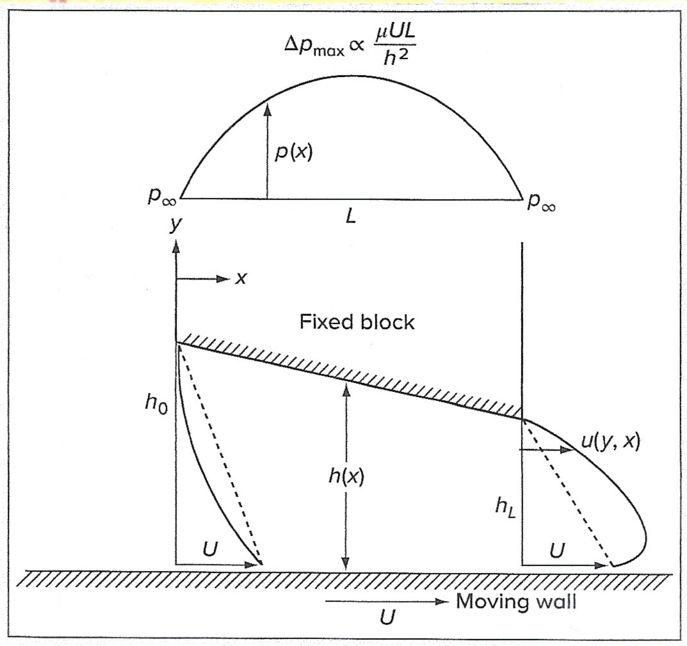
Lubrication Theory (also applicable porous media, filtration, adhesion, biological, and manufacturing flows)

Lubrication is friction reduction of two bodies in near contact is generally accomplished by a viscous fluid moving through a narrow but variable gap between the two bodies with one or both moving (Reynolds, 1886)



Assume 2D $\frac{\partial}{\partial z} = 0$ & Stokes flow such that inertia negligible ie $\rho u u_x \ll \mu u_{yy}$

$\rho U \frac{U}{L} \ll \mu \left(\frac{U}{h}\right)^2$

$\frac{\rho U L}{\mu}\left(\frac{h}{L}\right)^2 \ll 1$ Re can be large if h/L small

$U = 10\ m/s$, $L = 4\ cm$, $h = .1\ mm$
SAE 50 lubricating oil $\nu \approx 7 \times 10^{-4}\ m^2/s$
$Re_L = 570$ but $Re_L \left(\frac{h}{L}\right)^2 = .004$ so OK

FIGURE 3-48 Low Reynolds number Couette flow in a varying gap: To maintain continuity, the gap pressure rises to a maximum and superimposes Poiseuille flow toward both ends of the gap.

$\varepsilon = h/L \ll 1$ & Re moderate

Like Stokes flow since Re fairly small and inertia negligible. Usually quasi-steady.

Like BL flow in that $\frac{\partial p}{\partial y} = 0$ & $\frac{\partial p}{\partial x}$ important

However, the pressure & viscous stress scale differently than Stokes or BL flows.

Reynolds Equation for Bearing Theory

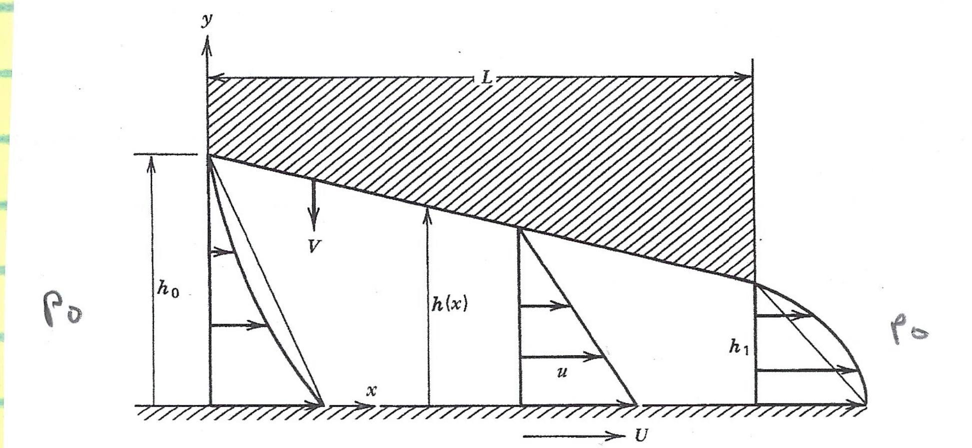


Figure 22.3 Flow in a slipper pad bearing is locally the sum of a Couette flow and a Poiseuille flow.

Events:

1. Moving wall sweeps fluid into narrowing passage due to viscous shear forces, which induces Couette flow $u = Uy/h$ and $Q_c = \frac{1}{2} U h$

2. Continuity requires $Q = \text{constant}$; therefore, p_x required that induces Poiseuille u component that redistributes fluid and maintains $Q = \text{constant}$

In general $h(x,t)$ and for simplicity upper wall vertical motion only V which may be $v(t)$, lower wall $U = \text{constant}$, and allow for w due p_z

$x, z \sim L$ $\quad y \sim h_0$ $\quad t \sim L/U$ $\quad u, w \sim U$ $\quad v \sim \dfrac{U h_0}{L}$ $\quad p \sim \dfrac{\mu U L}{h_0^2}$

y-momentum $\lim_{h_0/L \to 0} \Rightarrow \dfrac{\partial p}{\partial y} = 0$ ie $p = p(x,z)$

x- and z-momentum equations yield quasi-steady equations (See Appendix)

$$0 = -p_x + \mu u_{yy}$$
$$0 = -p_z + \mu w_{yy}$$

$u(y=0) = U$ $\quad v(y=0) = 0$ $\quad w(y=0) = 0$

$u(y=h) = 0$ $\quad v(y=h) = V$ $\quad w(y=h) = 0$

Partial integration over y & application BC:

$$u = \frac{1}{2\mu} p_x (y^2 - yh) + (1 - y/h) U$$

$$w = \frac{1}{2\mu} p_z (y^2 - yh)$$

Combination Couette & Poiseuille flow function pressure gradient, h, and U

Need to determine pressure distribution that will support the load with the bearing.

The Reynolds equation for pressure is derived by integrating the continuity equation over the y-direction.

