Axisymmetric round/circular jet

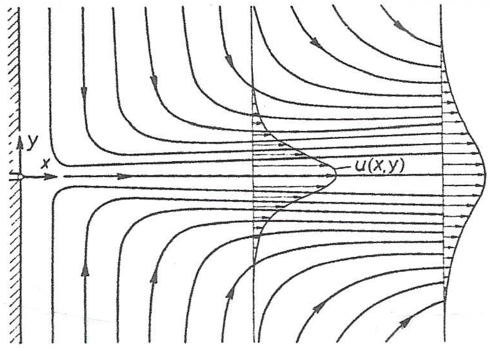


Fig. 11.5. Streamline pattern for a circular laminar jet

$Re = \frac{u_0 r_0}{\nu}$ large $\quad \frac{\partial}{\partial r} \gg \frac{\partial}{\partial x} \quad$ (x,r) coordinate and $\underline{u} = (u, v)$

p = constant and jet narrow with slow growth rate

As with 2D jet: $J = 2\pi\rho \int_0^\infty u^2 r \, dr$ = constant (1)

$u_x + \frac{1}{r}(rv)_r = 0$ (2) $\qquad u_x + v_r + \frac{v}{r} = 0$

$u u_x + v u_r = \frac{\nu}{r}(r u_r)_r$ (3) $\qquad$ RHS $= \frac{\nu}{r}(u_r + r u_{rr})$

$= \nu(u_r/r + u_{rr})$

$u_y = 0$ ($u = u_{max}$) and $v = 0$ $\qquad r = 0$

$u = 0$ $\qquad r = \infty$

Assume similarity solution for $u(x,y)$ with jet width $\propto x^n$ and stream function

$\psi \propto x^p F(\eta)$ and $\eta = r/x^n = r x^{-n}$

$r = \eta x^n \propto x^n$

p and n determined $J \neq f(x)$ and LHS and RHS (3) same order of magnitude.

Note: $u = \frac{1}{r}\psi_r$ and $v = -\frac{1}{r}\psi_x$ $\qquad m^{-1} \times \psi_r = \frac{m}{s}$

$u \propto x^{p-2n}$ ie $r^{-1}\psi_r \propto \frac{x^p}{x^n} x^{-n}$

$u_x \propto x^{p-2n-1}$ $u_r \propto x^{p-2n} x^{-n} = x^{p-3n}$ $u_{rr} \propto x^{p-3n} x^{-n}$

$\propto x^{p-4n}$

From (1) $u^2 v u_r \propto$ $2p-4n+2n=0$ $p=n$

From (3) $uu_x \propto r^{-1}(ru_r)_r$ $2p-4n-1 = p-4n$

$-2p-1 = -3p$ $p=1$

$\frac{m3}{5}$ So $\psi = \nu x f(\eta)$ $\eta = r/x$ $\eta_r = x^{-1}$ $\eta_{rr}=0$

$r = x\eta$ $\eta_x = -rx^{-2}$

$u = r^{-1}\psi_r = r^{-1}\nu x f' x^{-1}$ $= -\eta/x$

$= \frac{\nu f'}{r} = \nu f'/x\eta$

$v = -r^{-1}\psi_x = -r^{-1}\nu\left[f - xf'\eta/x\right] = \frac{-\nu}{r}\left[f-\eta f'\right]$

$= \nu\left[\frac{\eta}{r}f' - f/r\right]$

$u_x = \nu r^{-1} f''(-rx^{-2}) = -\nu f'' x^{-2}$ $= \frac{\nu}{x}\left[f' - f/\eta\right]$

$u_r = \nu\left[-f' r^{-2} + r^{-1} f'' x^{-1}\right]$

$= \frac{\nu}{r}\left[-f'/r + f''/x\right] = \nu\left[-f'/x^2\eta^2 + f''/x^2\eta\right]$

$u_{rr} = \nu\left[-r^{-2}(-f'/r + f''/x) + r^{-1}(f'/r^2 - r^{-1}f''x^{-1} + f'''x^{-2})\right]$

$= \nu\left[f'/r^3 - f''/xr^2 + f'/r^3 - f''/xr^2 + f'''/rx^2\right]$

$= \nu\left[2f'/r^3 - 2f''/xr^2 + f'''/rx^2\right]$

$= \nu\left[2f'/x^3\eta^3 - 2f''/x^3\eta^2 + f'''/x^3\eta\right]$

✓ ✓ ✓ ✓

$\frac{\nu f'}{x\eta}\left[-\nu f''/x^2\right] + \frac{\nu}{x}\left[f' - f/\eta\right]\left[\nu(-f'/x^2\eta^2 + f''/x^2\eta\right]$

