, y\right)$ wave

$\sqrt{\frac{\omega}{2\nu}}\left(\sqrt{2\nu\omega}\, t - y\right) = k(ct - y)$

$k = \frac{2\pi}{\lambda}$ = wave number $\lambda = \frac{2\pi}{k}$ = wave length

$\nu = m^2/s$ $\nu/\omega = m^2$ $\lambda = 2\pi\sqrt{\frac{2\nu}{\omega}}$

$\omega = 1/s$

Damped means wave travels away from the wall. Effect wall motion delayed as per phase difference τ_w and u_{max}

$c = \sqrt{2\nu\omega}$ = wave speed

$\exp\left(-\sqrt{\frac{\omega}{2\nu}}\, y\right)$ exponentially damped

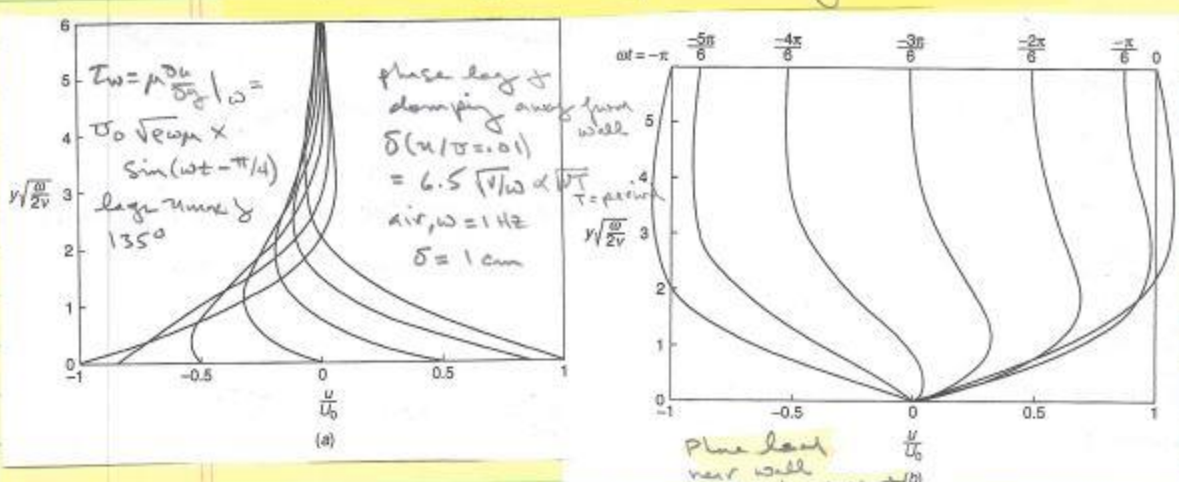


FIGURE 3-19
Stokes' second problem: (*a*) flow above an oscillating infinite plate, Eq. (3-111); (*b*) an oscillating stream above a fixed plate, Eq. (3-112). Velocity profiles shown for 30° increments over a half period.

Stokes 2nd problem can alternately be transformed as flow over fixed wall due to oscillating outer flow caused by constant pressure gradient

$$p_x = \rho U_0 \omega \sin \omega t$$

$\rho u_t = -p_x$ Euler equation

$$u_t = -p_x/\rho = -U_0 \omega \sin \omega t$$

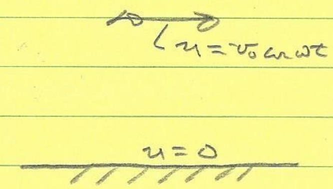
$$u = U_0 \cos \omega t$$

$u = U_0 \sin(\omega t + \pi/2)$ leads p_x by 90°

$$\rho u_t = -p_x + \mu u_{yy}$$

$$u(0,t) = 0$$

$$u(\infty,t) = U_0 \cos \omega t$$



Solution same as negative previous solution plus outer flow

$a \sin(bx + c) = p \sin bx + q \cos bx$

$a = (p^2+q^2)^{1/2}$

$c = \tan^{-1} q/p$

$$u = \underbrace{U_0 \cos \omega t}_{\text{inviscid part}} - \underbrace{U_0 \exp\left(-\sqrt{\tfrac{\omega}{2\nu}}\, y\right) \cos\left(\omega t - \sqrt{\tfrac{\omega}{2\nu}}\, y\right)}_{\text{viscous part}}$$

Can be written as single wave with different amplitude and phase

relative $u = U_0 \cos \omega t$

near $\sqrt{\frac{\omega}{2\nu}}\, y$ = 1.57

near $\frac{\lambda}{4}$

Solution shows $\underbrace{\text{phase leads}}_{\text{low momentum fluid near wall responds 1st to } p_x}$ and overshoot at $0, \pi$

At $y=0$ μu_{yy} counteracts p_x
so that $u=0$ satisfying no slip condition

Away from wall μu_{yy} decays
exponentially.

Intermediate region p_x & μu_{yy} in
phase such that overshoot occurs

p_x transmitted instantaneously vs.
μu_{yy} transmitted by viscous diffusion

called Richardson overshoot as per pipe
flow with oscillating p_x

Stern F., Hwang W. S., Jaw S. Y., "Effects of Waves on the Boundary Layer of a Surface-Piercing Flat Plate: Experiment and Theory", Journal of Ship Research, Vol. 33, No. 1, March 1989, pp. 63-80.