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx = \int_a^b \frac{\partial f}{\partial t}\,dx + \frac{db}{dt} f(b,t) - \frac{da}{dt} f(a,t)$$

$$\int_0^h u_x\,dy + \int_0^h w_z\,dy = -\int_0^h v_y\,dy = -V = +\frac{\partial h}{\partial t}$$

$$\frac{\partial}{\partial x}\underbrace{\int_0^h u\,dy}_{Q_x} = \int_0^h \frac{\partial u}{\partial x}\,dy + \frac{dh}{dx} u(y=h) - \frac{d0}{dx} u(y=0) = \int_0^h \frac{\partial u}{\partial x}\,dy$$

$\bar{u} = Q_x/h$

$\bar{w} = Q_z/h$

$$\frac{\partial}{\partial z}\underbrace{\int_0^h w\,dy}_{Q_z} = \int_0^h \frac{\partial w}{\partial z}\,dy + \frac{dh}{dz} w(y=h) - \frac{d0}{dz} w(y=0) = \int_0^h \frac{\partial w}{\partial z}\,dy$$

$$u(y) = \frac{1}{2\mu} p_x (y^2 - yh) + (1 - y/h) U$$
$$= a(y^2 - yh) + U - Uy/h$$

$$\int_0^h \left[a(y^2 - yh) + U - Uy/h\right] dy$$

$$= \left[\frac{ay^3}{3} - \frac{ahy^2}{2} + Uy - \frac{Uy^2}{2h}\right]_0^h = \frac{ah^3}{3} - \frac{ah^3}{2} + Uh - \frac{Uh}{2}$$

$$= -\frac{ah^3}{6} + \frac{Uh}{2}$$

$$w(y) = \frac{1}{2\mu} p_z (y^2 - yh) = b(y^2 - yh)$$

$$b\int_0^h (y^2 - yh)\,dy = b\left[\frac{y^3}{3} - \frac{y^2 h}{2}\right]_0^h = -\frac{bh^3}{6}$$

$$\frac{\partial}{\partial x}\left[-\frac{ah^3}{6} + \frac{Uh}{2}\right] + \frac{\partial}{\partial z}\left[-\frac{bh^3}{6}\right] = -\frac{\partial h}{\partial t}$$

$$\frac{\partial}{\partial x}\left[-ah^3 + 3Uh\right] + \frac{\partial}{\partial z}\left[-bh^3\right] = -6\frac{\partial h}{\partial t}$$

$$\frac{\partial}{\partial x}\left[-\frac{1}{2\mu} p_x h^3 + 3Uh\right] + \frac{\partial}{\partial z}\left[\frac{1}{2\mu} p_z h^3\right] = -6\frac{\partial h}{\partial t}$$

$$\frac{1}{\mu}\left[\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial z}\left(h^3\frac{\partial p}{\partial z}\right)\right] = 6U\frac{\partial h}{\partial x} + 12\frac{\partial h}{\partial t}$$

Reynolds equation lubrication in channel $h(x,t)$ with lower wall moving speed U. $p(x,z,t)$ found as f (geometry motion walls). Once pressure known proportion Poiseuille vs. Couette in the velocity profiles (u,w).

Slipper Bearing

Non-dimensional variables:

$$p^* = \frac{p-p_0}{\mu U L/h_0^2} \qquad x^* = x/L \quad h^* = \frac{h}{h_0} = 1 - Ax^*$$

$$\frac{\mu U L}{h_0^2} p^* + p^0 = p$$

$$A = \frac{\alpha L}{h_0} = \frac{h_0 - h_1}{h_0}$$

Assume 1D flow, ie, $\frac{\partial}{\partial z} = 0$

$w = 0$ $\qquad \alpha = \frac{h_0 - h_1}{L}$

$\frac{\partial}{\partial t} = 0$ ie $V = 0$

$$\frac{\partial}{\partial x}\left(h^3 \frac{\partial p}{\partial x}\right) = 6U\mu \frac{\partial h}{\partial x}$$

$$h^3 \frac{\partial p}{\partial x} = 6U\mu h + C$$

$$C = -6U\mu h_m \qquad \text{where } h_m = h\left(\frac{\partial p}{\partial x} = 0\right)$$

ie where Poiseuille flow = 0

$$h^3 \frac{\partial p}{\partial x} = 6U\mu(h - h_m) \qquad h^3 = h_0^3 h^{*3}$$

$$p^*_{x^*} \cancel{h_0^3} h^{*3} \frac{\mu U}{\cancel{h_0^2}} = 6\cancel{U\mu}(\cancel{h_0} h^* - \cancel{h_0} h^*_m) \qquad \frac{\partial p}{\partial x} = \frac{\partial x^*}{\partial x}\frac{\partial p}{\partial p^*}\frac{\partial p^*}{\partial x^*}$$

$$h^{*3} p^*_{x^*} = 6(h^* - h^*_m) \qquad = \frac{1}{L} \times \frac{\mu U L}{h_0^2} p^*_{x^*}$$

Write at h^*_m $\quad u = (1 - y/h)U = \frac{U}{h}(h - y)$

$$h^*_m = \frac{2Q}{U h_0} = Q^* \qquad Q = \int_0^{h_m} u\,dy = \frac{U}{h}\left[hy - y^2/2\right]_0^{h_m} = \frac{U h_m}{2}$$

$$dp^* = \frac{6}{h^{*3}} (h^* - h_m^*) dx^*$$

$$= 6 (h^{*-2} - h_m^* h^{*-3}) dx^*$$

$$dp^* = 6 [(1 - Ax^*)^{-2} - h_m^* (1 - Ax^*)^{-3}] dx^*$$

$x = 1 - Ax^* = h^*$

$dx = -A dx^*$

$$= \frac{-6}{A} [x^{-2} - h_m^* x^{-3}] dx \qquad \int x^n dx = \frac{x^{n+1}}{n+1} \quad n \neq -1$$

$$p^* = \frac{-6}{A} \left[x^{-1}/-1 \quad - h_m^* \frac{x^{-2}}{-2} \right] + c$$

$$= 6A^{-1} \left(+ h^{*-1} - \frac{h_m^*}{2} h^{*-2} \right) + c$$

$$h_0 h^* = h = h_0 (1 - Ax^*) = h_0 \left[1 - \frac{h_0 - h_1}{h_0} \frac{x}{L} \right]$$

$h(0) = h_0$ $\qquad h = h_0 - (h_0 - h_1) \frac{x}{L}$

$h(L) = h_1$

$h^* = 1 = h = h_0$ at $p = p_0$ $\therefore$ $p^* = 0$

$$0 = 6A^{-1} (+1 - h_m^*/2) + c$$

$$c = 6A^{-1} (-1 + h_m^*/2)$$

$$p^* = 6A^{-1} \left[1 - h^{*-1} + \frac{h_m^*}{2} (-h^{*2} + 1) \right]$$

$$= 6A^{-1} (h^{*-1} - 1) - 3A^{-1} h_m^* (h^{*-2} - 1)$$

$p^* = 0$ at $h^* = 1 - A$ ie $h = h_0 - (h_0 - h_1)\frac{x}{L}$

$= a$

$0 = 6A^{-1}\left(\frac{1}{a} - 1\right) - 3A^{-1} h_m^* \left(\frac{1}{a^2} - 1\right)$ $\quad h(L) = h_1 \quad x^* = 1$