$= \nu\left[\nu(-f'/x^3\eta^3 + f''/x^3\eta^2) + \nu(2f'/x^3\eta^3 - 2f''/x^3\eta^2 + f'''/x^3\eta)\right]$

$$F'F''\left[-\nu^2/\cancel{x^3}\eta + \nu^2/\cancel{x^3}\eta\right] + F'^2\left[-\cancel{\nu^2}/\cancel{x^3}\eta^2\right] + FF'\left[\cancel{\nu^2}/\cancel{x^3}\eta^3\right]$$

$$- FF''\left[\cancel{\nu^2}/\cancel{x^3}\eta^2\right] = -F'\left[\cancel{\nu^2}/\cancel{x^3}\eta^3\right] + F''\left[\cancel{\nu^2}/\cancel{x^3}\eta^2\right]$$

$$+ F'\left[2\cancel{\nu^2}/\cancel{x^3}\eta^3\right] - F''\left[2\cancel{\nu^2}/\cancel{x^3}\eta^2\right] + F'''\left[\cancel{\nu^2}/\cancel{x^3}\eta\right]$$

$$-F'^2/\eta^2 + FF'/\eta^3 - FF''/\eta^2 = -F'/\eta^3 + F''/\eta^2 + 2F'/\eta^3$$
$$-2F''/\eta^2 + F'''/\eta$$

$$FF'/\eta^2 - F'^2/\eta - FF''/\eta = F'/\eta^2 - F''/\eta + F''' = \frac{d}{d\eta}\left(F'' - F'/\eta\right)$$

$$\frac{d}{d\eta}\left(-FF'/\eta\right) = \eta^{-2}FF' - \eta^{-1}\left(F'^2 + FF''\right)$$

$u = u_{max} \wedge v = 0 \quad y = 0$ $\qquad \frac{d}{d\eta}\left(-F'/\eta\right) = -F''/\eta + F'/\eta^2$

$F'(0) = 0 \quad F(0) = 0$

Since u even $f(\eta)$, F'/η = even, F' = odd, F = even

1st integration $\frac{d}{d\eta}\left(-FF'/\eta\right) = \frac{d}{d\eta}\left(F'' - F'/\eta\right)$

$$FF' = F' - \eta F'' + \cancel{c_1} \qquad F(0) = F'(0) = 0$$
$$\therefore c_1 = 0$$

rewrite equation: $\frac{d}{d\eta}\left(\frac{F^2}{2}\right) = 2F' - \frac{d}{d\eta}\left(\eta F'\right)$

$$\int \underbrace{\frac{dF}{\frac{1}{2}(4F - F^2)}} \qquad F^2/2 = 2F - \eta F' + \cancel{c}$$

$dF \div dx$

$X = a + bx + cx^2$ $\qquad$ ie $\quad (2F - F^2/2)^{-1} F' = \eta^{-1}$

$a = 0 \quad b = 2 \quad c = -\frac{1}{2}$

$q = 4ac - b^2 = -4 \quad \sqrt{-q} = 2$ $\qquad \frac{1}{2}\ln\left(\frac{F}{4-F}\right) = \ln\eta + c$

$$\frac{1}{2}\ln\frac{-F}{-F+4} = \frac{1}{2}\ln\frac{F}{4-F} \qquad \ln\left(\frac{F}{4-F}\right) = 2\ln\eta + 2c = \ln\eta^2 + 2c = \ln\eta^2 + \ln a$$
$$= \ln a\eta^2$$

$$F/(4-F) = a\eta^2$$
$$F = a\eta^2(4-F) = 4a\eta^2 - a\eta^2 F$$
$$F = 4a\eta^2/(1+a\eta^2)$$

$2\sqrt{a}\,\eta = \xi \qquad \xi^2 = 4a\eta^2 \qquad \gamma = 2\sqrt{a}$ ie $\xi = \gamma\eta = \gamma\, r/x$