Choi, J.-E., Sreedhar, M., and Stern, F., "Stokes Layers in Horizontal-Wave Outer Flows," ASME J. Fluids Eng., Vol. 118, September 1996, pp. 537 – 545.

Paterson, E.G. and Stern, F., "Computation of Unsteady Viscous Marine-Propulsor Blade Flows - Part 1: Validation and Analysis," ASME J. Fluids Eng., Vol. 119, March 1997, pp. 145 – 154.

Paterson, E.G. and Stern, F., "Computation of Unsteady Viscous Marine-Propulsor Blade Flows - Part 2: Parametric Study," ASME J. Fluids Eng, Vol. 121, March 1999, pp. 139 – 147.

Additional discussion Stokes 2nd Problem

(1) moving plate

$$u = U e^{-y\sqrt{\omega/2\nu}} \underbrace{\cos\left(\omega t - y\sqrt{\tfrac{\omega}{2\nu}}\right)}$$

$$\cos\left[k(ct - y)\right]$$

$k = 2\pi/\lambda$

$\lambda = 2\pi\sqrt{\frac{2\nu}{\omega}}$

$c = \sqrt{2\nu\omega}$

vs. Stokes 1st problem where viscous diffusion penetrates into flow $\delta = 2.76\sqrt{\nu t}$

ie damped viscous wave traveling away from the wall. Note how effect of wall motion is delayed. When wall motion changes sign (τ_w changes sign) net acceleration in flow still in opposite direction and only after time delay flow is decelerated and reverses direction

(2) oscillating outer flow

$$u = U_0 \cos\omega t - U_0 e^{-y\sqrt{\omega/2\nu}} \cos\left(\omega t - y\sqrt{\omega/2\nu}\right)$$

note over shoot at $y\sqrt{\frac{\omega}{2\nu}} = 3.2$ ie near $\lambda/2$

$$\frac{\partial u}{\partial t} = -\omega U_0 \sin\omega t + \omega U_0 e^{-y\sqrt{\omega/2\nu}} \sin\left(\omega t - y\sqrt{\tfrac{\omega}{2\nu}}\right)$$

$$= -p_x/\rho + \nu u_{yy}$$

$$-p_x/\rho = -U_0 \omega \sin\omega t$$

$$u_y = -U\left(-\sqrt{\omega/2\nu}\right) e^{-y\sqrt{\omega/2\nu}} \cos\left(\omega t - y\sqrt{\omega/2\nu}\right)$$

$$+ U e^{-y\sqrt{\omega/2\nu}} \left(-\sqrt{\omega/2\nu}\right) \sin\left(\omega t - y\sqrt{\omega/2\nu}\right)$$

$$u_{yy} = -U\left(\frac{\omega}{2\nu}\right) e^{-y\sqrt{\omega/2\nu}} \cos\left(\omega t - y\sqrt{\omega/2\nu}\right) + U e^{-y\sqrt{\frac{\omega}{2\nu}}} \left(\omega/2\nu\right) \sin\left(\omega t - y\sqrt{\omega/2\nu}\right)$$
$$+ U\left(\frac{\omega}{2\nu}\right) e^{-y\sqrt{\omega/2\nu}} \sin\left(\omega t - y\sqrt{\omega/2\nu}\right) + U e^{-y\sqrt{\frac{\omega}{2\nu}}} \left(\frac{\omega}{2\nu}\right) \cos\left(\omega t - y\sqrt{\omega/2\nu}\right)$$

$u_{yy} = +2 U_0 \left(\frac{\omega}{2\nu}\right) e^{-y\sqrt{\omega/2\nu}} \sin\left(\omega t - y\sqrt{\omega/2\nu}\right)$

$\nu u_{yy} = +\omega U_0 e^{-y\sqrt{\omega/2\nu}} \sin\left(\omega t - y\sqrt{\omega/2\nu}\right)$

large y recover Euler equation

at $y=0$

$$-\omega U_0 \sin \omega t + \omega U_0 \sin \omega t = -\omega U_0 \sin \omega t + \omega U_0 \sin \omega t$$

$$0 = 0$$

at the wall $u_t = 0$ & pressure gradient counter balanced by viscous force

Intermediate region pressure & viscous combine and accelerate fluid to higher velocities than that due to pressure forces alone. The net viscous stress created at the wall diffuses & attenuates into the flow and about half cycle later combines with pressure gradient to create overshoot.

Results highlight: p_x transmitted instantaneously, whereas viscous forces are transmitted by viscous diffusion ie have different time scales. Viscous forces are not always damping effect:

(1) μ leads to instability flat plate BL

(2) Many 3D skewed BL have u_{max} within BL.