$\frac{3h_m^*}{\cancel{A}}\left(\frac{1-a^2}{a^2}\right) = \frac{\cancel{6}^{\,2}}{\cancel{A}}\left(\frac{1-a}{a}\right)$ $\quad h^* = 1 - A, \; p = p_0$

$p^* = 0$

$h_m^*(\cancel{1-a})(1+a) = 2(\cancel{1-a})a$ $\quad (1-a)(1+a) =$

$h_m^* = \frac{2a}{1+a}$ $\quad 1 + a - a - a^2 = 1 - a^2$

$(1-A)(1-A) = 1 - 2A + A^2$

$= \frac{2(1-A)}{1+1-A} = \frac{2(1-A)}{2-A}$

$p^* = 6A^{-1}\left(\frac{1-h^*}{h^*}\right) - 3A^{-1}h_m^*\left(\frac{1-h^{*2}}{h^{*2}}\right)$

$= 6A^{-1}h^*(1-h^*) - 3A^{-1}h_m^* h^{*2}(1-h^{*2})$

$h^* = 1 - Ax^*$

$\frac{h^* - h^{*2}}{A}$ $\quad = 6A^{-1}h^*\left[(1-h^*) - \frac{h_m^* h^*}{2}(1-h^{*2})\right]$

$\frac{1-h^*}{h^*A} = \frac{Ax^*}{Ah^*}$ $\quad = 6A^{-1}h^*(1-h^*)\left[1 - \frac{h_m^* h^*}{2}(1+h^*)\right]$

$= \frac{x^*}{1-Ax^*}$ $\quad p^*(x^*) = \frac{6x^*}{1-Ax^*}\left[1 - \frac{1-A}{2-A} \times \frac{2-Ax^*}{1-Ax^*}\right]$

$h^*(1+h^*)$ $\quad h_m^* = 1 - Ax_m^* = 2\,\frac{1-A}{2-A}$

$= h^* + h^{*2}$

$= \frac{1+h^*}{h^*} = \frac{2-Ax^*}{1-Ax^*}$

$h^* = 1 - A x^*$ and $h^{*3} p^*_{x^*} = 6(h^* - h^*_m)$

and $h^*_m = 2\dfrac{1-A}{2-A}$ from 1st integration equation $p^*_{x^*} = 0$ when $h^* = h^*_m$

$\therefore \quad 2\dfrac{1-A}{2-A} = 1 - A x^*_m$

$$A x^*_m = 1 - 2\frac{1-A}{2-A} = \frac{2 - A - 2(1-A)}{2-A} = \frac{\cancel{2} - A - \cancel{2} + 2A}{2-A}$$

$$= \frac{A}{2-A} \qquad \text{ie} \quad x^*_m = \frac{1}{2-A}$$

$$1 - A x^*_m = 1 - \frac{A}{2-A} = \frac{2 - A - A}{2-A} = \frac{2(1-A)}{2-A}$$

$$2 - A x^*_m = 2 - \frac{A}{2-A} = \frac{4 - 2A - A}{2-A} = \frac{4-3A}{2-A}$$

$$p_m^* = \frac{6}{\cancel{(2-A)}\, 2(1-A)} \cancel{(2-A)} \left[1 - \frac{\cancel{1-A}}{\cancel{2-A}} \frac{4-3A}{2-A} \frac{\cancel{2-A}}{2(\cancel{1-A})} \right]$$

$\cancel{4} - 2A - \cancel{4} + 3A = A$

$$p_m^* = \frac{3}{(1-A)}\left[1 - \frac{4-3A}{2(2-A)} \right] = \frac{3}{(1-A)} \left[\frac{2(2-A) - 4 + 3A}{2(2-A)} \right]$$

$$p_m^* = 3A / 2(1-A)(2-A)$$

$A = 0$ walls $\parallel$

Couette flow $p = \text{const} = p_0$

$A > 0$ but very small

$$p^*_m = \frac{3A}{4\left(1-\frac{A}{2}\right)(1-A)}$$

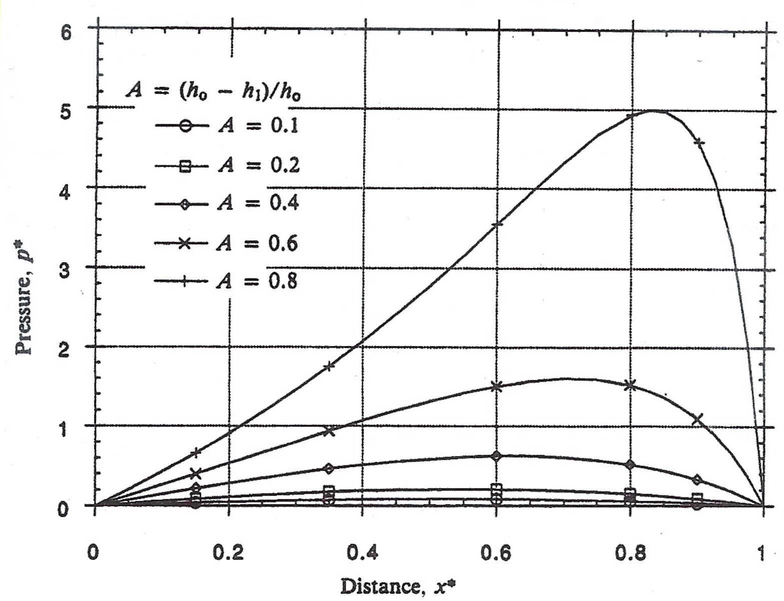


Figure 22.4 Pressure in a slipper pad bearing.

$$= \frac{3A}{4}\left(1-\frac{A}{2}\right)^{-1}(1-A)^{-1}$$

$$= \frac{3A}{4}\left(1+\frac{A}{2}\right)(1+A) \approx \frac{3A}{4} \qquad x^*_m = \frac{1}{2}\left(1-\frac{A}{2}\right)^{-1} = \frac{1}{2}\left(1+\frac{A}{2}\right) \approx \frac{1}{2}$$

$$\underbrace{\frac{dp^*}{dx^*} > 0 \quad x^* < x^*_m}_{\text{Poiseuille flow opposes Couette flow \& vice versa}} \quad \& \quad \frac{dp^*}{dx^*} < 0 \quad x^* > x^*_m$$

Process: fluid dragged into converging channel via viscous shear force piles up to create high pressure $x^* = \frac{1}{2}$ after which $\frac{dp^*}{dx^*}$ changes sign & pushes flow towards exit. $\frac{dp^*}{dx^*}$ between center & either end induces Poiseuille flow towards both ends of the bearing ∴ subtracts Couette flow first half & adds second half

$$\frac{p_m - p_0}{\mu U L / h_0^2} = \frac{3A}{4} + O[A^2]$$

$$p_m - p_0 = \frac{\mu U L}{h_0^2} \times \frac{3A}{4}$$

$$= \frac{3\mu U L}{4}\left(\frac{h_0 - h_1}{h_0^3}\right) = \frac{3\mu U L}{4}\,\frac{1 - h_1/h_0}{h_0^2}$$

Shows importance h_0

SAE oil with $U = 10$ m/s, $L = 4$ cm, & $h_0 = .1$ mm

$p_m - p_0$ of order $\frac{\mu U L}{h_0^2} \simeq 2.5 \times 10^7$ Pa or 250 atm

ie very high force to slipper block, which enables it to support large load without block touching wall.