$\xi^2/4 = a\eta^2 \qquad d\xi = \gamma\, d\eta \quad \xi_\eta = \gamma$

$$F = \xi^2/(1+\tfrac{1}{4}\xi^2) = \xi^2(1+\tfrac{1}{4}\xi^2)^{-1}$$

$$dF/d\xi = \left[2\xi(1+\tfrac{1}{4}\xi^2)^{-1} - \xi^2\left(1+\tfrac{\xi^2}{4}\right)^{-2}\left(\tfrac{\xi}{2}\right)\right]$$

$\eta = r/x \qquad u = \frac{\nu}{x}\frac{F'}{\eta}$

$$= \left[2\xi(1+\tfrac{1}{4}\xi^2)^{-1} - \tfrac{\xi^3}{2}\left(1+\tfrac{\xi^2}{4}\right)^{-2}\right]$$

$$= \frac{\nu}{x\eta}\gamma\frac{dF}{d\xi} \qquad = \frac{2\xi(1+\xi^2/4) - \xi^3/2}{\left(1+\frac{\xi^2}{4}\right)^2} = 2\xi + \cancel{\xi^3/2} - \cancel{\xi^3/2}$$

$$= \frac{\nu}{x\xi}\gamma^2\frac{dF}{d\xi}$$

$$u = \frac{\nu}{x}\frac{2\gamma^2}{(1+\xi^2/4)^2}$$

$$v = \frac{\nu}{x}(F' - F/\eta) = \frac{\nu}{x}\gamma\left(\frac{dF}{d\xi} - \frac{F}{\xi}\right) = \frac{\nu}{x}\gamma\left[\frac{2\xi}{\left(1+\frac{\xi^2}{4}\right)^2} - \frac{\xi}{\left(1+\frac{\xi^2}{4}\right)}\right]$$

$$J = 2\pi\rho\int_0^\infty u^2 r\,dr \qquad v = \frac{\nu}{x}\gamma\left[\frac{\xi - \xi^3/4}{\left(1+\frac{\xi^2}{4}\right)^2}\right]$$

$X = a + bx + cx^2 \qquad = 2\pi\rho\int_0^\infty\left[\frac{\nu}{x}\frac{2\gamma^2}{\left(1+\frac{\xi^2}{4}\right)^2}\right]^2 r\,dr \qquad \xi = \gamma r/x \qquad r = \frac{x}{\gamma}\xi$

$a=1 \quad b=0 \quad c=\frac{1}{4} \qquad d\xi = \frac{\gamma}{x}dr \qquad dr = \frac{x}{\gamma}d\xi$

$q = 4ac - b^2 = 1 \qquad = 2\pi\rho\frac{\nu^2}{x^2}4\gamma^4\frac{x^2}{\gamma^2}\int_0^\infty\frac{\xi\, d\xi}{\left(1+\frac{\xi^2}{4}\right)^4}$

$\int\frac{x}{X^4}dx = \qquad r\,dr = \frac{x^2}{\gamma^2}\xi\, d\xi$

$-\frac{2a+bx}{3qX^3} - \frac{3b}{3q}\int\frac{dx}{X^3} \qquad = 8\pi\rho\nu^2\gamma^2\left[\frac{-2}{3\left(1+\frac{\xi^2}{4}\right)^3}\right]_0^\infty$

$= \frac{-2}{3\left(1+\frac{\xi^2}{4}\right)^3} \qquad J = \frac{16}{3}\pi\rho\gamma^2\nu^2 \qquad$ Kinematic momentum $= K = J/\rho$

$$\left[\frac{3J}{16\pi\rho\nu^2}\right]^{1/2} = \gamma = \left[\frac{3K}{16\pi\nu^2}\right]^{1/2}$$

$$u = \frac{2\nu}{x}\left[\frac{3K}{16\pi\nu^2}\right]\left(1+\frac{\xi^2}{4}\right)^{-2} = \underbrace{\frac{3}{8\pi}\,\frac{K}{\nu x}}_{u_{max}}\underbrace{\left(1+\frac{\xi^2}{4}\right)^{-2}}_{\text{jet profile differs 2D jet}} \qquad u_{max} \propto x^{-1}$$

$$v = \underbrace{\frac{\nu}{x}\left[\frac{3K}{16\pi\nu^2}\right]^{1/2}}_{\frac{1}{4x}\left[\frac{3K}{\pi}\right]^{1/2}}\left(\xi - \xi^3/4\right)\left(1+\frac{\xi^2}{4}\right)^{-2}$$

