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Flow on an Inclined Open Channel

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Abstract

Fluid flow on an inclined channel is considered. The effect of gravity and the friction of the fluid to the bottom wall is included in the model, so that we have a system of partial differential equations of fluid depth and averaged depth velocity. From the balancing between those two forces, the model is derived and is then solved analytically for kinematic wave and numerically. Both types of solution can be compared, and the kinematic wave is a special case of the numerical solution, i.e. for Froude number $F = 2$.

Keywords: Inclined wall, uniform flow, Froude number, kinematic wave

1 Introduction

Fluid flow is a physical phenomena that is often seen in our daily life such as in channel. Gravity is one of forces that can make the fluid flows on an inclined channel. In this paper we concern a mathematical model of that flow. Based on mass and momentum conservations, we formulate the 2-D flow into a system of partial differential equations. The effect of the gravity and the bottom friction of the wall are investigated in propagating waves on the fluid surface.

The derivation of the model can be formulated basically from a control volume, such as given in Whitham [1], or can be obtained from Euler equation, see for example Chow [2]. The conventional model such problem is shallow water equation. The derivation of that model can be seen in Chaudhry [3], the theory is so named because the flow in which the vertical dimensions are small compared to the horizontal dimensions.

Meanwhile, for flow on an inclined wall the vertical velocity can not be ignored as the gravity plays an important roll in accelerating the flow, but it is also important to include the resistant effect from the wall. These two forces should be involved in

shallow water model, and we concern to the augmented model of shallow water equation. The Chezy formula is used to represent the resisted force, proportional to the horizontal velocity. Dressler [4] worked on that model to show that the uniform flow becomes unstable when the Froude number F exceeds 2, i.e. waves are formed on the surface for that Froude number. Dressler was able to show that condition and found a one-parameter family of solutions. This was then extended by Needham & Merkin [5] for the model by including the effect of the energy dissipation. They obtained periodic solutions.
Since the model is strongly nonlinear, it is not easy to be solved analytically and numerically. To simplify the problem, solution near the constant one is considered so that the model becomes linear for the first order, and it is solved numerically by a finite difference method. Similar problem has been done by Wiryanto & Mungkasi [6, 7], but for wave generation phenomena. Our numerical solution confirms to the condition for Froude number greater than 2, and for tends to kinematic wave as Froude number is two.

2 Formulation

We consider fluid flow on an inclined wall of angle α to the horizontal line, illustrated in Figure 1. We choose Cartesian coordinates with x-axis along the bottom and the y-axis perpendicular to x-axis, so that the fluid surface is $y = h(x, t)$, measured from the bottom.

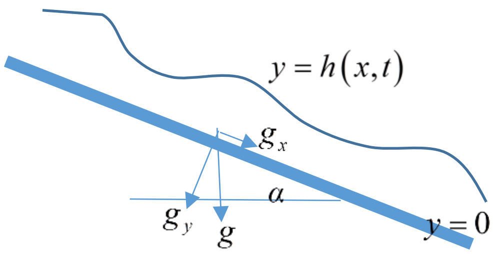

Figure 1. Sketch of the flow and coordinate

Now, we take a small fluid element $x \in [x_1, x_2]$. The mass conservation of fluid for that element can be written

$$\frac{d}{dt}\int_{x_1}^{x_2} h\, dx + [hv]_{x_1}^{x_2} = 0, \tag{1}$$

and the momentum conservation

$$\frac{d}{dt}\int_{x_1}^{x_2} hv\, dx + \left[hv^2\right]_{x_1}^{x_2} + \left[\frac{1}{2}gh^2\cos\alpha\right]_{x_1}^{x_2} = \int_{x_1}^{x_2} gh\sin\alpha\, dx - \int_{x_1}^{x_2} C_f v^2\, dx \tag{2}$$

We do not write the fluid density ρ in the equations as it is a factor in each term, so that we can cancel it. The third term of (2) represents the net total pressure force, the first term in the right hand side is the gravitational force down the incline and the second in the right term is the frictional effect of the bottom with friction coefficient C_f. Note that v is the mean velocity.

In this model we assume that h and v are continuously differentiable, so that when we take the limit $x_2 - x_1 \to 0$, and we use the relation in the mass conservation to the other equation, so that (1) and (2) become

$$\begin{cases} h_t + (hv)_x = 0, \\ v_t + vv_x + g' h_x = g' S - C_f \dfrac{v}{h}, \end{cases} \tag{3}$$

where $g' = g\cos\alpha$ and $S = \tan\alpha$.

In case the left hand side of the momentum conservation is neglected, we obtain the relation between h and v

$$v = \left(\frac{g'S}{C_f}\right)^{1/2} h^{1/2}.$$

This is then substituted to the first equation giving

$$h_t + \tfrac{3}{2} k\, h^{\frac{1}{2}}\, h_x = 0 \tag{4}$$

where $k = \left(\dfrac{g'S}{C_f}\right)^{1/2}$, as the equation of kinematic wave approximation. This equation is still difficult to be solved, but we can approximate the solution near the constant h, by writing $h(x,t) = h_0 + \varepsilon h_1(x,t)$ for constant h_0 and it is perturbed by small term containing $h_1(x,t)$. So that we have

$$h_{1t} + \frac{3}{2} k h_0^{1/2} h_{1x} = O(\varepsilon) \tag{5}$$

Analytically, we can obtain the solution, depending on the initial condition. If at the beginning it is given $h(x,0) = f(x)$, say $f(x) = h_0 + \varepsilon f_1(x)$, the solution of (5) is

$$h_1(x,t) = f_1\left(x - \frac{3}{2} k h_0^{1/2} t\right)$$

The wave profile does not change by increasing time t, but it just travels with wave speed $c = \frac{3}{2} k h_0^{1/2}$ to the right as the coefficient $\frac{3}{2} k h_0^{1/2}$ is positive. This type of solution is then compared to the solution of full equation (3), but it is required to be solved numerically, presented below.