Stokes flow is linear and ∴ reversible. If reverse wall motion $U < 0$ then $\Delta p < 0$ ie will cavitate and form vapor void in gap (G.I. Taylor film low Re hydrodynamics) ∴ flow in expanding narrow gap may not support large loads and provide good lubrication. Issue for rotating journal bearing where gap contracts / expands and often leads to partial cavitation.

Squeeze film lubrication: viscous adhesion

Wringing together smooth surfaces relationship

Crankshaft bearing

$\Uparrow$ power stroke piston causes $V(t)$ which dominates over hydrodynamic journal-bearing effect, whereas separating smooth surfaces by pulling normal to increase gap height is difficult, although sliding is easy

Assume $V = -\frac{\partial h}{\partial t}$ and $U = 0$ Also ω and $\frac{\partial}{\partial z} = 0$

Take $x = 0$ such that bearing pad ends at $\pm L/2$

Reynolds pressure equation: $\frac{1}{\mu}\frac{d}{dx}\left(h^3 \frac{dp}{dx}\right) = 12\frac{\partial h}{\partial t} = 12V(t)$

$\therefore \frac{dp}{dx} = \frac{12\mu}{h^3}\frac{dh}{dt}x$ $\qquad u = \frac{1}{2\mu}\frac{\partial p}{\partial x}(y^2 - yh)$ at $x = 0$ $= 0$ and maximum at $x = \pm L/2$

$$p - p_0 = \frac{12\mu}{h^3}\frac{\partial h}{\partial t}\left.\frac{x^2}{2}\right|_{-L/2}^{x}$$

thus all flow into or out gap must cross the ends

$$= \frac{12\mu}{h^3}\frac{\partial h}{\partial t}\left[\frac{x^2}{2} - \frac{L^2/4}{2}\right] = \frac{12\mu}{h^3}\frac{\partial h}{\partial t}\left[\frac{x^2}{2} - L^2/8\right] = \frac{12\mu}{h^3}\frac{\partial h}{\partial t}\frac{1}{2}\frac{L^2}{4}\left(\frac{x^2}{L^2/4} - 1\right)$$

$$p - p_0 = -\frac{3\mu L^2}{2h^3}\frac{\partial h}{\partial t}\left[1 - \left(\frac{x}{L/2}\right)^2\right]$$

$p \propto h^{-3}$

very large pressures

$$= \frac{3\mu L^2}{2h^3}V\left[1 - \left(\frac{x}{L/2}\right)^2\right]$$

$$\bar{W} = \int_{-L/2}^{L/2} (p - p_0)\, dx$$ load capacity per unit span

$$= \frac{3\mu L^2}{2h^3} V \int_{-L/2}^{L/2} \left[1 - \left(\frac{x}{L/2}\right)^2\right] dx$$

$x' = x / L/2$

$dx' = dx / L/2$

$L/2\, dx' = dx$

$$\int_{-1}^{1} (1 - x'^2)\, dx' = \left(x' - \frac{x'^3}{3}\right)\Bigg|_{-1}^{1} = \frac{2}{3} - \underbrace{\left(-1 + \frac{1}{3}\right)}_{-2/3} = 4/3 \times \frac{L}{2} = \frac{2L}{3}$$

$$\bar{W} = \frac{\cancel{3}\mu L^2}{\cancel{2}h^3} V \times \frac{\cancel{2}L}{\cancel{3}} = \mu \left(\frac{L}{h}\right)^3 V$$

$$= -\mu \left(\frac{L}{h}\right)^3 \frac{dh}{dt}$$

$$\mu = \frac{Ws}{m^2} \times \frac{m}{s} = \frac{W}{m}$$

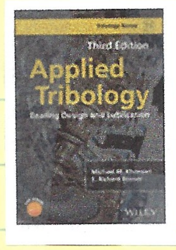
$$\int_{t_1}^{t_2} dt = -\frac{\mu L^3}{\bar{W}} \int_{h_1}^{h_2} \frac{dh}{h^3}$$

$$\int h^{-3} dh = -\frac{1}{2} h^{-2}$$

$$\Delta t = \frac{-\mu L^3}{\bar{W}} \left(-\frac{1}{2} h^{-2}\right)\Bigg|_{h_1}^{h_2}$$

$$\Delta t = \frac{\mu L^3}{2\bar{W}} \left(\frac{1}{h_2^2} - \frac{1}{h_1^2}\right)$$

Δt = time of approach for film gap to reduce from h_2 to h_1. $\Delta t \to \infty$ as $h_2 \to 0$ ie takes infinite time squeeze out all the fluid!



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9.4 Generalization for planar squeeze film

For planar squeeze-film problems, the time of approach has the following form (Moore, 1993):

$$\Delta t = K\frac{\mu A^2}{W}\left(\frac{1}{h_2^2}-\frac{1}{h_1^2}\right) \tag{9.13}$$

where K is the shape function. Taking the time derivative of Equation (9.13), the surface approach velocity can be obtained:

$$V_s = \frac{1}{2K}\frac{h^3 W}{\mu A^2} \tag{9.14}$$

where A is the plate area and h is the squeeze-film thickness.

Forms of the function K for a series of planar squeeze-film geometries are shown in Table 9.1 and plotted in Figure 9.5 for convenience (Khonsari and Jang, 1997). Using either the table or the figure, one can readily evaluate K. Then, for a given load, W, Equation (9.13) gives the time of approach for the film thickness to drop from an initial h_1 to a final h_2.

Example 9.1 Estimate the time of approach in a wet clutch system modeled as two rigid concentric annuli with $R_i = 0.047$ m and $R_o = 0.06$ m submerged in a lubricant with viscosity of $\mu = 0.006$ Pa s. Hydraulic pressure is $P = 1.25$ MPa, and the initial separation gap is $h_1 = 25\times10^{-6}$ m. Estimate the initial squeeze velocity and the length of time necessary for the film thickness to drop to $h_2 = 5\times10^{-6}$ m.

