6.108

6.108 An incompressible Newtonian fluid flows steadily between two infinitely long, concentric cylinders as shown in Fig. P6.108. The outer cylinder is fixed, but the inner cylinder moves with a longitudinal velocity V_0 as shown. The pressure gradient in the axial direction is $-\Delta p/\ell$. For what value of V_0 will the drag on the inner cylinder be zero? Assume that the flow is laminar, axisymmetric, and fully developed.

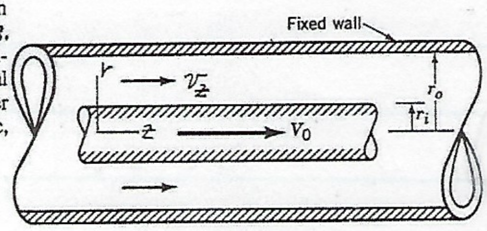

FIGURE P6.108

Assume $v_r = v_\theta = 0$ then use continuity

$$\frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

ie $\frac{\partial v_z}{\partial z} = 0$ at fully developed flow

z momentum: $0 = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right]$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{1}{\mu}\frac{\partial p}{\partial z}$$

$$r\frac{\partial v_z}{\partial r} = \frac{r^2}{2\mu}\frac{\partial p}{\partial z} + c_1$$

Same solution circular pipe but different BC (circular pipe no annulus: $v_z(r_i = 0) \neq \infty$ and $v_z(r_o) = 0$)

$$v_z = \frac{r^2}{4\mu}\frac{\partial p}{\partial z} + c_1 \ln r + c_2$$

$$v_z(r_i) = V_0 = \frac{r_i^2}{4\mu}\frac{\partial p}{\partial z} + c_1 \ln r_i + c_2$$

$$v_z(r_o) = 0 = \frac{r_o^2}{4\mu}\frac{\partial p}{\partial z} + c_1 \ln r_o + c_2$$

subtract : $V_0 = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)(r_i^2 - r_0^2) + c_1 \ln r_i/r_0$

$$c_1 = \frac{V_0 - \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)(r_i^2 - r_0^2)}{\ln r_i/r_0}$$

add : $V_0 = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)(r_i^2 + r_0^2) + c_1 \ln r_i r_0 + 2c_2$

$$c_2 = \left[V_0 - \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)(r_i^2 + r_0^2) - c_1 \ln r_i r_0\right]/2$$

the drag on the inner cylinder will be zero if :

$$\tau_{rz}(r_i) = 0$$

$$\tau_{rz} = \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)$$

$$= \mu \frac{\partial v_z}{\partial r}$$

$$= \mu\left[\frac{r}{2\mu}\frac{\partial p}{\partial z} + c_1/r\right]$$

$$\Rightarrow \tau_{rz}(r_i) = 0 = \mu\left[\frac{r_i}{2\mu}\frac{\partial p}{\partial z} + c_1/r_i\right]$$

$$V_0 = -\frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left[2r_i^2 \ln\frac{r_i}{r_0} - (r_i^2 - r_0^2)\right]$$