$$\xi = \gamma y = \left[\frac{3K}{16\pi}\right]^{1/2}\frac{y}{\nu x}$$

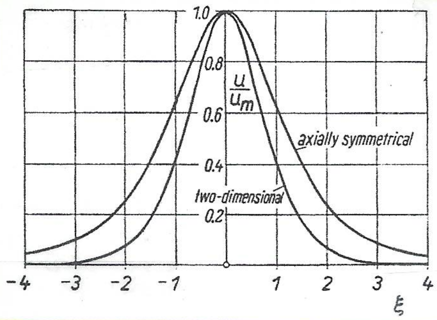


Fig. 9.13. Velocity distribution in a two-dimensional and circular free jet from eqns. (9.44) and (11.15) respectively. For the two-dimensional jet $\xi = 0{\cdot}275\, K^{1/3}\, y/(\nu x)^{2/3}$, and for the circular jet $\xi = 0{\cdot}244\, K'^{1/2}\, y/\nu x$. K and K' denote the kinematic momentum J/ϱ

$$Q = 2\pi\int_0^\infty u r\,dr = \frac{\Psi}{S} = 2\pi\int_0^\infty \frac{\nu}{x}\,\frac{2\gamma^2}{\left(1+\frac{\xi^2}{4}\right)^2}\, r\,dr$$

$$-\frac{3x+2a}{8x} - \frac{3}{8}\int\frac{x\,dx}{x} \qquad = \underbrace{4\pi\gamma^2\frac{\nu}{x}\,\cancel{\frac{x^2}{\gamma^2}}}_{4\pi\nu x}\int_0^\infty \frac{\xi\,d\xi}{\left(1+\frac{\xi^2}{4}\right)^2} = 4\pi\nu x\left[\frac{-2}{\left(1+\frac{\xi^2}{4}\right)}\right]_0^\infty$$

$$= \frac{-2a}{x} = \frac{-2}{\left(1+\frac{\xi^2}{4}\right)} \qquad Q = 8\pi\nu x \neq f(K)$$

2D jet $Q = 3.3019\,(K\nu x)^{1/3}$ $\qquad \dot m = 8\pi\mu x$

Interesting axisymmetric jet $Q = f(x)$ ~~but~~ $\neq f(J)$ ie jet with large J/K has large γ is narrower vs jet with small J/K and small γ. Small γ jet carries more external flow such that both have same Q if both have same ν.

$$\xi = \gamma y = \gamma\, r/x \Rightarrow r = \frac{x\xi}{\gamma}$$

large γ small r
vs small γ large r

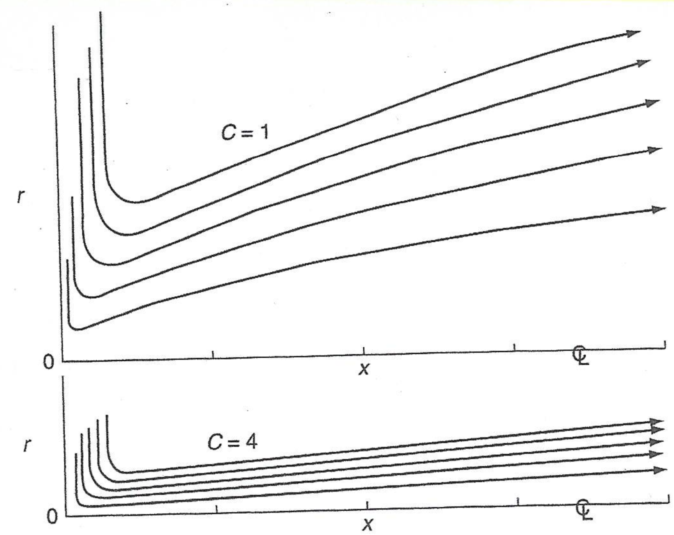


FIGURE 4-37
Streamlines of a round jet for two cases, from the boundary-layer theory of Eq. (4-205).

$C = \gamma$: $\gamma = 1$ not narrow so BL assumptions weak

$\gamma \geq 4$ narrow & BL assumptions OK

$Re = J/\rho\nu^2 > 250$ White

$Re = \frac{1}{\nu}\left(\frac{J}{2\pi\rho}\right)^{1/2} = \frac{1}{\nu}\left(\frac{K}{2\pi}\right)^{1/2}$ Schlichting

$\gg 1$

Landau–Squire jet

In fluid dynamics, **Landau–Squire jet** or **Submerged Landau jet** describes a round submerged jet issued from a point source of momentum into an infinite fluid medium of the same kind. This is an exact solution to the incompressible form of the Navier-Stokes equations, which was first discovered by Lev Landau in 1944[1][2] and later by Herbert Squire in 1951.[3] The self-similar equation was in fact first derived by N. A. Slezkin in 1934,[4] but never applied to the jet. Following Landau's work, V. I. Yatseyev obtained the general solution of the equation in 1950.[5]

Yih (1969)

Limiting behaviors

When the force becomes large: he jet in this limiting case is called the Schlichting jet.