Table 9.1 Types of planar squeeze

Type	Ratio, r	Configuration	Constant, K
Circular section	—	R	$\frac{3}{4\pi}$
Elliptical section	$\Re = \frac{b}{a}$	a, b	$\frac{3\Re}{2\pi(1+\Re^2)}$
Rectangular section	$\Re = \frac{B}{L}$	L, B	$\frac{1}{2\Re}\left[1-\frac{192}{\pi^5\Re}\sum_{n=1,3,5,\ldots}^{\infty}\frac{\tanh(n\pi\Re/2)}{n^5}\right]$
Triangular section	—	a	$\frac{\sqrt{3}}{10}$
Circular sector	$\Re = \frac{\alpha}{2\pi}$	α	$\sum_{n=1,3,5,\ldots}^{\infty}\frac{24}{n^2\pi^3\Re\left[2+\left(\frac{n}{2\Re}\right)\right]^2}$
Concentric annulus	$\Re = \frac{D_i}{D_o}$	D_o, D_i	$\frac{3}{4\pi}\left[\frac{\ln\Re-\Re^4\ln\Re+(1-\Re^2)^2}{(1-\Re^2)^2\ln\Re}\right]$

The shape factor K for the concentric annulus can be evaluated from Table 9.1:

$$A = \pi\left(R_o^2 - R_i^2\right) = 0.0044\,\mathrm{m}^2$$

$$\Re = \frac{R_i}{R_o} = \frac{0.047}{0.06} = 0.783$$

$$K = \frac{3}{4\pi}\frac{\ln(\Re)-\Re^4\ln\Re+(1-\Re^2)^2}{(1-\Re^2)^2\ln(\Re)}$$

$$= \frac{3}{4\pi}\frac{\ln(0.783)-(0.783)^4\ln(0.783)+(1-0.783^2)^2}{(1-0.783^2)^2\ln(0.783)} = 0.019$$

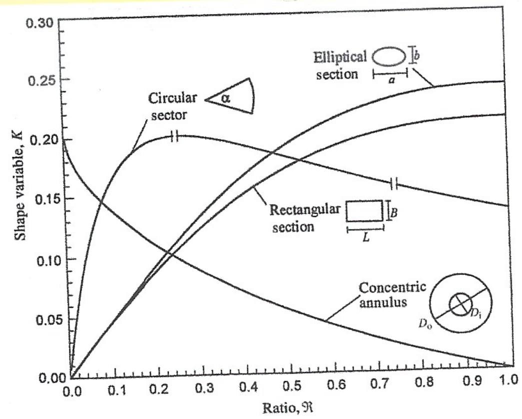


Figure 9.5 Variation of constant K with shape ratio $\Re$

Using Equation (9.13), the time of approach is

$$\Delta t = K\frac{\mu A^2}{W}\left(\frac{1}{h_2^2}-\frac{1}{h_1^2}\right) = K\frac{\mu A}{P}\left(\frac{1}{h_2^2}-\frac{1}{h_1^2}\right)$$

$$= \frac{0.019(0.006)(0.0044)}{1.25\times10^6}\left[\frac{1}{(5\times10^{-6})^2}-\frac{1}{(25\times10^{-6})^2}\right] = 0.02\,\mathrm{s}$$

The squeeze velocity can be estimated using Equation (9.14):

$$V_s = \frac{1}{2K}\frac{h_1^3 W}{\mu A^2} = \frac{1}{2K}\frac{h_1^3 P}{\mu A}$$

$$= \frac{1}{2(0.019)}\frac{(25\times10^{-6})^3 1.25\times10^6}{(0.006)(0.0044)} = 0.02\,\mathrm{m/s}$$

The time of approach predicted above is a representation of the first stage of the engagement duration while the clutch operates in the hydrodynamic regime. This engagement begins when pressure is applied hydraulically by means of a piston and hydrodynamic pressure is developed in the ATF as a result of squeeze action which supports most of the applied load. Since the surfaces are separated by a relatively thick film of fluid, behavior of the clutch is governed by the theory of hydrodynamic lubrication. This period lasts only 0.02 s.

During engagement, the fluid film thickness drops to the extent that surface asperities come into contact. As a result, contact pressure at the asperity level begins to support a major portion of the imposed load, significantly influencing the behavior of the wet clutch. The film thickness is further reduced as the friction-lining material is compressed and deforms elastically. The surfaces are subsequently pressed together and 'locked' when their relative speed drops to zero. The timescale of the engagement process is typically of the order of 1 s, during which the squeeze action is of paramount importance. It follows, therefore, that in a typical engagement cycle, the lubrication regime undergoes a transition from hydrodynamic to mixed or boundary lubrication. This shift in the lubrication regime has important implications on the signature of the total torque, i.e. the combination of the viscous torque and contact torque. The total torque reaches a peak value when the film thickness drops to a minimum. After this peak value, the torque initially remains relatively flat for a short period of time and then begins to rise gradually as the relative speed between the clutch disks decreases. This increase in the torque is a direct consequence of the change in the coefficient of friction as a function of speed, in accordance with the Stribeck friction curve. The most interesting torque signature is a highly undesirable sudden spike or 'rooster's tail' toward the end of the engagement. Its occurrence can be predicted analytically (Jang and Khonsari, 1999) and can be treated to minimize its effect by altering the friction behavior.

Note that the friction material is rough, porous, and deformable. Also, the large disk diameters used in wet clutch systems may require consideration of the centrifugal forces. These elements can affect the engagement time and torque-transfer characteristics, as well as the temperature field in the ATF and on the surface of the separator. The interested reader may refer to Jang and Khonsari (1999) for detailed analysis of automotive wet clutch systems. More recent studies of wet clutches have included parametric analysis of variance and experimental results (Mansouri *et al.*, 2001, 2002; Marklund *et al.*, 2007).

Journal Bearing

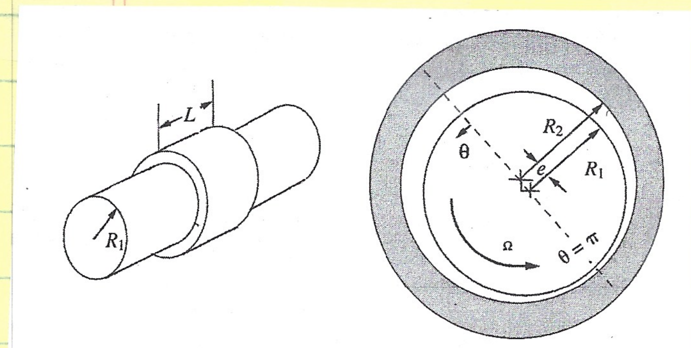


Figure 22.5 Journal bearing.

