Free

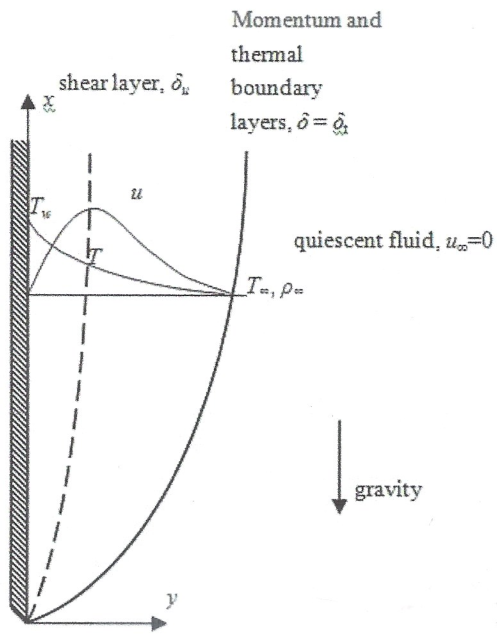
P5.46 If a vertical wall at temperature T_w is surrounded by a fluid at temperature T_o, a natural convection boundary layer flow will form. For laminar flow, the momentum equation is

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = \rho\beta(T-T_o)g + \mu\frac{\partial^2 u}{\partial y^2}$$

to be solved, along with continuity and energy, for (u, v, T) with appropriate boundary conditions. The quantity β is the thermal expansion coefficient of the fluid. Use ρ, g, L, and $(T_w - T_o)$ to nondimensionalize this equation. Note that there is no "stream" velocity in this type of flow.

Using ρ, g, L, and $(T_w - T_o)$

$\Rightarrow U = \sqrt{gL}$



so $u^* = \frac{u}{\sqrt{gL}}$ $\quad v^* = \frac{v}{\sqrt{gL}}$ $\quad x^* = \frac{x}{L}$ $\quad y^* = y/L$

$T^* = T - T_o / T_w - T_o$

$u = u^*\sqrt{gL}$ $\quad v = v^*\sqrt{gL}$ $\quad x = x^*L$ $\quad y = y^*L$

$T = T^*(T_w - T_o) + T_o$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x^*}\frac{\partial x^*}{\partial x} = \frac{1}{L}\frac{\partial}{\partial x^*}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y^*}\frac{\partial y^*}{\partial y} = \frac{1}{L}\frac{\partial}{\partial y^*} \qquad \frac{\partial^2}{\partial y^2} = \frac{1}{L^2}\frac{\partial^2}{\partial y^{*2}}$$

Substitute into momentum equation:

(1) Convection

$$\rho\left[\frac{u^*\sqrt{gL}}{L}\frac{\partial(u^*\sqrt{gL})}{\partial x^*} + \frac{v^*\sqrt{gL}}{L}\frac{\partial(u^*\sqrt{gL})}{\partial y^*}\right] =$$

$$\frac{\rho g L}{L}u^*\frac{\partial u^*}{\partial x} + \frac{\rho g L}{L}v^*\frac{\partial u^*}{\partial y} = \rho g\left[u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right]$$

(2) Bouyancy

$\rho\beta[T^*(T_w-T_0)]g$

(3) viscous

$$\frac{\mu}{L^2}\frac{\partial^2}{\partial y^{*2}}(u^*\sqrt{gL}) = \frac{\mu\sqrt{gL}}{L^2}\frac{\partial^2 u^*}{\partial y^{*2}}$$

Combine:

$$\rho g\left[u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right] = \rho g\beta\left[T^*(T_w-T_0)\right] + \frac{\mu\sqrt{gL}}{L^2}\frac{\partial^2 u^*}{\partial y^{*2}}$$

$$\frac{\mu g^{1/2}L^{1/2}}{\rho g L^2} = \frac{\mu}{\rho\sqrt{g}L\,L}$$

or $$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = \underbrace{[\beta(T_w-T_0)]}_{1}T^* + \underbrace{\frac{\mu}{\rho\sqrt{g}L\,L}}_{2}\frac{\partial^2 u^*}{\partial y^{*2}}$$

two nondimensional parameters (1) + (2) w/o name

4-13.3 Free Convection along a Vertical Isothermal Plate

For a vertical plate with $g_x = g$, let us assume constant T_w. If we ignore the anomalous case of water near freezing, where β may be negative, all common fluids have low density at a hot wall. Buoyant motion and free convection are both upward along a hot vertical plate, and the leading edge $x = 0$ starts at the bottom. For a cold plate, buoyancy is downward and $x = 0$ is located at the top of the plate. The coordinate y is normal to the plate.

