Velocity Potential

If $\underline{\omega} = \nabla \times \underline{V} = 0$ then $\underline{V} = \nabla \varphi$

Cartesian coordinates $\underline{V} = u\hat{i} + v\hat{j} + w\hat{k}$

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\underline{\omega} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

$$\nabla \varphi = \frac{\partial \varphi}{\partial x}\hat{i} + \frac{\partial \varphi}{\partial y}\hat{j} + \frac{\partial \varphi}{\partial z}\hat{k}$$

ie $u = \frac{\partial \varphi}{\partial x}$, $v = \frac{\partial \varphi}{\partial y}$ and $w = \frac{\partial \varphi}{\partial z}$ if $\underline{\omega} = 0$

Cylindrical coordinates $\underline{V} = v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z$

$$\nabla = \frac{\partial}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{e}_\theta + \frac{\partial}{\partial z}\hat{e}_z$$

$$\underline{\omega} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\hat{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{e}_\theta + \left[\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]\hat{e}_z$$

$$\nabla \varphi = \frac{\partial \varphi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial \varphi}{\partial \theta}\hat{e}_\theta + \frac{\partial \varphi}{\partial z}\hat{e}_z$$

ie $v_r = \frac{\partial \varphi}{\partial r}$, $v_\theta = \frac{1}{r}\frac{\partial \varphi}{\partial \theta}$ and $v_z = \frac{\partial \varphi}{\partial z}$ if $\underline{\omega} = 0$

For incompressible flow: $\nabla \cdot \underline{V} = 0 \Rightarrow \nabla^2 \varphi = 0$

Stream function

Restricted 2D flow

Cartesian coordinates: $w = 0$, $\frac{\partial}{\partial z} = 0$

$u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$ $\quad \underline{\omega} = \nabla \times \underline{V} = (0, 0, \omega_z)$ $\quad \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

$\rho\left[\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2 \psi) - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2 \psi)\right] = \mu \nabla^4 \psi$ $\quad \omega_z = 0 \Rightarrow \nabla^2 \psi = 0$

$u = \frac{\partial \psi}{\partial y} = \frac{\partial \varphi}{\partial x}$ $\quad v = -\frac{\partial \psi}{\partial x} = \frac{\partial \varphi}{\partial y}$

Cylindrical Coordinates: $v_\theta = 0$, $\frac{\partial}{\partial \theta} = 0$

$v_r = \frac{1}{r}\frac{\partial \psi}{\partial z}$ $\quad v_z = \frac{1}{r}\frac{\partial \psi}{\partial r}$ $\quad \underline{\omega} = (0, \omega_\theta, 0)$ $\quad \omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}$

$\omega_\theta = 0$ $\quad v_r = \frac{1}{r}\frac{\partial \psi}{\partial z} = \frac{\partial \varphi}{\partial r}$ $\quad v_z = \frac{1}{r}\frac{\partial \psi}{\partial r} = \frac{\partial \varphi}{\partial z}$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$