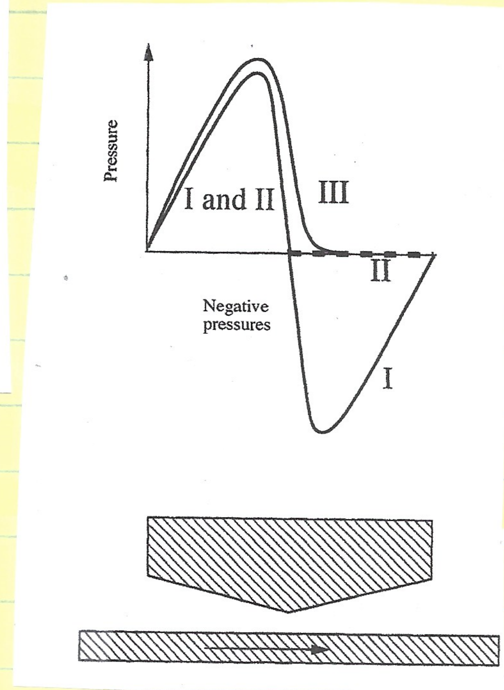


Figure 22.6 Similarity of journal bearing and double slider block. Pressure distribution I, full Sommerfeld; pressure distribution II, half Sommerfeld; pressure distribution III, Swift–Steiber.

Rotating shaft, journal, radius R_1, in bearing housing radius R_2. $L/R_1 < .5$ = short
> 2 = long
aspect ratio

journal offset bearing distance
e = eccentricity
$c = R_2 - R_1$ = clearance
$\varepsilon = e/c$ = eccentricity ratio with $\min = c - e$
= measure change film height $\max = c + e$

rotation ccw such that drags fluid wide to narrow passage, which creates high pressure support journal. Amount of load determines e and c. Physics similar double slider. Converging/diverging passages similar double slider with reversed motion expanding half and since lubrication equations reversible solution is valid for proper reversal of $p = -p$ however large $-p$ implies cavitation. Hence, p assumed constant or other approximations.

Appendix

$u_x + v_y + w_z = 0$

$u_t + uu_x + vu_y + wu_z = -p_x/\rho + \nu(u_{xx} + u_{yy} + u_{zz})$

$v_t + uv_x + vv_y + wv_z = -p_y/\rho + \nu(v_{xx} + v_{yy} + v_{zz})$

$w_t + uw_x + vw_y + ww_z = -p_z/\rho + \nu(w_{xx} + w_{yy} + w_{zz})$

$(x^*, z^*) = (x,z)/L \quad y^* = y/h_0 \quad t^* = Ut/L$

$(u^*, w^*) = (u,w)/U \quad v^* = vL/Uh_0 \quad p^* = p/p_0$

$\frac{\partial}{\partial t} = \frac{\partial}{\partial t^*}\frac{\partial t^*}{\partial t} = \frac{U}{L}\frac{\partial}{\partial t^*} \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial x^*}\frac{\partial x^*}{\partial x} = \frac{1}{L}\frac{\partial}{\partial x^*} \quad \frac{\partial}{\partial z} = \frac{1}{L}\frac{\partial}{\partial z^*} \quad \frac{\partial}{\partial y} = \frac{1}{h_0}\frac{\partial}{\partial y^*}$

$(u,w) = U(u^*, w^*) \quad v = \frac{Uh_0}{L}v^* \quad p = p_0 p^* \quad t = \frac{L}{U}t^*$

$= U\varepsilon v^* \quad \varepsilon = h_0/L << 1$

$\frac{1}{L}\frac{\partial}{\partial x^*}Uu^* + \frac{1}{\cancel{h_0}}\frac{\partial}{\partial y^*}\frac{U\cancel{h_0}}{L}v^* + \frac{1}{L}\frac{\partial}{\partial z^*}Uw = 0$

$\frac{U}{L}\left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*}\right) = 0$ ie no change continuity

$Re_L = \frac{\rho UL}{\mu}$

$\varepsilon = h_0/L$

$\varepsilon^2 Re_L = \frac{h_0^2}{L^2}\frac{\rho UL}{\mu} = \frac{h_0^2 \rho U}{L\mu}$

$\Lambda = \frac{\mu UL}{p_0 h_0^2}$

$\frac{U}{L}\frac{\partial}{\partial t^*}Uu^* + \frac{U^2}{L}\left(u^*\frac{\partial u^*}{\partial x^*}\right) + \frac{Uh_0}{L}v^*\frac{1}{h_0}\frac{\partial}{\partial y^*}Uu^* + Uw^*\frac{1}{L}\frac{\partial}{\partial z^*}Uu^*$

$= -\frac{p_0}{\rho L}\frac{\partial p^*}{\partial x^*} + \nu\left(\frac{1}{L^2}\frac{\partial^2}{\partial x^{*2}}Uu^* + \frac{1}{h_0^2}\frac{\partial^2}{\partial y^{*2}}Uu^* + \frac{1}{L^2}\frac{\partial^2}{\partial z^{*2}}Uu^*\right)$

$\frac{U^2}{L}\left(\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} + w^*\frac{\partial u^*}{\partial z^*}\right) = -\frac{p_0}{\rho L}\frac{\partial p^*}{\partial x^*} + \nu\left(\frac{U}{L^2}\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{U}{h_0^2}\frac{\partial^2 u^*}{\partial y^{*2}} + \frac{U}{L^2}\frac{\partial^2 u^*}{\partial z^{*2}}\right)$

$\frac{\rho U^2}{L}\frac{Du^*}{Dt} = -\frac{p_0}{L}\frac{\partial p^*}{\partial x^*} + \mu\frac{U}{L^2}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{L^2}{h_0^2}\frac{\partial^2 u^*}{\partial y^{*2}} + \frac{\partial^2 u^*}{\partial z^{*2}}\right)$

$\frac{\rho UL}{\mu}\frac{Du^*}{Dt} = \frac{-p_0 L}{\mu U}\frac{\partial p^*}{\partial x^*} + \frac{\partial^2 u^*}{\partial x^{*2}} + \varepsilon^{-2}\frac{\partial^2 u^*}{\partial y^{*2}} + \frac{\partial^2 u^*}{\partial z^{*2}}$

$\varepsilon^2 Re_L \frac{Du^*}{Dt} = \underbrace{\frac{-p_0 L h_0^2}{\mu U L^2}}\frac{\partial p^*}{\partial x^*} + \varepsilon^2\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial z^{*2}}\right) + \frac{\partial^2 u^*}{\partial y^{*2}}$

$\frac{-p_0 h_0^2}{\mu U L} = \Lambda^{-1} \qquad \Lambda = \frac{\mu UL}{p_0 h_0^2} = \frac{\text{viscous}}{\text{pressure}}$ forces

bearing #

$\varepsilon \ll 1 \quad Re_L$ moderate $\quad \Lambda \sim$ order unity

$$\therefore \qquad 0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \qquad \text{drop } *$$

