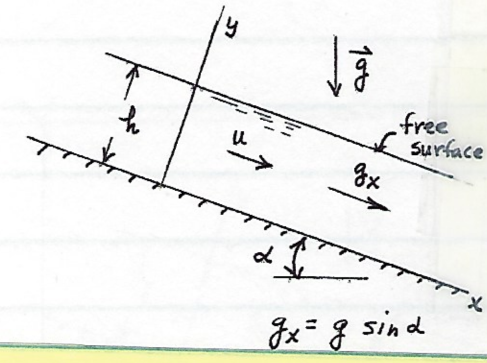
6.76

6.76 A layer of viscous liquid of constant thickness (no velocity perpendicular to plate) flows steadily down an infinite, inclined plane. Determine, by means of the Navier–Stokes equations, the relationship between the thickness of the layer and the discharge per unit width. The flow is laminar, and assume air resistance is negligible so that the shearing stress at the free surface is zero.



$v = 0 \quad w = 0 \Rightarrow u_x = 0$ i.e. $u = u(y)$ fully developed flow

$$\rho[u_t + uu_x + vu_y + wu_z] = -\frac{\partial p}{\partial x} + \rho g \sin\alpha + \mu[u_{xx} + u_{yy} + u_{zz}]$$

$\frac{Du}{Dt} = 0$

$$u_{yy} = -\frac{\rho g}{\mu}\sin\alpha$$

$$u_y = -\frac{\rho g}{\mu}\sin\alpha\, y + c_1$$

$$u = -\frac{\rho g}{2\mu}\sin\alpha\, y^2 + c_1 y + c_2$$

$u(0) = 0 \qquad u_y(h) = 0$ since $\tau(h) = \mu u_y = 0$

$u_y(h) = 0 = -\frac{\rho g}{\mu}\sin\alpha\, h + c_1 \qquad \tau_{yx} = \mu(u_y + v_x)$

$$c_1 = \frac{\rho g}{\mu}\sin\alpha\, h$$

$u(0) = 0 \Rightarrow c_2 = 0$

$$u(y) = -\frac{\rho g}{\mu}\sin\alpha\,(y^2/2 - hy)$$

$$q = \int_0^h u\,dy = \frac{\rho g h^3 \sin\alpha}{3\mu}$$ per unit width