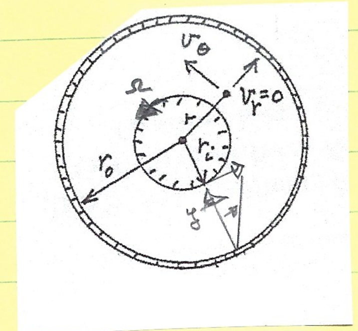
1.49 An amazing number of commercial and laboratory devices have been developed to measure fluid viscosity, as described in Ref. 27. Consider a concentric shaft, as in Prob. 1.47, but now fixed axially and rotated inside the sleeve. Let the inner and outer cylinders have radii r_i and r_o, respectively, with total sleeve length L. Let the rotational rate be Ω (rad/s) and the applied torque be M. Using these parameters, derive a theoretical relation for the viscosity μ of the fluid between the cylinders.



Assume linear velocity profile

$y = \dfrac{r_o - r}{r_o - r_i}$ wall coordinate $dy = -dr$

$$u(r) = \Omega r_i \left[\frac{r_o - r}{r_o - r_i} \right]$$

narrow gap solution

wide gap: $u = Ar + B/r$

$A/B = f(r_i, r_o, \omega_i, \omega_o)$

$$\tau = \mu \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} = \mu \frac{\Omega r_i}{r_o - r_i} \qquad \frac{\partial r}{\partial y} = -1$$

$$M = \int_{r=0}^{2\pi} r_i \, dF$$

moment about shaft axis

$dF = \tau \, dA$
$= \tau \, r_i \, d\theta \, L$

force on inner shaft element dA

$$= \int_0^{2\pi} r_i \, \mu \frac{\Omega r_i}{r_o - r_i} r_i \, d\theta \, L$$

$$= 2\pi \mu \Omega r_i^3 L / (r_o - r_i)$$

$$\mu = \frac{M(r_o - r_i)}{2\pi \, \Omega \, r_i^3 L}$$

However subject centrifugal instability / Taylor vortices

Inviscid stability theory: Rayleigh

$$\frac{d(rV_\theta)^2}{dr} > 0 \qquad \Omega_o r_o^2 > \Omega_i r_i^2 \qquad \Omega_o = 0 \text{ unstable}$$

$r_2 > r_1$

Viscous stability theory: $T_a = r_i (r_o - r_i)^3 (\Omega_i^2 - \Omega_o^2)/\nu^2$

$c = r_o - r_i \ll r_i$ $\qquad T_{a\,crit} = 1708$ narrow gap

Solution: Assuming a linear velocity distribution in the annular clearance, the shear stress is

$$\tau = \mu \frac{\Delta V}{\Delta r} \approx \mu \frac{\Omega r_i}{r_o - r_i}$$

This stress causes a force $dF = \tau\, dA = \tau (ri\; d\theta) L$ on each element of surface area of the inner shaft. The moment of this force about the shaft axis is $dM = ri\; dF$. Put all this together:

$$M = \int r_i \, dF = \int_0^{2\pi} r_i \mu \frac{\Omega r_i}{r_o - r_i} r_i \, L \, d\theta = \frac{2\pi \mu \Omega r_i^3 L}{r_o - r_i}$$

Solve for the viscosity: $\boldsymbol{\mu \approx M(r_o - r_i) / \{2\pi\Omega r_i^3 L\}}$ *Ans.*

$rV_\theta = r^2 \Omega$ = reduced circulation

Rayleigh criterion is that reduced circulation must increase radially outward