Similarly for z $\quad 0 = -\frac{\partial p}{\partial z} + \mu \frac{\partial^2 w}{\partial z^2}$

$$\frac{U}{L}\frac{\partial}{\partial t^*}\frac{U h_0}{L}v^* + \frac{U u^*}{L}\frac{\partial}{\partial x}\frac{U h_0}{L}v^* + \frac{U h_0}{L}v^*\frac{1}{h_0}\frac{\partial}{\partial y^*}\frac{U h_0}{L}v^* + \frac{U w^*}{L}\frac{\partial}{\partial z^*}\frac{U h_0}{L}v^*$$
$$= -\frac{p_0}{h_0}\frac{\partial p^*}{\partial y^*} + \nu\left(\frac{1}{L^2}\frac{\partial^2}{\partial x^{*2}}\frac{U h_0}{L}v^* + \frac{1}{h_0^2}\frac{\partial^2}{\partial y^{*2}}\frac{U h_0}{L}v^* + \frac{1}{L^2}\frac{\partial^2}{\partial z^{*2}}\frac{U h_0}{L}v^*\right)$$

$$\frac{U^2 h_0}{L^2}\left[\frac{\partial v^*}{\partial t} + u^*\frac{\partial v^*}{\partial x} + v^*\frac{\partial v^*}{\partial y^*} + w^*\frac{\partial v^*}{\partial z^*}\right] = \frac{-p_0}{\rho h_0}\frac{\partial p^*}{\partial y^*} + \nu\frac{U}{h_0 L}\frac{\partial^2 v^*}{\partial y^{*2}}$$
$$+ \nu\frac{U h_0}{L^3}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial z^{*2}}\right)$$

$$\frac{\rho U L}{\mu}\frac{Dv^*}{Dt} = \frac{-p_0 L^3}{\mu U h_0^2} + \frac{L^2}{h_0^2}\frac{\partial^2 v^*}{\partial y^{*2}} + \left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial z^{*2}}\right)$$

$$\varepsilon^4 Re_L \frac{Dv^*}{Dt} = \underbrace{\frac{-p_0 L^3}{\mu U h_0^2}\frac{h_0^4}{L^4}}\frac{\partial p^*}{\partial y^*} + \varepsilon^2\frac{\partial^2 v^*}{\partial y^{*2}} + \varepsilon^4\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial z^{*2}}\right)$$

$$\frac{-p_0 h_0^2}{\mu U L} = \Lambda^{-1}$$

$$\therefore \qquad 0 = -\frac{\partial p^*}{\partial y^*}$$

note $\varepsilon^2 Re_L \approx .001$

room temperature 30-weight oil

with $\nu = 4 \times 10^{-4}$ m²/s

and $h_0 = .1$ mm $\quad L = 25$ cm

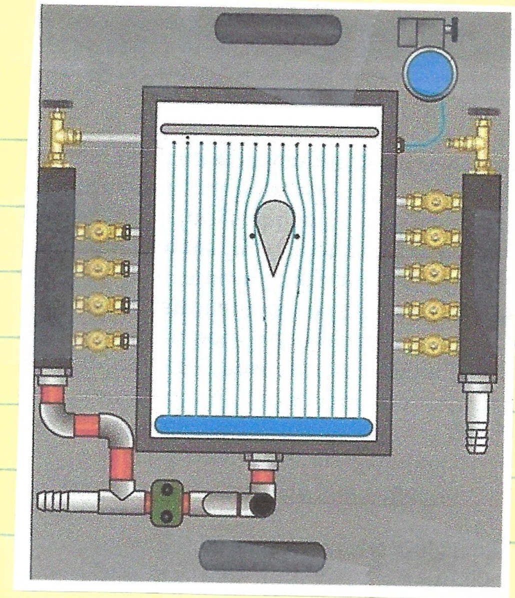
$\Delta U = 10$ m/s

If external time

Scale τ = period

mechanical oscillation,

$\rho h^2/\mu\tau \ll 1$



Hele-Shaw Flow

Table/cell = thin flat chamber, constant h driven by p or g

Originally for study lubrication by Hele-Shaw inventor fluid clutch & variable pitch propeller

2D viscous dominated flow with potential flow streamlines. Wall shapes and obstacles can be used to visualize ideal flow patterns, including singularities ie point sources/sinks.

Consider lubrication approximation with x, y main flow scaled L, z = width scaled h, & $\varepsilon = h/L \to 0$.

$$0 = -p_x + \mu u_{zz} \qquad 0 = -p_y + \mu v_{yy} \qquad 0 = -p_z$$

$$z = 0 \quad (u, v, w) = 0$$
$$z = h \quad (u, v, w) = 0$$

$$u = \frac{h^2}{2\mu} p_x \left(\left(\frac{z}{h}\right)^2 - \frac{z}{h} \right)$$
$$v = \frac{h^2}{2\mu} p_y \left(\left(\frac{z}{h}\right)^2 - \frac{z}{h} \right)$$
$$w = 0$$
$$p = p(x, y)$$

$\chi \neq \chi(z)$ & $\left.\frac{dx}{dy}\right|_\chi = \frac{u}{v} = p_x / p_y = f(x,y)$

& in z direction χ are
uniform on top of each other

$\omega_z = -u_y + v_x = \left[-p_{xy} + p_{yx}\right] \frac{h^2}{2\mu} \left[\left(\frac{z}{h}\right)^2 - \frac{z}{h}\right]$
$= 0$

& continuity $u_x + v_y = 0$ ie $\nabla \cdot \underline{V} = 0$

& $\underline{V} = \nabla \varphi$

However, p not governed
Bernoulli equation.

Such that $\nabla^2 \varphi = 0$
$=$ ideal flow

$u_x = a\, p_{xx}$ & $v_y = a\, p_{yy}$ $a = \frac{h^2}{2\mu}\left[\left(\frac{z}{h}\right)^2 - \frac{z}{h}\right]$
$\therefore$ $p_{xx} + p_{yy} = 0$ as per Stokes flow

$\frac{1}{4} - \frac{1}{2}$
$= -\frac{1}{4}$

$\underline{V} = u\hat{\imath} + v\hat{\jmath}$
$u = a p_x$
$v = a p_y$

$\nabla p = p_x \hat{\imath} + p_y \hat{\jmath} = a(u\hat{\imath} + v\hat{\jmath})$
ie $\nabla p \propto \underline{V}$ and aligned $\underline{V}$
$\underline{V} \times \nabla p = 0$
$(u\hat{\imath} + v\hat{\jmath}) \times (p_x\hat{\imath} + p_y\hat{\jmath}) = 0$ $\hat{\imath}\hat{\jmath}\hat{k}\hat{\imath}\hat{\jmath}\hat{k}$

