Flow Between Rotating Cylinders

An incompressible Newtonian liquid of density ρ and dynamic viscosity μ is sheared between concentric cylinders with radius R_1 (inner) and R_2 (outer) rotating at angular velocity ω_1 and ω_2. (a) Simplify the momentum equations and solve the differential equations to get the velocity u_θ and pressure p profiles using the appropriate boundary conditions. (b) For what angular velocity must the outer cylinder rotate for stability assuming the inviscid stability criterion is sufficient, i.e., $d\gamma^2/dr > 0$ where $\gamma = u_\theta r$.

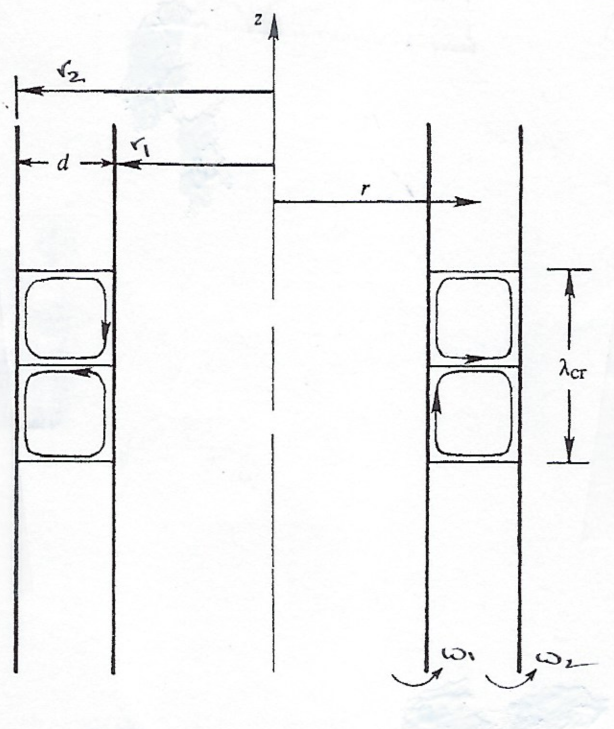

Fig. 11.10 Definition sketch of instability in rotating Couette flow.

(a) $\underline{V} = u_\theta(r)\,\hat{e}_\theta \qquad p = p(r)$

$u_r = u_z = 0 \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial \theta} = \frac{\partial}{\partial z} = 0$

$u_\theta(r_1) = r_1 \omega_1$

$u_\theta(r_2) = r_2 \omega_2$

Continuity: $\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta) + \frac{\partial}{\partial z}(u_z) = 0$

$$\boxed{\begin{aligned}&\frac{1}{r}\frac{d}{dr}\left[r\frac{du_\theta}{dr}\right] - \frac{u_\theta}{r^2} = \\ &\frac{1}{r}\frac{du_\theta}{dr} + \frac{d^2 u_\theta}{dr^2} - \frac{u_\theta}{r^2} \\ &\frac{d^2 u_\theta}{dr^2} + \frac{d}{dr}\left(\frac{u_\theta}{r}\right) = \frac{d}{dr}\left[\frac{du_\theta}{dr} + \frac{u_\theta}{r}\right] \\ &= \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r u_\theta)\right]\end{aligned}}$$

$r u_r = \text{Constant} \qquad u_r = \frac{c}{r}$

$u_r(r_1) = u_r(r_2) = 0 \implies c = 0$

momentum r: $-\frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{dp}{dr}$

outward force
centrifugal a balanced ∇p
$p(r)\uparrow$ due a
Centripetal force

circular Couette flow

① $u_\theta(r)$ driven
no slip satisfies
viscous diffusion

② $\frac{dp}{dr} = \frac{\rho u_\theta^2}{r}$

θ: $0 = \mu\left[\nabla^2 u_\theta - \frac{u_\theta}{r^2}\right]$

$0 = \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) - \frac{u_\theta}{r^2}\right]$

$0 = \mu \frac{d}{dr}\left[\underbrace{\frac{1}{r}\frac{d}{dr}(r u_\theta)}_{=2A}\right]$

$u_\theta(r) = Ar + B/r$

Ar = rigid body rotation
B/r = potential vortex

$\frac{1}{\rho}\frac{\partial p}{\partial r} = u_\theta^2/r$

$$u_\theta(r_1) = Ar_1 + B/r_1 = r_1\omega_1$$
$$u_\theta(r_2) = Ar_2 + B/r_2 = r_2\omega_2$$

