Planar Jet

Width h

$Re = u_0 h/\nu$ large

At sufficient distance orifice all jets decay with similarity independent $u_0(x)$

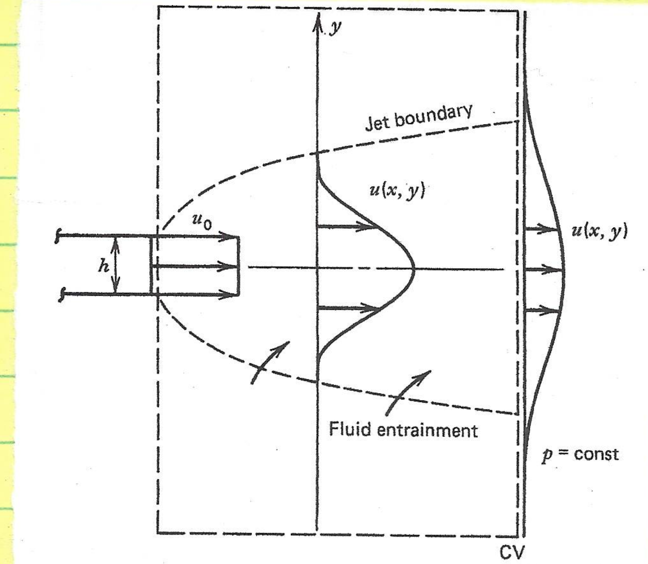

Figure 20.21 Plane laminar jet into an infinite medium.

Assume $p_{inlet} = p_{exit}$

ie p = constant & $p_x = 0$; and $p_y = 0$ along with BL assumptions

$$\int_{cs} \rho \underline{V}\cdot\underline{n}\, dA = 0 = -\rho u_0 h + \int_{-\infty}^{\infty} \rho u\, dy + \dot m_t + \dot m_b$$

$$\dot m_t + \dot m_b = \rho u_0 h - \int_{-\infty}^{\infty} \rho u\, dy$$

$$\int_{cs} \rho u \underline{V}\cdot\underline{n}\, dA = 0 = -\rho u_0^2 h + \int_{-\infty}^{\infty} \rho u^2 dy + u_{t,b}(\dot m_t + \dot m_b) \quad u_{t,b} = 0$$

$$M = \rho u_0^2 h = \int_{-\infty}^{\infty} \rho u^2 dy = \text{constant}$$

Since jet entrains ambient fluid, flow rate $f(x)$

Assume $\psi(x,y)$ exists with $u = \psi_y$ $v = -\psi_x$

$$u u_x + v u_y = \nu u_{yy}$$

$$\psi_y \psi_{yx} - \psi_x \psi_{yy} = \nu \psi_{yyy}$$

Ax^p scales χ and Bx^q scales y

$\chi = Ax^p f(\eta)$ $\qquad \eta = y/Bx^q$ $\qquad$ p, q part of solution

$\quad = a(x) f(\eta)$ $\qquad \eta = y/b(x)$ $\qquad$ & A, B make f & η dimensionless

$$(a_x b - a b_x) f'^2 - a_x b f f'' - \nu f''' = 0$$

$a_x = Ap x^{p-1}$ $\qquad a_x b = Ap x^{p-1} B x^q = ABp x^{p+q-1}$

$b_x = Bq x^{q-1}$ $\qquad a b_x = Ax^p B q x^{q-1} = ABq x^{p+q-1}$

$$f'^2 (ABp x^{p+q-1} - ABq x^{p+q-1}) - ABp x^{p+q-1} f f'' - \nu f''' = 0$$

$$\frac{AB x^{p+q-1}}{\nu} \left[(p-q) f'^2 - p f f'' \right] = f'''$$

$u = 0 \quad \eta = \pm\infty$ $\qquad u_y = 0 \quad \eta = 0$ $\qquad v = 0 \quad \eta = 0$

$f'(\pm\infty) = 0$ $\qquad f''(0) = 0$ $\qquad f(0) = 0$

For similarity $\dfrac{AB x^{p+q-1}}{\nu} \neq f(x) \Rightarrow p + q = 1$

$$M = \text{constant} = \int_{-\infty}^{\infty} \rho u^2 dy = \int_{-\infty}^{\infty} \rho \chi_y^2 dy$$

$\chi_y = Ax^p f' / Bx^q = \dfrac{Ax^p}{Bx^q} f'(\eta)$ $\qquad \eta = \dfrac{y}{Bx^q}$

