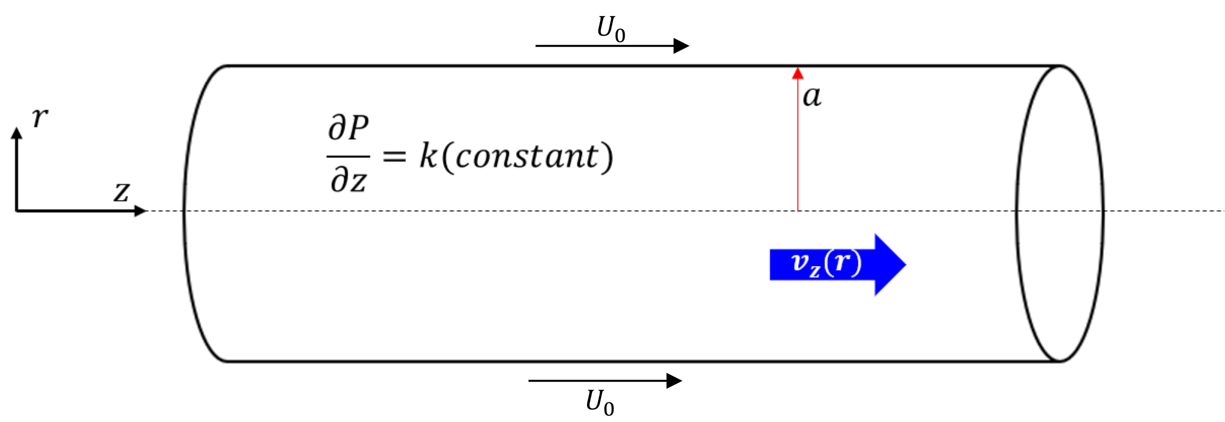
Name: ----------------- **Quiz 7** **Time: 20 minutes**

ME:5160 **Fall 2023**

The exam is closed book and closed notes.

The viscous oil in below Figure is set into steady motion by a constant pressure gradient $\frac{\partial P}{\partial z}$. The pipe radius is a and the pipe moves with constant velocity U_0. Assuming fully developed flow, constant density, circumferentially symmetric flow, and a purely axial fluid motion: (a) Simplify the governing equations (continuity and momentum) with these given conditions. (b) Apply appropriate boundary condition and derive the fluid velocity distribution $v_z(r)$. (c) Calculate wall shear stress at pipe wall.



The equations of motion of an incompressible Newtonian fluid with constant density and viscosity in cylindrical coordinates (*r, θ, z*) with velocity components (*vr, vθ, vz*):

Continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0$$

r-momentum:

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = \rho g_r - \frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right]$$

θ-momentum:

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = \rho g_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right]$$

z-momentum:

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right]$$

Boundary condition Hint

- At the pipe wall, the velocity is U_0
- At the pipe center, the velocity gradient should be zero

Wall shear stress Hint

- $\tau_{wall} = \mu\left.\frac{\partial v_z}{\partial y}\right|_{y=0} = -\mu\left.\frac{\partial v_z}{\partial r}\right|_{r=a}$, $y = a - r$ where a: *Radius of pipe*

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Solution:

ASSUMPTIONS:

1. Steady flow ($\frac{\partial}{\partial t}$=0)
2. Incompressible flow (ρ=constant)
3. Purely axial flow (vr=vθ=0) **(+3)**
4. Circumferentially symmetric flow, so properties do not vary with θ ($\frac{\partial}{\partial \theta}$=0)
5. Constant pressure gradient ($\partial p/\partial$z =k)
6. Horizontal cylinders (gz=0)

(a)

Continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0$$

$$0(3) + 0(3) + \frac{\partial v_z}{\partial z} = 0 \qquad \textbf{(+1)}$$

z-momentum:

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] \qquad \textbf{(+1.5)}$$

$$\rho(0(1) + 0(3) + 0(3,4) + 0(continuity)) = 0(6) - k(5) + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + 0(4) + 0(continuity)\right]$$

$$\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = k \qquad \textbf{(+1)}$$

(b) Integrate

$$\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{k}{\mu}r$$

$$r\frac{\partial v_z}{\partial r} = \frac{k}{2\mu}r^2 + C_1$$

$$\frac{\partial v_z}{\partial r} = \frac{k}{2\mu}r + \frac{C_1}{r}$$

$$\therefore v_z(r) = \frac{k}{4\mu}r^2 + C_1\ln(r) + C_2 \qquad \textbf{(+1)}$$

Apply two boundary conditions

$$v_z(a) = U_0 \;\rightarrow\; \frac{k}{4\mu}a^2 + C_1 \ln(a) + C_2 = U_0 \qquad \textbf{(+0.5)}$$

$$\left.\frac{\partial v_z}{\partial r}\right|_{r=0} = 0 \;\rightarrow\; \frac{k}{2\mu}(0) + \frac{C_1}{(0)} = 0 \qquad \textbf{(+0.5)}$$

$$\therefore C_1 = 0$$

$$\therefore C_2 = U_0 - \frac{k}{4\mu}a^2$$

Hence,

$$\therefore v_z(r) = U_0 + \frac{k}{4\mu}r^2 - \frac{k}{4\mu}a^2 = U_0 + \frac{k}{4\mu}(r^2 - a^2) = U_0 + \frac{1}{4\mu}\frac{\partial P}{\partial z}(r^2 - a^2) \quad \textbf{(+1)}$$

(c) Wall shear stress at the pipe wall

$$\tau_{wall} = -\mu \left.\frac{\partial v_z}{\partial r}\right|_{r=a}$$

$$\mu\frac{\partial v_z}{\partial r} = -\frac{k}{2}r$$

Apply

$$r = a$$

$$\therefore \tau_{wall} = -\frac{ka}{2} = -\frac{a}{2}\frac{\partial P}{\partial z} \qquad \textbf{(+0.5)}$$