$Re = \frac{u_{max} R}{\nu}$ $\qquad u_{max} = \frac{3K}{8\pi\nu x} = \frac{3J}{8\pi\mu x}$ $\qquad \gamma = \left[\frac{3K}{16\pi\nu^2}\right]^{1/2}$

$\xi = \gamma\eta = \gamma r/x \Rightarrow r = \frac{x\xi}{\gamma}$

$Re = \nu^{-1}\left[\frac{3K}{8\pi\nu x}\right] x\xi\nu\left[\frac{16\pi}{3K}\right]^{1/2}$ $\qquad = x\xi\left[\frac{16\pi\nu^2}{3K}\right]^{1/2}$

$= \nu^{-1}\left[\frac{9K^2}{64\pi^2}\,\frac{16\pi}{3K}\right]^{1/2}\xi$

$= \nu^{-1}\left[\frac{3K}{4\pi}\right]^{1/2}\xi = \nu^{-1}\left[\frac{3K\xi^2}{4\pi}\right]^{1/2}$ $\qquad \left[\frac{3}{4}\right]^{1/2}\xi = \left[\frac{1}{2}\right]^{1/2}$ $\qquad \frac{3}{4}\xi^2 = \frac{1}{2}$

$\xi^2 = \frac{4}{6} = \frac{2}{3}$ $\quad \xi = \frac{2}{\sqrt{6}}$ $\quad .8165$

$u/u_{max} = \frac{1}{2} = \left(1+\frac{\xi^2}{4}\right)^{-2}$ $\qquad \frac{u}{u_{max}} = \left(1+\frac{\xi^2}{4}\right)^{-2}$ $\qquad \frac{\frac{4}{6}}{4} = \frac{1}{6}$

$\left(1+\frac{\xi^2}{4}\right)^2 = 2$ $\qquad = \left(1+\frac{1}{6}\right)^{-2}$

$Re = \nu^{-1}\left[\frac{6\xi^2}{4}\right]^{1/2}\left[\frac{K}{2\pi}\right]^{1/2}$ $\qquad \left(1+\frac{\xi^2}{4}\right) = \sqrt{2}$ $\qquad = 1/\left(\frac{7}{6}\right)^2 = 36/49 = .7347$

For Schlichting

$\frac{\xi^2}{4} = \sqrt{2}-1$

$\frac{6}{4}\,4(\sqrt{2}-1)$ $\qquad \xi^2 = 4(\sqrt{2}-1)$ $\qquad Re = \nu^{-1}\left[\frac{3K}{4\pi}\xi^2\right] = \left[6(\sqrt{2}-1)\frac{K}{2\pi}\right]^{1/2}$

$= 6(\sqrt{2}-1)$ $\qquad \xi = 2(\sqrt{2}-1)^{1/2} = 2\sqrt{.4142}$

$Re = [\underbrace{6(\sqrt{2}-1)}_{2.4856}]^{1/2} Re_{Schlichting}$ $\qquad = 1.2872$

$= 1.5765\, Re_S$

$\sim \pi/2\; Re_{Schlichting}$

inlet kinematic momentum $J/\rho = \int_0^{R_0} u^2(2\pi r)\,dr = u^2(2\pi)\frac{r^2}{2}\Big|_0^{R_0} = u^2\pi R_0^2 = u^2 A_0$

$A_0 = \pi D^2/4$

$J \quad \frac{m^2}{s^2}\times m^2\,\rho$ $\qquad K = u^2 A_0 \Rightarrow u = (K/A_0)^{1/2}$

$\rho = \frac{kg}{m^3}$ $\qquad Re = uD/\nu = \left(\frac{K}{A_0}\right)^{1/2} D/\nu = \frac{K^{1/2}D}{\left(\frac{\pi D^2}{4}\right)^{1/2}\nu} = \frac{2}{\sqrt{\pi}}\,\frac{K^{1/2}}{\nu}$