Stokes' second problem: analysis of the extrema of the velocity field

Velocity field:

$$U = U_0 \cos(\omega t) - U_0 e^{-\sqrt{\frac{\omega}{2\nu}}y} \cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}y\right)$$

Assume $-\sqrt{\frac{\omega}{2\nu}}y = x$

$$U = U_0 \cos(\omega t) - U_0 e^{x} \cos(\omega t + x)$$

$$\frac{dU}{dx} = -U_0 e^{x} \cos(\omega t + x) + U_0 e^{x} \sin(\omega t + x) = 0$$

Divide by $U_0 e^{x}$:

$$-\cos(\omega t + x) + \sin(\omega t + x) = 0$$

$$\tan(\omega t + x) = 1$$

$$\omega t + x = \frac{\pi}{4} + k\pi \rightarrow x = \frac{\pi}{4} + k\pi - \omega t$$

Where $k = 0, \pm 1, \pm 2 \ldots \pm \infty$. Substitute back for $\sqrt{\frac{\omega}{2\nu}}y$:

$$-\sqrt{\frac{\omega}{2\nu}}y = \frac{\pi}{4} + k\pi - \omega t$$

The condition for the location of the extrema is:

$$\sqrt{\frac{\omega}{2\nu}}y = \omega t - \frac{\pi}{4} - k\pi$$

Therefore, the velocity field has multiple local maxima/minima. For example, when $\omega t = 0$, the locations of the local extrema are:

$$\sqrt{\frac{\omega}{2\nu}}y = -\frac{\pi}{4} - k\pi$$

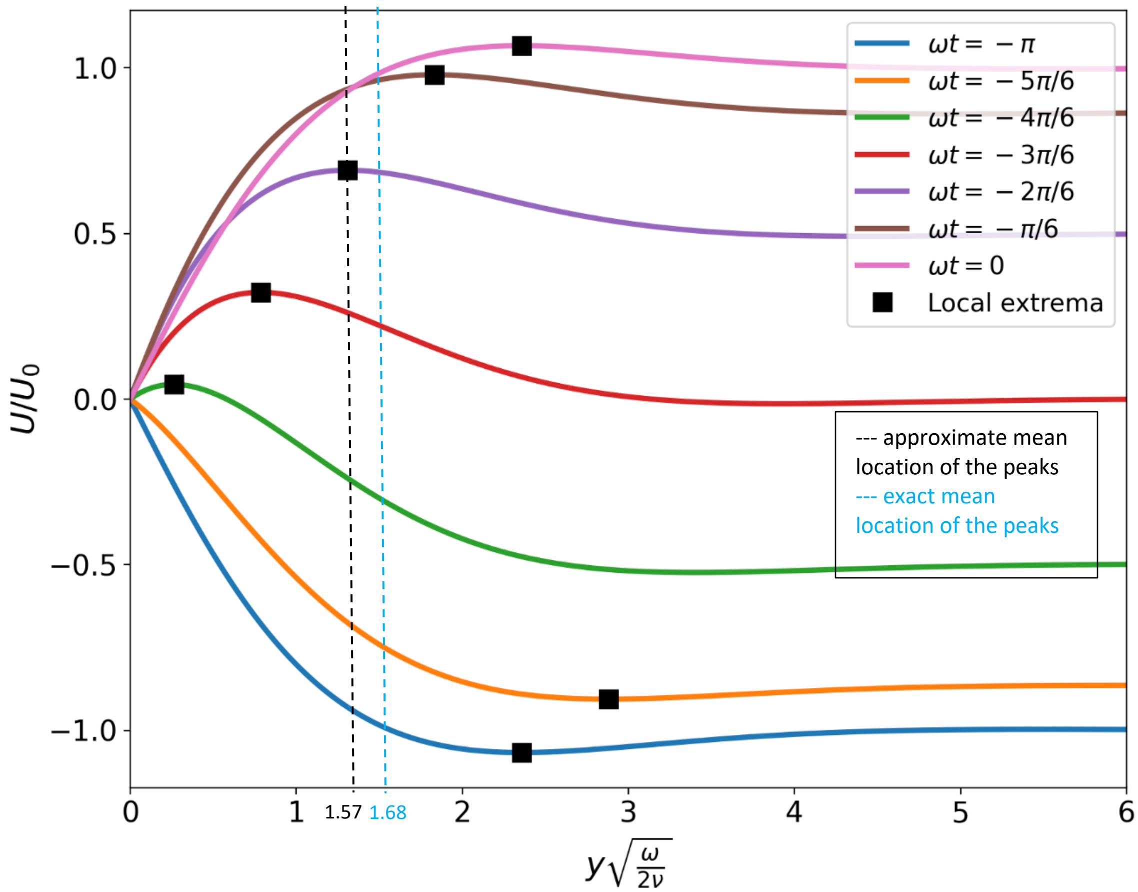
i.e.,

$$\sqrt{\frac{\omega}{2\nu}}y = +\infty, \ldots, \frac{7}{4}\pi, \frac{3}{4}\pi, -\frac{\pi}{4}, -\frac{5}{4}\pi, -\frac{9}{4}\pi, \ldots, -\infty$$

For $k = -\infty, \ldots, -2, -1, 0, 1, 2, \ldots, +\infty$.

The extrema for $\sqrt{\frac{\omega}{2\nu}}y > 2\pi$ are difficult to see due to the damping effect of the exponential function.

If we only consider $0 < \sqrt{\frac{\omega}{2\nu}}y < 6$, the local extrema are shown in the figure below.



For $-\frac{4}{6}\pi < \omega t < 0$, the velocity field shows a local maximum, which moves towards smaller y when ωt increases its absolute value. For $-\pi < \omega t < \frac{4}{6}\pi$, the local maximum moves to negative values of y, i.e., a region which is not physically interesting. Therefore, the next extremum is a minimum, as shown in the figure.