$(u,v)_{max}$ @ $z = h/2$

$u p_y \hat{k} - v p_x \hat{k} = 0$
$a\, p_x p_y - a\, p_y p_x = 0 = a(p_x p_y - p_y p_x)$

$p_x = -\frac{8\mu}{h^2} u_0$ $p_y = -\frac{8\mu}{h^2} v_0$
$\underline{V_0} = u_0 \hat{\imath} + v_0 \hat{\jmath}$

$\Delta p = \int (p_x dx + p_y dy) = -\frac{8\mu}{h^2}\left[\int u_0\, dx + \int v_0\, dy\right]$

$\hat{e}_s = dx\hat{\imath} + dy\hat{\jmath} = \underline{ds}$

$p(x,y) - p_0 = -\frac{8\mu}{h^2} \int_{\chi_{\text{midplane}}} V_0\, ds$

$\underline{V_0} = u_0\hat{\imath} + v_0\hat{\jmath} = V_0\, \underline{ds}$
$\underline{V_0} \cdot \underline{ds} = u_0\, dx + v_0\, dy$

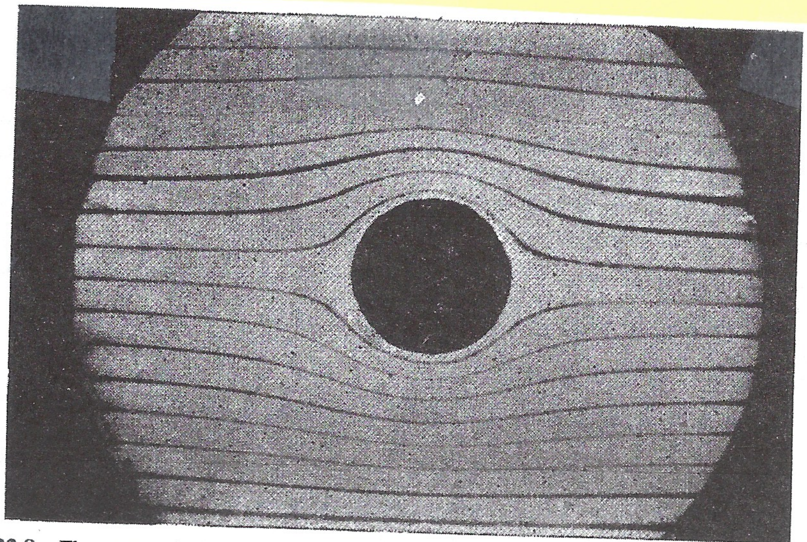


Figure 22.8 Flow around circular cylinder model in a Hele-Shaw apparatus. Streamlines form the potential flow pattern. Photograph courtesy of Professor Bloor, University of Edinburgh, is attributed to Institute of Naval Architects in Bloor (2008).

Example : circular cylinder

potential flow solution $\underline{V} = u_r \hat{e}_r + u_\theta \hat{e}_\theta$

$u_r = U(1 - r_0^2/r^2)\cos\theta$ $\quad u_\theta = -U(1 + r_0^2/r^2)\sin\theta$

U = midplane velocity far from cylinder along

Stagnation streamline intersects body

at $r = R$ & $\theta = \pi$ where $V_0 = u_r$

$\uparrow$ far from cylinder

$$p(r,\pi) - p(R,\pi) = -\frac{8\mu U}{h^2}\left(R + \frac{r_0^2}{R} - r - \frac{r_0^2}{r}\right)$$

Then from $r = r_0$ to $\theta = \pi$ $\quad V_0 = u_\theta$

$$p(r_0,\theta) - p(r_0,\pi) = \frac{+16\mu U}{h^2}(1 - \cos\theta)$$

Along the stagnation point & around the cylinder the pressure decreases

$$p(x,y) - p_0 = -\frac{8\mu}{h^2}\int_{x} V_0 \, ds$$

$$p(r,\pi) - p(R,\pi) = -\frac{8\mu}{h^2}\int_{r=R}^{r} \bar{v}\left(1-\frac{r_0^2}{r^2}\right)\cos\theta \, dr$$

$$= \frac{8\mu\bar{v}}{h^2}\int_{R}^{r} \left(1 - r_0^2 r^{-2}\right) dr$$

$$= \frac{8\mu\bar{v}}{h^2}\left[r + r_0^2 r^{-1}\right|_{R}^{r}$$

$$= -\frac{8\mu\bar{v}}{h^2}\left[R + \frac{r_0^2}{R} - r - \frac{r_0^2}{r}\right]$$

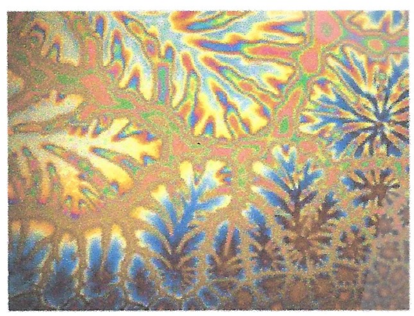
$$p(r_0,\theta) - p(r_0,\pi)$$

$$= +\frac{8\mu}{h^2}\int_{\theta=\pi}^{\theta} \bar{v}\left(1 + r_0^2/r_0^2\right)\sin\theta \, d\theta$$

$$= \frac{16\mu\bar{v}}{h^2}\left(-\cos\theta\right|_{\pi}^{\theta}$$

$$= +\frac{16\mu\bar{v}}{h^2}\left(-\cos\theta + 1\right)$$

$$= \frac{-16\mu\bar{v}}{h^2}\left(\cos\theta - 1\right)$$



An example of viscous fingering in a Hele-Shaw cell.

The **Saffman–Taylor instability**, also known as **viscous fingering**, is the formation of patterns in a morphologically unstable interface between two fluids in a porous medium, described mathematically by Philip Saffman and G. I. Taylor in a paper of 1958.[1][2] This situation is most often encountered during drainage processes through media such as soils.[3] It occurs when a less viscous fluid is injected, displacing a more viscous fluid; in the inverse situation, with the more viscous displacing the other, the interface is stable and no instability is seen. Essentially the same effect occurs driven by gravity (without injection) if the interface is horizontal and separates two fluids of different densities, the heavier one being above the other: this is known as the Rayleigh-Taylor instability. In the rectangular configuration the system evolves until a single finger (the Saffman–Taylor finger) forms, whilst in the radial configuration the pattern grows forming fingers by successive tip-splitting.[4]

Most experimental research on viscous fingering has been performed on Hele-Shaw cells, which consist of two closely spaced, parallel sheets of glass containing a viscous fluid. The two most common set-ups are the channel configuration, in which the less viscous fluid is injected at one end of the channel, and the radial configuration, in which the less viscous fluid is injected at the centre of the cell. Instabilities analogous to viscous fingering can also be self-generated in biological systems.[5]