$\nu = \frac{m^2}{s}$

$w = kg\,\frac{m}{s^2}$

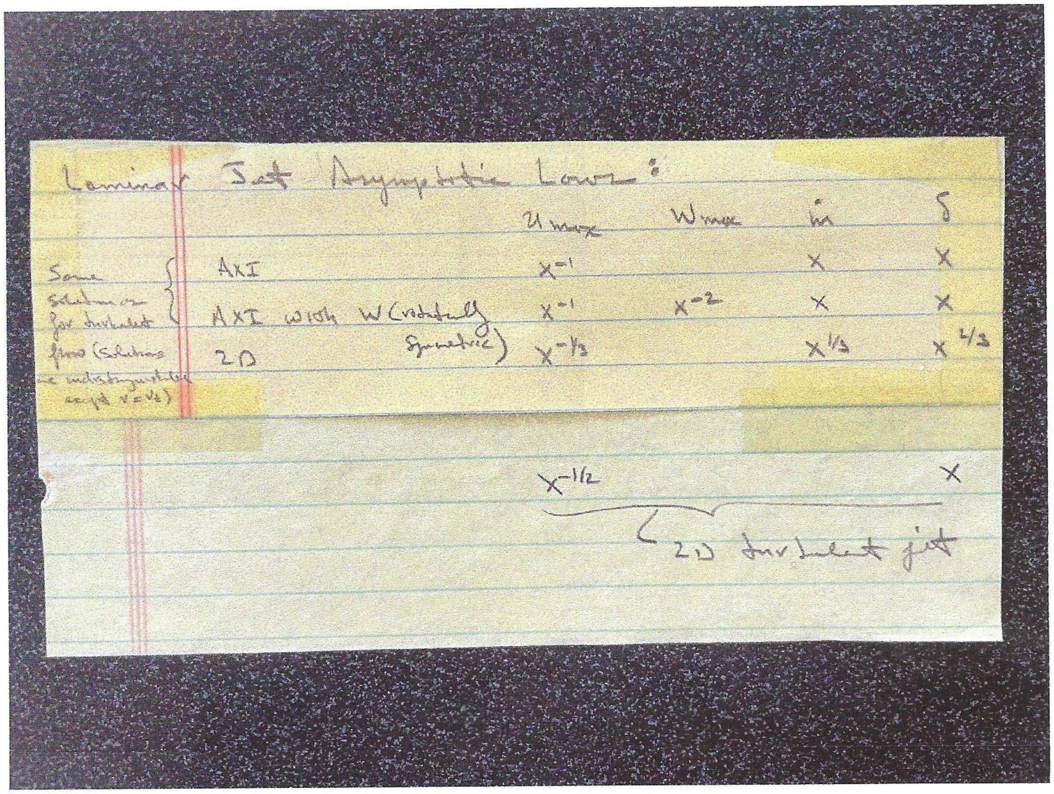
$\frac{ws^2}{m} = kg$

$\rho = \frac{ws^2}{m^4}$

$K = \frac{J}{\rho} \quad \frac{m^4}{s^2} \quad \sqrt{K}/\nu$ = OK

$$\boxed{Re = \frac{J}{\rho\nu^2} = \frac{K}{\nu^2} \qquad Re^{1/2} = K^{1/2}/\nu}$$

White



Journal of Engineering Mathematics **26**: 339–348, 1992.

The swirling round laminar jet

WEI-SHIEN HWANG* and ALLEN T. CHWANG

Department of Mechanical Engineering, Institute of Hydraulic Research, The University of Iowa, Iowa City, IA 52242, U.S.A.

Received 2 November 1990; accepted in revised form 7 July 1991

Abstract. The swirling round laminar jet in an unbounded viscous fluid is investigated in this paper. The axisymmetric laminar jet with a swirling velocity is simulated by a linear-momentum source and an angular-momentum source, both located at the origin. The first-order and the second-order solutions in the far field have been obtained by solving the complete Navier–Stokes equations. It is found that the first-order solution is the well-known round-laminar-jet solution without the swirling velocity obtained by Landau [2] and Squire [3]. The second-order solution represents a pure rotating flow. The swirling velocity predicted by the present solution is compared with that obtained by Loitsyanskii [15] and Görtler [16], who solved the corresponding boundary-layer equations. It is found that the swirling velocity predicted by the present theory is smaller than that obtained from the boundary-layer equations.