1) The overshoot is located where the pressure gradient and viscous term have the same sign.
2) The y −location and amount of the overshoot depends on the value of ωt.
3) The y-location should depend on the travelling wave concept.
4) Explain the physics of the y-location and the amount of the maximum.
5) All of the above need to be compared with Panton's discussion.

Explanation Richardson's annular effect, i.e., velocity overshoot

At $y = 0$, $u_t = 0$ and $-\frac{p_x}{\rho}$ balances νu_{yy}

Away from wall oscillating νu_{yy} decays $e^{-y\sqrt{\omega/2\nu}}$

Intermediate region complex, since p_x instantaneous and νu_{yy} has phase lag $-y\sqrt{\omega/2\nu}$ (at wall phase lag $= 0$ and terms cancel). At small y and later t νu_{yy} peaks and mismatch induces acceleration u_t. At some distance, the lag large enough that $-p_x/\rho$ and νu_{yy} add such that combined effect causes $u_t > -p_x/\rho$ alone. About half cycle after νu_{yy} generated, viscous diffusion carried away and has decayed but still strong enough to aid pressure which has changed direction: combination causes overshoot!

Not related annular.

Acoustic air waves near wall incompressible with Stokes layer: 200 Hz, $\nu = .15\ cm^2/s$, $\delta = 4.5\,(2\nu/\omega)^{1/2} = 1.5$ mm.

EXAMPLE 9.10

Show that $\dot{w}$ = the rate of work (per unit area) done on the fluid by the oscillating plate is balanced by $\dot{e}$ = the viscous dissipation of energy (per unit area) in the fluid above the plate.

Solution

The rate of work (per unit area) done on the fluid by the moving plate is the product of the shear stress on the fluid, τ_w, and the plate velocity, $U\cos(\omega t)$:

$$\dot{w} = \tau_w U\cos(\omega t) = -\mu(\partial u/\partial y)_{y=0}\cdot U\cos(\omega t).$$

The negative sign appears because the outward normal from the fluid points downward on the surface of the plate. Differentiating (9.38) with respect to y, leads to:

$$\frac{\partial u}{\partial y} = U\sqrt{\frac{\omega}{2\nu}}\exp\left\{-y\sqrt{\frac{\omega}{2\nu}}\right\}\left[-\cos\left(\omega t - y\sqrt{\frac{\omega}{2\nu}}\right) + \sin\left(\omega t - y\sqrt{\frac{\omega}{2\nu}}\right)\right],$$

and evaluating the result at $y = 0$ produces:

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = U\sqrt{\frac{\omega}{2\nu}}[-\cos(\omega t) + \sin(\omega t)].$$

Thus, the time-average rate of work (per unit area) done by the plate on the fluid is:

$$\overline{\dot{w}} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\tau_w U\cos(\omega t)dt = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} -\mu U\sqrt{\frac{\omega}{2\nu}}[-\cos(\omega t) + \sin(\omega t)]U\cos(\omega t)dt = \mu\frac{U^2}{2}\sqrt{\frac{\omega}{2\nu}}$$

where $2\pi/\omega$ is the period of the plate's oscillations.

From (4.58), the rate of dissipation of fluid kinetic energy per unit volume is $\tau_{ij}S_{ij}$, which reduces to $2\mu S_{ij}S_{ij}$ for an incompressible viscous fluid. Thus, the time-average energy dissipation rate (per unit area) above the plate will be:

$$\overline{\dot{e}} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\int_0^{\infty} 2\mu S_{ij}S_{ij}dydt = \frac{\omega}{\pi}\mu\int_0^{2\pi/\omega}\int_0^{\infty}\left(S_{xy}^2 + S_{yx}^2\right)dydt = \frac{\omega}{2\pi}\mu\int_0^{2\pi/\omega}\int_0^{\infty}\left(\frac{\partial u}{\partial y}\right)^2 dydt,$$

since the only strain-rate component in this flow is $S_{xy} = S_{xy} = (1/2)(\partial u/\partial y)$. The final result is easiest to obtain by performing the time average first:

$$\frac{\omega}{2\pi}\mu\int_0^{2\pi/\omega}\left(\frac{\partial u}{\partial y}\right)^2 dt = \mu U^2\frac{\omega}{2\nu}\exp\left\{-2y\sqrt{\frac{\omega}{2\nu}}\right\}.$$

This leaves the vertical integral:

$$\overline{\dot{e}} = \int_0^{\infty}\mu U^2\frac{\omega}{2\nu}\exp\left\{-y\sqrt{\frac{2\omega}{\nu}}\right\}dy = \mu U^2\frac{\omega}{2\nu}\sqrt{\frac{\nu}{2\omega}} = \mu\frac{U^2}{2}\sqrt{\frac{\omega}{2\nu}},$$

and this matches the time-averaged result for $\dot{w}$. Thus, the average rates of work input and energy dissipation are equal. They are not instantaneously equal, so the fluid's kinetic energy (per unit area) fluctuates, but it does not grow without bound.