Since the local Grashof number Gr_x plays the role of Re_x^2 in buoyant motion, we can guess (correctly) that the boundary-layer thickness δ/x will be proportional to $Gr_x^{1/4}$. Indeed, the "scale analysis" in the text by Bejan (1994) predicts that δ will be proportional to $x^{1/4}$. For vertical plate flow, velocity and thickness scales are prescribed by Schmidt et al. (1930) in an assortment of similarity variables:

more advanced analysis shows only single non dimensional parameter ie

$$\eta = \left(\frac{Gr_x}{4}\right)^{1/4}\frac{y}{x} \quad Gr_x = \frac{\beta g(T_w - T_\infty)x^3}{\nu^2} \quad u = 2\sqrt{x\beta g(T_w - T_\infty)}f' \quad v = \left[\frac{\beta g(T_w - T_\infty)\nu^2}{4x}\right]^{1/4}(\eta f' - 3f) \quad \Theta = \frac{T - T_\infty}{T_w - T_\infty} \tag{4-259}$$

$$Gr = \frac{\rho\beta(T_w-T_0)L^3}{\nu^2}$$

The reader can verify that u and v satisfy continuity and that these variables will reduce momentum and energy relations to the two coupled nonlinear ordinary differential equations

$$f''' + 3ff'' - 2f'^2 + \Theta = 0 \quad \text{and} \quad \Theta'' + 3Prf\Theta' = 0 \tag{4-260}$$

subject to the boundary conditions

$$f(0) = f'(0) = f'(\infty) = 0 \quad \text{and} \quad \Theta(0) = 1 \quad \Theta(\infty) = 0 \tag{4-261}$$

Note that Eqs. (4-260) are coupled and must be solved simultaneously, which is always the case in free-convection problems. Since no analytic solution is known, numerical integration is necessary. There are two unknown initial values at the wall. One must find the proper values of $f''(0)$ and $\Theta'(0)$, which cause the velocity and temperature to vanish for large η.

Two dimensionless parameters:

$\beta\,(T_w - T_o)$, $\dfrac{\mu}{\rho L \sqrt{gL}}$ $\quad Re^{-1}_{\sqrt{gL}}$

can be combined as

$Grashof\,\# = g\,\beta\,(T_w - T_o)\,L^3/\nu^2$

or any nondimensional coeff't had we use $\dfrac{\nu}{L}$ as velocity scale

$$\nu = \frac{\mu}{\rho} = \frac{\frac{M}{LT}}{\frac{M}{L^3}} = \frac{M}{LT}\times\frac{L^3}{M} = \frac{L^2}{T}$$

$$u^* u^*_{x^*} + v^* u^*_{y^*} = Gr\,T^* + u^*_{y^*y^*}$$

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to be solved, along with continuity and energy, for (u, v, T) with appropriate boundary conditions. The quantity β is the thermal expansion coefficient of the fluid. Use ρ, g, L, and ($T_w - T_o$) to nondimensionalize this equation. Note that there is no "stream" velocity in this type of flow.

Solution: For the given constants used to define dimensionless variables, there is only one pairing which will give a velocity unit: $(gL)^{1/2}$. Here are the writer's dimensionless variables:

$$u^* = \frac{u}{\sqrt{gL}};\quad v^* = \frac{v}{\sqrt{gL}};\quad x^* = \frac{x}{L};\quad y^* = \frac{y}{L};\quad T^* = \frac{T-T_o}{T_w-T_o}$$

Substitute into the momentum equation above and clean up so all terms are dimensionless:

$$\rho\left(u^*\frac{\partial u^*}{\partial x^*}\frac{gL}{L}\right) + \rho\left(v^*\frac{\partial u^*}{\partial y^*}\frac{gL}{L}\right) = \rho\beta g(T_w-T_o)T^* + \mu\frac{\partial^2 u^*}{\partial y^{*2}}\frac{\sqrt{gL}}{L^2}$$

$$\text{or:}\quad u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = [\beta(T_w-T_o)]T^* + \left[\frac{\mu}{\rho L\sqrt{gL}}\right]\frac{\partial^2 u^*}{\partial y^{*2}} \qquad \textit{Ans.}$$

There are two dimensionless parameters: $\beta(T_w - T_o)$ and $\mu/[\rho L\sqrt{gL}\,]$. Neither has a name, to the writer's knowledge, because a much cleverer analysis would result in only a single dimensionless parameter, the *Grashof number*, $g\,\beta(T_w - T_o)L^3/\nu^2$. (See, for example, White, *Viscous Fluid Flow*, 3rd edition, Section 4-14.3, page 323.)