$M = \displaystyle\int_{-\infty}^{\infty} \rho \frac{A^2 x^{2p}}{(Bx^q)^2} Bx^q f'^2 d\eta$ $\qquad Bx^q d\eta = dy$

$\quad = \rho A^2 B^{-1} x^{2p-q} \displaystyle\int_{-\infty}^{\infty} f'^2 d\eta \neq f(x) \Rightarrow 2p - q = 0$

$\qquad 2p = q$

$p + 2p = 1 \Rightarrow p = \frac{1}{3}$ $\qquad q = 2/3$

$$-f''' + \frac{AB}{\nu}\left[-\frac{1}{3}f'^2 - \frac{1}{3}ff''\right] = 0$$

$$f''' + \frac{AB}{3\nu}\underbrace{\left[f'^2 + ff''\right]}_{\frac{d}{d\eta}(f'f)} = 0$$

$$\frac{d}{d\eta}(f'')$$

integrate $\pm\infty$ $\quad f'' + \frac{AB}{3\nu}f'f = c_1 = 0 \qquad \eta = \pm\infty \quad f' = f'' = 0$

$$\frac{d}{d\eta}(f') + \frac{AB}{3\nu}\frac{d}{d\eta}(f^2/2) = 0$$

$$f' + \frac{AB}{6\nu}f^2 = c_2$$

$\eta = 0 \quad f(0) = 0$

$c_2 = f'(0)$

$$\frac{df}{d\eta} + af^2 = 1$$

$u = \frac{Ax^{1/3}}{Bx^{2/3}} = \frac{A}{B}x^{-1/3}f'$

$$df = (1 - af^2)\,d\eta$$

$u(0) = u_{max}f'(0)$

$f = a^{-\frac{1}{2}}\tanh\beta$

$df = a^{-\frac{1}{2}}\operatorname{sech}^2\beta\, d\beta$

$$\frac{df}{(1-af^2)} = d\eta$$

$f'(0) = 1$

$c_2 = 1$

$$\int \frac{a^{-1/2}\operatorname{sech}^2\beta\, d\beta}{(1-\tanh^2\beta)} = \int d\eta$$

$a = \frac{AB}{6\nu}$

$$\beta = \eta a^{1/2} + c_3 = \tanh^{-1} f a^{1/2}$$

$A = \left(\frac{9\nu M}{2e}\right)^{1/3}$

$$f = a^{-1/2}\tanh(\eta a^{1/2} + c_3)$$

$$f = \tanh(\eta + c_3)$$

$B = \left(\frac{48\nu^2 e}{M}\right)^{1/3}$

$$f(0) = 0 \Rightarrow c_3 = 0$$

$AB = \left(\frac{9\nu \cancel{M}}{2\cancel{e}} \times \frac{48\nu^2\cancel{e}}{\cancel{M}}\right)^{1/3}$

$$f = \tanh\eta$$

$= (216\nu^3)^{1/3} = 6\nu$

$\Rightarrow a = 1$

$$f' = \operatorname{sech}^2\eta$$

$$u = \frac{A}{B} x^{-1/3} f'(\eta) = u_{max} \operatorname{sech}^2 \eta$$

$\frac{3 \times 3}{2 \times 3 \times 16} = \frac{3}{32}$

$$AB^{-1} = \left(\frac{9 \nu m}{2 \rho} \frac{m}{48 \nu^2 \rho} \right)^{1/3} = \left(\frac{3}{32} \frac{m^2}{\nu \rho^2} \right)^{1/3}$$

$$\eta = y/b = y \left(\frac{m}{48 \nu^2 \rho} \right)^{1/3} x^{-2/3}$$

$u_{max} \propto x^{-1/3}$

$h(x) \propto u_{max}/e$ = locus points $y\, x^{-2/3}$ = const.

$h \propto x^{2/3}$

Viscous forces at jet edge accelerate ambient fluid & entrain fluid at rate:

$$Q = \int_{-\infty}^{\infty} u \, dy = \left(\frac{36 m \nu}{\rho} \right)^{1/3} x^{1/3} \quad \propto x^{1/3}$$

$x = 0$ flow singularity $\quad u_{max} = \infty$
due breakdown BL $\quad h = Q = 0$
assumptions, ie,
need replace $x = x - x_0$ $\quad x_0$ = effective origin