Unsteady Fully Developed Pipe Flow

$$u = u(r,t) \qquad v = w = 0$$

$$\rho u_t = -\hat{p}_x + \mu\left(u_{rr} + \frac{1}{r}u_r\right)$$

f(t) only

1. Starting flow (fluid accelerates from rest under action of constant $\hat{p}_x$)

no flow

initial cond. $\quad u(r,0) = 0$

large t fully developed flow $\quad u(r,\infty) = u_{max}(1-r^2)\quad$ ie parabolic profile

no-slip cond. $\quad u(r_0,t) = 0$

Solution for $u = u - u(r,\infty)$ can be obtained by separation of variables.

$$\frac{u}{u_{max}} = (1-r^2) - \sum_{n=1}^{\infty} \frac{8 J_0(\lambda_n r)}{\lambda_n^3 J_1(\lambda_n)} \exp\left(-\lambda_n^2 \frac{\nu t}{r_0^2}\right)$$

$$= \frac{r_0^2}{4\mu}\left(-\hat{p}_x\right)$$

roots of Bessel function

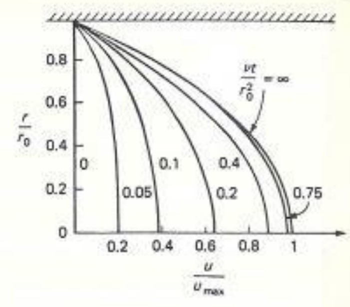


FIGURE 3-14
Instantaneous velocity profiles for starting flow in a pipe, Eq. (3-102). [*After Szymanski (1932).*]

(1) initially BL effects occurs near wall and central core accelerates uniformly
potential flow

(2) $\underbrace{\text{for } t^* = .75}$ flow is essentially parabolic $\left(t^* = \frac{\nu t}{r_0^2}\right)$

can be used to estimate time for flow
to respond to sudden change as $f(\nu, r_0^2)$

small r_0 large
ν develop
rapidly

$\therefore$ for laminar small r_0 pipe
flow with $\hat{p}_x = f(t)$ may use
quasisteady assumption.

$D = 1 \text{cm}$
$\nu = 1.5E\text{-}5 \frac{m^2}{s}$
air
$t^* = .75$
$t = 1.25 s$
~ SAE oil
$t = .06 s$

2. Oscillatory pipe flow

$-\frac{1}{\rho}\hat{p}_x = Ke^{i\omega t}$ $\quad e^{i\omega t} = \cos\omega t + i\sin\omega t \quad i = \sqrt{-1}$

no slip at steady oscillation condition, ie, neglect transient / start up

$i\, i^{1/2} = i^{3/2}$

$$u(r,t) = \frac{K}{i\omega}\left[1 - \frac{J_0(ir\sqrt{i\omega/\nu})}{J_0(ir_0\sqrt{i\omega/\nu})}\right]e^{i\omega t}$$

$\underbrace{\frac{K}{i\omega} = \frac{-iK}{\omega}}_{\text{as per Kundu}}$

$$J_0(z) \approx 1 - \frac{z^2}{4} + \frac{z^4}{64} + \cdots \qquad z \ll 1$$

$$\sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\pi}{4}\right) \qquad z \gg 1$$

$r' = r/r_0 \quad \underbrace{\omega' = \frac{\omega r_0^2}{\nu}} \quad u' = u/u_{max} \quad u_{max} = \frac{Kr_0^2}{4\nu}$ & u for steady Poiseuille flow $\hat{p}_x = -\rho K$

$\omega' > 2000$ transition turbulence

Re_k = kinetic Re = measure viscous effects in oscillatory flow

related $\lambda_s = \frac{1}{2}\sqrt{\omega'}$ Stokes #

using $J_0(z)$ expansions: $\quad \alpha = \sqrt{\omega'}$ Womersley #

$\omega' \ll 1 \quad u(r',t)/u_{max} = (1-r'^2)\cos\omega t + \frac{\omega'}{16}(r'^4 + 4r'^2 - 5)\sin\omega t + O(\omega'^2)$

$\omega' \gg 1 \quad = \frac{4}{\omega'}\left[\sin\omega t - \frac{e^{-B}}{\sqrt{r'}}\sin(\omega t - B)\right] + O(\omega'^2)$

$$B = (1-r')\sqrt{\omega'/2}$$

$\omega' \ll 1$ since $\hat{p}_x \propto \cos \omega t$, $u \sim$ quasi static
i.e. Poiseuille flow

1st term in phase slowly varying $\hat{p}_x$

2nd term lag which reduces the velocity : $r'^4 + 4r'^2 < -5$

$\omega' \gg 1$ u lags $\hat{p}_x$ by $\pi/2$ and $\underbrace{u_{\mathbb{C}} < u_{max}}_{\text{plot profile}}$

mean square $\overline{u^2} / K^2/(2\omega^2) = 1 - \frac{2}{\sqrt{r'}} e^{-B} + \frac{e^{-2B}}{r'}$

$\omega' \gg 1$ equation average over one cycle

Overshoot occurs when

$\cos B + \sin B = e^{-B}$

ie $B = 2.2841$

$r' = 1 - 3.23021/\sqrt{\omega'}$

Speed overshoot ratio:

$\overline{u^2} / (8 u_{max}^2 / \omega'^2) = \frac{\overline{u^2}}{K^2/(2\omega^2)} =$