$$A = (r_2^2\omega_2 - r_1^2\omega_1)/(r_2^2 - r_1^2)$$

$$B = r_1^2 r_2^2 (\omega_1 - \omega_2)/(r_2^2 - r_1^2)$$

$$p(r)/\rho = \frac{A^2 r^2}{2} + 2AB \ln r - \frac{B^2}{4r^4} + c$$

$$p(r = r_1 \text{ or } r_2) \Rightarrow c$$

$$u_\theta(r) = Ar + B/r$$

(1) rigid body rotation: $\omega_1 = 0$, $r_1 = 0$

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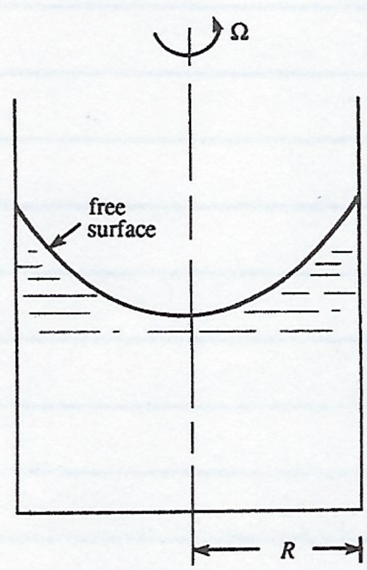
$$u_\theta(r) = \omega_2 r$$



igure 9.8 Steady rotation of a tank containing viscous fluid. The shape of the free surface is also ndicated.

(2) potential vortex : $\omega_2 = 0$, $r_2 \to \infty$

$$u_\theta(r) = \frac{r_1^2 \omega_1}{r}$$

equivalent potential flow vortex driven by inner rotating cylinder with no-slip condition, as per chapter 8

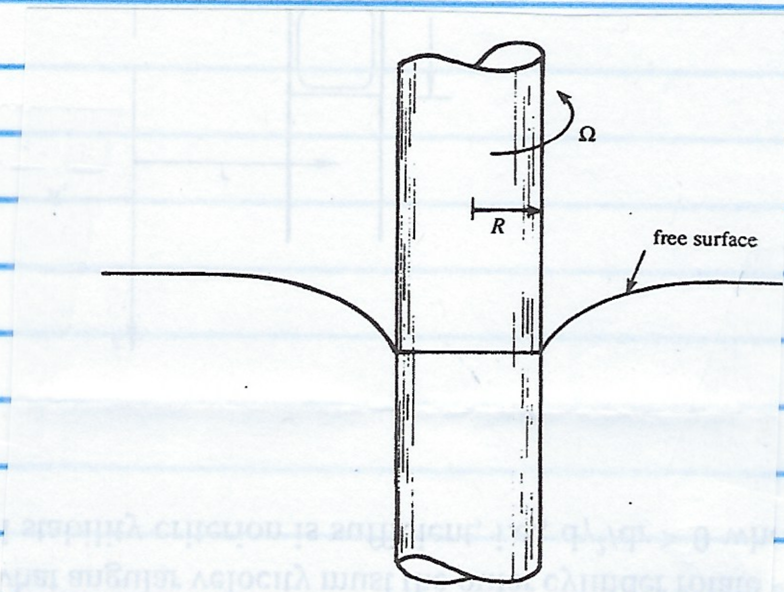


Figure 9.7 Rotation of a solid cylinder of radius R in an infinite body of viscous fluid. The flow field is viscous but irrotational.

8.14 A tornado may be modeled as the circulating flow shown in Fig. P8.14, with $v_r = v_z = 0$ and $v_\theta(r)$ such that

$$v_\theta = \begin{cases} \omega r & r \le R \\ \dfrac{\omega R^2}{r} & r > R \end{cases}$$

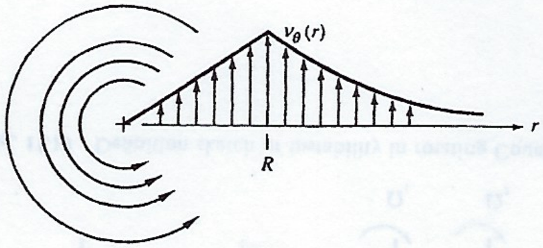
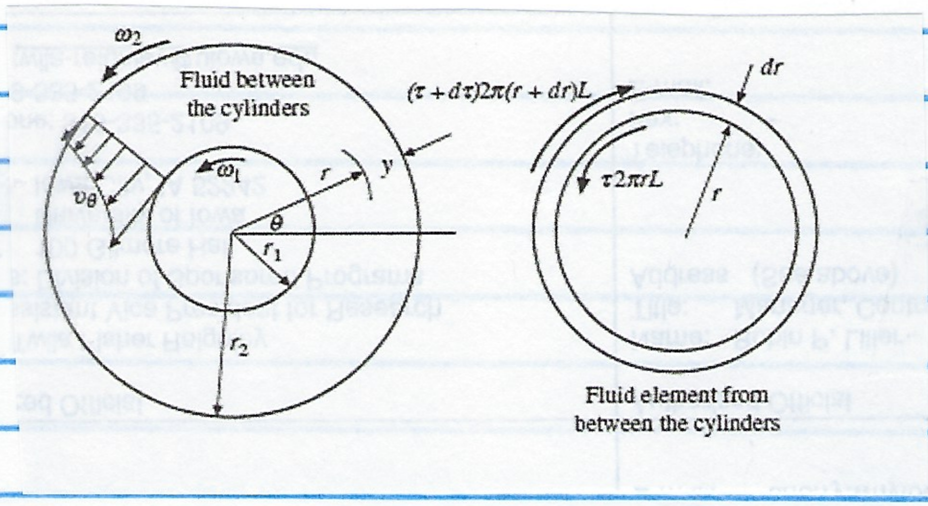