$1.143685 + \frac{.248826}{\sqrt{\omega'}} + O(\omega'^{-1})$

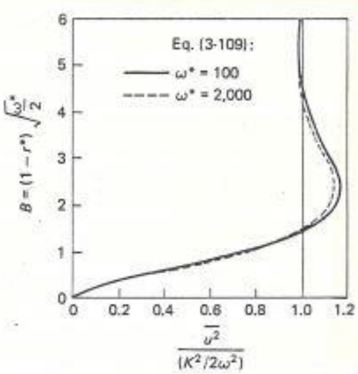


FIGURE 3-15
The near-wall velocity overshoot (Richardson's annular effect) due to oscillating pressure gradient.

overshoot reduces slightly as ω' increases from 100 to 2000

overshoot = Richardson annular effect, Verified via large ω' $u(r', t)/u_{max}$ equation

The **Stokes number** (Stk), named after George Gabriel Stokes, is a dimensionless number characterising the behavior of particles suspended in a fluid flow. The Stokes number is defined as the ratio of the characteristic time of a particle (or droplet) to a characteristic time of the flow or of an obstacle, or

$$\mathrm{Stk} = \frac{t_0\, u_0}{l_0}$$

where t_0 is the relaxation time of the particle (the time constant in the exponential decay of the particle velocity due to drag), u_0 is the fluid velocity of the flow well away from the obstacle, and l_0 is the characteristic dimension of the obstacle (typically its diameter) or a characteristic length scale in the flow (like boundary layer thickness).[1] A particle with a low Stokes number follows fluid streamlines (perfect advection), while a particle with a large Stokes number is dominated by its inertia and continues along its initial trajectory.

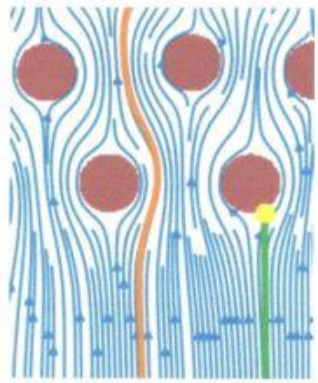


Illustration of the effect of varying the Stokes number. Orange and green trajectories are for small and large Stokes numbers, respectively. Orange curve is trajectory of particle with Stokes number less than one that follows the streamlines (blue), while green curve is for a Stokes number greater than one, and so the particle does not follow the streamlines. That particle collides with one of the obstacles (brown circles) at point shown in yellow.

In the case of Stokes flow, which is when the particle (or droplet) Reynolds number is less than unity, the particle drag coefficient is inversely proportional to the Reynolds number itself. In that case, the characteristic time of the particle can be written as

$$t_0 = \frac{\rho_p d_p^2}{18\mu_g}$$

where ρ_p is the particle density, d_p is the particle diameter and μ_g is the fluid dynamic viscosity.[2]

In experimental fluid dynamics, the Stokes number is a measure of flow tracer fidelity in particle image velocimetry (PIV) experiments where very small particles are entrained in turbulent flows and optically observed to determine the speed and direction of fluid movement (also known as the velocity field of the fluid). For acceptable tracing accuracy, the particle response time should be faster than the smallest time scale of the flow. Smaller Stokes numbers represent better tracing accuracy; for $\mathrm{Stk} \gg 1$, particles will detach from a flow especially where the flow decelerates abruptly. For $\mathrm{Stk} \ll 1$, particles follow fluid streamlines closely. If $\mathrm{Stk} < 0.1$, tracing accuracy errors are below 1%.[3]

The **Womersley number** (α or Wo) is a dimensionless number in biofluid mechanics and biofluid dynamics. It is a dimensionless expression of the pulsatile flow frequency in relation to viscous effects. It is named after John R. Womersley (1907–1958) for his work with blood flow in arteries.[1] The Womersley number is important in keeping dynamic similarity when scaling an experiment. An example of this is scaling up the vascular system for experimental study. The Womersley number is also important in determining the thickness of the boundary layer to see if entrance effects can be ignored.

The square root of this number is also referred to as **Stokes number**, $\mathrm{Stk} = \sqrt{\mathrm{Wo}}$, due to the pioneering work done by Sir George Stokes on the Stokes second problem.

Pipe flow with oscillating pressure gradient

$u_t = -\frac{p_z}{\rho} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$ $\quad$ $u\,(0,0,u_z(r,t))$

$p_z = \mathrm{Re}\left[(\Delta P/L)\,e^{i\omega t}\right]$ $\quad$ Δp = pressure fluctuation amplitude between

Assume $u(r,t) = \mathrm{Re}\left[f(r)e^{i\omega t}\right]$ $\quad$ pipe length L ends

$f'' + r^{-1}f' - i\frac{\omega}{\nu}f = \frac{\Delta p}{\mu L}$ $\quad$ 0^{th} order Bessel equation

$f(r) = A\,J_0(i^{3/2}r') + B\,Y_0(i^{3/2}r') + i\frac{\Delta p}{\omega\rho L}$ $\quad$ $r' = r/\sqrt{\nu/\omega}$

0^{th} order Bessel functions of 1^{st} kind complex argument with r scaled by diffusion distance $\sqrt{\nu/\omega}$. A + B constants. Since $f(0)$ finite $B=0$ $(Y_0(0)\to\infty)$.

a = radius pipe $\quad$ $f(r=a) = 0 = A\,J_0\left(\frac{i^{3/2}a}{\sqrt{\nu/\omega}}\right) + i\frac{\Delta p}{\omega\rho L}$

$a' = a/\sqrt{\nu/\omega}$ $\quad$ $A = -\frac{i\Delta p}{\omega\rho L} \Big/ J_0\left(\frac{i^{3/2}a}{\sqrt{\nu/\omega}}\right)$

$$u = \mathrm{Re}\left[i\frac{\Delta p}{\omega\rho L}\left[1 - J_0(i^{3/2}r')/J_0(i^{3/2}a')\right]e^{i\omega t}\right]$$

Special techniques Bessel functions complex plane.

$\omega\to 0$ $\quad$ $u(r) = \frac{r^2-a^2}{4\mu}\frac{dp}{dz}$ $\quad$ steady Poiseuille Flow

$\omega\to\infty$ $\quad$ = profile similar Stokes 2^{nd} problem

Exercise 9.32. a) When z is complex, the small-argument expansion of the zeroth-order Bessel function $J_o(z) = 1 - \frac{1}{4}z^2 + ...$ remains valid. Use this to show that (9.43) reduces to (9.6) as $\omega \to 0$ when $dp/dz = \Delta p/L$. The next term in the series is $\frac{1}{64}z^4$. At what value of $a/\sqrt{\nu/\omega}$ is the magnitude of this term equal to 5% of the second term.
b) When z is complex, the large-argument expansion of the zeroth-order Bessel function $J_o(z) \cong (2/\pi z)^{1/2}\cos\left[z - \frac{1}{4}\pi\right]$ remains valid for $|\arg(z)| < \pi$. Use this to show that (9.43) reduces to the velocity profile of a viscous boundary layer on a plane wall beneath an oscillating flow as $\omega \to \infty$:

$$u_z(y,t) = -\frac{\Delta p}{\rho\omega L}\left[\sin(\omega t) - \exp\left\{-y\sqrt{\frac{\omega}{2\nu}}\right\}\sin\left(\omega t - y\sqrt{\frac{\omega}{2\nu}}\right)\right],$$

where y is the distance from the tube wall, $R = a - y$, $y << a$, and $dp/dz = \Delta p/L$.

Solution 9.32. a) Start from (9.43):

$$u_z(R,t) = \mathrm{Re}\left\{i\frac{\Delta p}{\omega\rho L}\left[1 - J_o\left(\frac{i^{3/2}R}{\sqrt{\nu/\omega}}\right)\bigg/ J_o\left(\frac{i^{3/2}a}{\sqrt{\nu/\omega}}\right)\right]e^{i\omega t}\right\}, \text{ and}$$

use the small argument form of J_o for the limit $\omega \to 0$:

$$\lim_{\omega\to 0} u_z(R,t) = \lim_{\omega\to 0}\mathrm{Re}\left\{i\frac{\Delta p}{\omega\rho L}\left[1 - \frac{1 - \frac{1}{4}\left(\frac{i^{3/2}R}{\sqrt{\nu/\omega}}\right)^2 + ...}{1 - \frac{1}{4}\left(\frac{i^{3/2}a}{\sqrt{\nu/\omega}}\right)^2 + ...}\right]e^{i\omega t}\right\} = \lim_{\omega\to 0}\mathrm{Re}\left\{i\frac{\Delta p}{\omega\rho L}\left[1 - \frac{1 + \frac{i\omega R^2}{4\nu} + ...}{1 + \frac{i\omega a^2}{4\nu} + ...}\right]e^{i\omega t}\right\}.$$

Continue simplifying:

$$\lim_{\omega\to 0} u_z(R,t) = \lim_{\omega\to 0}\mathrm{Re}\left\{i\frac{\Delta p}{\omega\rho L}\left[1 - \left(1 + \frac{i\omega R^2}{4\nu} - \frac{i\omega a^2}{4\nu} + ...\right)\right]e^{i\omega t}\right\}$$

$$= \lim_{\omega\to 0}\mathrm{Re}\left\{i\frac{\Delta p}{\omega\rho L}\left[-\frac{i\omega R^2}{4\nu} + \frac{i\omega a^2}{4\nu}\right]e^{i\omega t}\right\}$$

$$= \mathrm{Re}\left\{\frac{\Delta p}{\rho L}\left[+\frac{R^2}{4\nu} - \frac{a^2}{4\nu}\right]\right\} = \frac{1}{4\mu}\left(-\frac{\Delta p}{L}\right)(a^2 - R^2)$$

and this is the same as (9.6) when the pressure gradient is $\Delta p/L$.