Fig. P8.14

Determine whether this flow pattern is irrotational in either the inner or outer region. Using the r-momentum equation (D.5) of App. D, determine the pressure distribution $p(r)$ in the tornado, assuming $p = p_\infty$ as $r \to \infty$. Find the location and magnitude of the lowest pressure.

$$u_\theta(r) = Ar + B/r$$



1888 Rayleigh's rotating inviscid flow stability criterion
circular streamlines

$$\theta \text{ momentum } \times r = r\left[\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_r u_\theta}{r} + u_z\frac{\partial u_\theta}{\partial z}\right] = 0$$

$$= \frac{D\gamma}{Dt} = \frac{\partial \gamma}{\partial t} + \underline{v}\cdot\nabla\gamma = 0 \quad \gamma = r u_\theta$$

ie axisymmetric disturbance must maintain
γ = constant = reduced circulation

recall definition $\Gamma = \oint \underline{v}\cdot d\underline{x} = \int_0^{2\pi} u_\theta(r)\, r\, d\theta$
circulation

$\Gamma/2\pi = \gamma$ $\quad = 2\pi r u_\theta$

$KE = \frac{1}{2}\rho u_\theta^2 = \frac{1}{2}\rho \gamma^2/r^2$ $\quad = 2\pi r^2\omega \quad u_\theta = r\omega$

before disturbance = exchange $KE_A = \rho/2\left[\left(\frac{\gamma_1}{r_1}\right)^2 + \left(\frac{\gamma_2}{r_2}\right)^2\right]$ fluid rings at r_1 and r_2
where r_1, r_2 arbitrary

after $KE_B = \rho/2\left[\left(\frac{\gamma_1}{r_2}\right)^2 + \left(\frac{\gamma_2}{r_1}\right)^2\right]$ but $r_2 > r_1$

$$\Delta KE = KE_B - KE_A = \frac{1}{2}\rho\left(\gamma_2^2 - \gamma_1^2\right)\left(r_1^{-2} - r_2^{-2}\right)$$

$\Delta E > 0$, $\gamma_2^2 > \gamma_1^2$ Stable ie requires E for exchange

$\Delta E < 0$ $\gamma_2^2 < \gamma_1^2$ unstable ie E released al disturbance can grow : centrifugal force > centripetal $\frac{\partial p}{\partial r}$ force

$\frac{d\gamma^2}{dr} > 0$ stability

$(r_2 u_{\theta_2})^2 > (r_1 u_{\theta_1})^2$

$(r_2^2 \omega_2)^2 > (r_1^2 \omega_1)^2$

$\omega_2 > \left(\frac{r_1}{r_2}\right)^2 \omega_1$ $r^2\omega$ must increase radially for stability

1923 Taylor rotating viscous flow stability criterion

Taylor Vortices

axisymmetric r, θ, z NS equations with

$\underline{V} = \overline{\underline{V}} + \underline{u}$ = mean + disturbance

$P = \overline{P} + p$

mean solution given $u_\theta(r) = Ar + B/r$

Substitute $\underline{V}$, P into NS, linearize

assume disturbance $e^{\sigma t + ikz}$ normal modes

Narrow gap solution

$$T_a = \frac{r_1 (r_2 - r_1)^3 (\omega_1^2 - \omega_2^2)}{\nu^2}$$

$= \frac{\text{centrifugal outward force}}{\text{viscous force}}$

$Ta_{crit} = 1708$ $0 \le \omega_2/\omega_1 \le 1$

$\lambda_{crit} = 2\pi d / k_{crit}$

$k_{crit} = 3.12 \Rightarrow \lambda_{cr} \sim 2d$

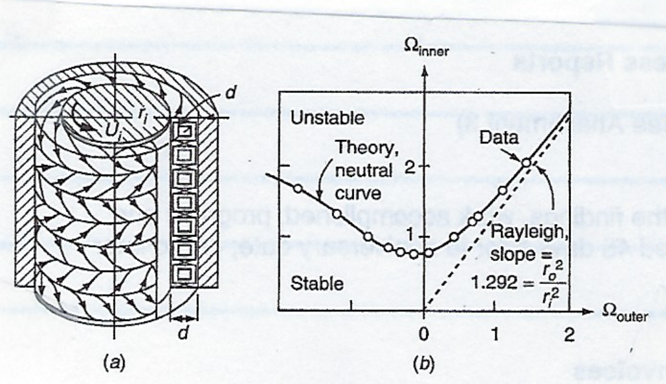


FIGURE 5-22
Theory and experiment by Taylor (1923) for the instability of Couette flow between rotating cylinders: (*a*) Taylor vortices; (*b*) theory and experiment for $r_o = 4.035$ cm, $r_i = 3.55$ cm.