To determine when $\frac{1}{64}z^4$ is 5% of $\frac{1}{4}z^2$, set $(0.05)\frac{1}{4}z^2 = \frac{1}{64}z^4$ and determine z. The solution is $|z| = a/\sqrt{\nu/\omega} = \sqrt{0.05(64)/4} = 0.894$.

b) Here, $z = \dfrac{i^{3/2}R}{\sqrt{\nu/\omega}} = \left(-\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}\right)\dfrac{R}{\sqrt{\nu/\omega}}$, so

$$\cos\left(z-\frac{\pi}{4}\right)=\frac{1}{2}\left[\exp\left\{i\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\frac{R}{\sqrt{\nu/\omega}}-i\frac{\pi}{4}\right\}+\exp\left\{-i\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\frac{R}{\sqrt{\nu/\omega}}+i\frac{\pi}{4}\right\}\right]$$
$$=\frac{1}{2}\left[\exp\left\{\left(-\frac{i}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)\frac{R}{\sqrt{\nu/\omega}}-i\frac{\pi}{4}\right\}+\exp\left\{\left(\frac{i}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)\frac{R}{\sqrt{\nu/\omega}}+i\frac{\pi}{4}\right\}\right]$$

When $\omega\to\infty$, the first term becomes exponentially small, so

$$\cos\left(z-\frac{\pi}{4}\right)\cong\frac{1}{2}\exp\left\{\left(\frac{i}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)\frac{R}{\sqrt{\nu/\omega}}+i\frac{\pi}{4}\right\}\quad\text{as}\quad\omega\to\infty.$$

Now use $R=a-y$ in the above expression and collect like factors:

$$\cos\left(z-\frac{\pi}{4}\right)\cong\frac{1}{2}\exp\left\{i\left(\frac{a-y}{\sqrt{2\nu/\omega}}+\frac{\pi}{4}\right)+\frac{a-y}{\sqrt{2\nu/\omega}}\right\}\quad\text{as}\quad\omega\to\infty.$$

or:

$$\cos\left(z-\frac{\pi}{4}\right)\cong\frac{e^{i\pi/4}}{2}\exp\left(\frac{a(1+i)}{\sqrt{2\nu/\omega}}\right)\exp\left(\frac{-(1+i)}{\sqrt{2\nu/\omega}}y\right)\quad\text{as}\quad\omega\to\infty.$$

So, in this limit:

$$\frac{J_o\left(\dfrac{i^{3/2}R}{\sqrt{\nu/\omega}}\right)}{J_o\left(\dfrac{i^{3/2}a}{\sqrt{\nu/\omega}}\right)}=\frac{\sqrt{\dfrac{2}{\pi}\dfrac{\sqrt{\nu/\omega}}{i^{3/2}(a-y)}}\dfrac{e^{i\pi/4}}{2}\exp\left(\dfrac{a(1+i)}{\sqrt{2\nu/\omega}}\right)\exp\left(\dfrac{-(1+i)}{\sqrt{2\nu/\omega}}y\right)}{\sqrt{\dfrac{2}{\pi}\dfrac{\sqrt{\nu/\omega}}{i^{3/2}a}}\dfrac{e^{i\pi/4}}{2}\exp\left(\dfrac{a(1+i)}{\sqrt{2\nu/\omega}}\right)}$$
$$=\sqrt{\frac{a}{a-y}}\exp\left(\frac{-(1+i)}{\sqrt{2\nu/\omega}}y\right)$$
$$\cong\exp\left(\frac{-(1+i)}{\sqrt{2\nu/\omega}}y\right)$$

where the final approximate equality holds when $y<<a$. Now substitute this approximate ratio of Bessel functions into (9.43) to find:

$$u_z(y,t)=\mathrm{Re}\left\{\frac{i\Delta p}{\omega\rho L}\left[1-\exp\left(\frac{-(1+i)y}{\sqrt{2\nu/\omega}}\right)\right]e^{i\omega t}\right\}=\mathrm{Re}\left\{\frac{i\Delta p}{\omega\rho L}\left[e^{i\omega t}-\exp\left(\frac{-y}{\sqrt{2\nu/\omega}}\right)\exp\left(i\omega t-\frac{iy}{\sqrt{2\nu/\omega}}\right)\right]\right\}.$$

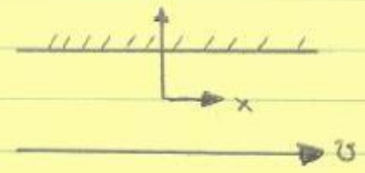
Take the real part to reach:

$$u_z(y,t)=-\frac{\Delta p}{\omega\rho L}\left[\sin(\omega t)-\exp\left(\frac{-y}{\sqrt{2\nu/\omega}}\right)\sin\left(\omega t-\frac{y}{\sqrt{2\nu/\omega}}\right)\right],$$

and this is the desired result.

As a final example we consider developing Couette flow:

3. Unsteady flow between two infinite planes



$u(0,t) = U$ sudden acceleration

$u(h,t) = 0$

$\Rightarrow$ find solution $u(y,\infty) \to U_0(1-y/h)$ ie linear Couette flow

Solution can be obtained using a Fourier series technique

$$u/U_0 = \underbrace{(1-y/h) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp(-n^2\pi^2 t^*)\sin\frac{n\pi y}{h}}_{\text{of form } u = u(y,\infty) - u}$$

$t^* = \nu t/h^2$

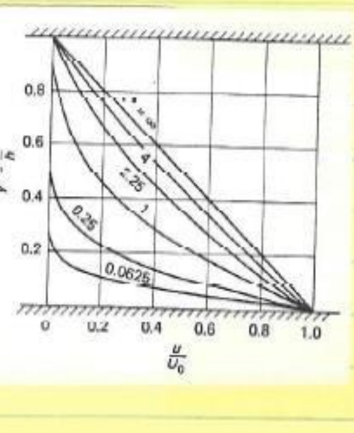


FIGURE 3-17
The development of plane Couette flow due to a suddenly accelerated lower wall, Eq. (3-120).