3 Numerical Procedure

We solve (3) in this section. To do so, we determine the solution near the constant solution, namely h_0 and v_0, satisfying

$$v_0{}^2 = \frac{g' S h_0}{C_f} \tag{6}$$

We write similar to the kinematic wave approximation

$$v = v_0 + \varepsilon v_1(x,t)$$
$$h = h_0 + \varepsilon h_1(x,t)$$

for small parameter ε, and we substitute to (3) so that we obtain

$$\begin{aligned} &h_{1_t} + v_0 h_{1_x} + h_0 v_{1_x} = 0 \\ &v_{1_t} + v_0 v_{1_x} + g' h_{1_x} = -2g' S \frac{v_1}{v_0} + g' S \frac{h_1}{h_0} \end{aligned} \tag{7}$$

Note that the last term of the second equation of (3) is approximated by Taylor series.

The next step is we non-dimensionalize the variables in (7) based on the constant solution, by defining

$$\eta = \frac{h_1}{h_0},\ w = \frac{v_1}{v_0},\ \bar{x} = \frac{x}{h_0},\ \bar{t} = \frac{v_0 t}{h_0}$$

When we apply this to (7) we have

$$\begin{aligned} &\eta_t + \eta_x + w_x = 0 \\ &w_t + w_x + \frac{1}{F^2}\eta_x = \frac{S}{F^2}(\eta - 2w) \end{aligned} \tag{8}$$

where $F = \dfrac{v_0}{\sqrt{g' h_0}}$ is Froude number, and we write x and t without bar to simplify in writing.

In solving (8) we discretize the space $x_j = j\Delta x,\ \mathrm{j} = 0,1,2,\cdots,\mathrm{J}$ and time $t_n = n\Delta t,\ n = 0,1,2,\cdots$, and we use notation $\eta_j^n = \eta(x_j, t_n),\ w_j^n = w(x_j, t_n)$. Equation (8) is discretized by forward time backward space, so that we have explicit formulation

$$\eta_j^{n+1} = \eta_j^n - \Delta t \left[\frac{\eta_j^n - \eta_{j-1}^n}{\Delta x} + \frac{w_j^n - w_{j-1}^n}{\Delta x} \right]$$
$$w_j^{n+1} = w_j^n - \Delta t \left[\frac{w_j^n - w_{j-1}^n}{\Delta x} + \frac{1}{F^2}\frac{\eta_j^n - \eta_{j-1}^n}{\Delta x} - \frac{S}{F^2}\left(\eta_j^n - 2w_j^n\right) \right]$$

As the initial condition we use $\eta_j^0 = 0,\ w_j^0 = 0$ indicating at the beginning the flow is uniform. Beside that we need boundary condition for η_0^n and w_0^n. Based on the kinematic wave approximation, they satisfy

$$w_0^n = \tfrac{1}{2}\eta_0^n$$

so that to simulate the wave propagation, we need only the boundary condition for η_0^n.

4 Numerical Simulation

The numerical procedure described above is used to simulate the wave propagation at the fluid surface on an inclined wall. Most of our calculations use $J = 1000$, $dx = 0.1$ and $dt = 2dx/3$ based on the kinematic wave approximation, where we found the characteristic line is $x = \frac{3}{2}t + \text{constant}$, after scaling the variables. This is important to get the stable calculation.

Figure 2 is typical wave propagation for $\alpha = 0.2$ radian $(S = 0.0035)$, $F = 3.5$. The boundary condition is

$$\eta(0,t) = \begin{cases} 0.1\sin(0.157t), & 0 < t < 10 \\ 0.1, & t > 10 \end{cases} \tag{9}$$

We plot $\eta(x,t)$ for some values t at the same plane by shifting upward for greater t. So, we can see as the surface disturbed by a part of a sinusoidal wave, it propagates to the right. Even the maximum elevation of the incoming wave is 0.1, but the front wave reaches greater than that.

We can compare the wave propagation for different F. We plot the previous result at a certain time $\eta(x,53.6)$ together with another calculation for $F = 6.5$. We found that greater Froude number produces higher front wave, but propagates slower. We show in Figure 3. For other values Froude number, we observe and found that the numerical procedure is able to calculate solutions for $F \ge 2$, and is fail below that number. This is indicated by the front wave that is no greater than the maximum amplitude.

We can also compare with different angle α. Steeper wall produces higher front wave and propagating faster, the effect of gravity playing on important role. Meanwhile the friction coefficient C_f is similar to the Froude number as it is proportional and presented as F, following (6).

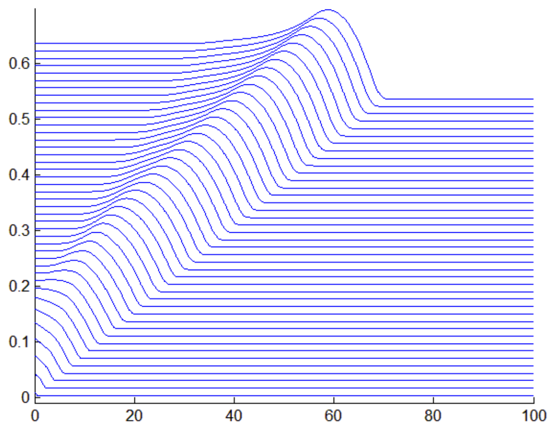

Fig. 2. Plot of wave propagation $\eta(x,t)$ as the solution of (8) for $F = 3.5$ and $\alpha = 0.2$, using boundary condition (9).

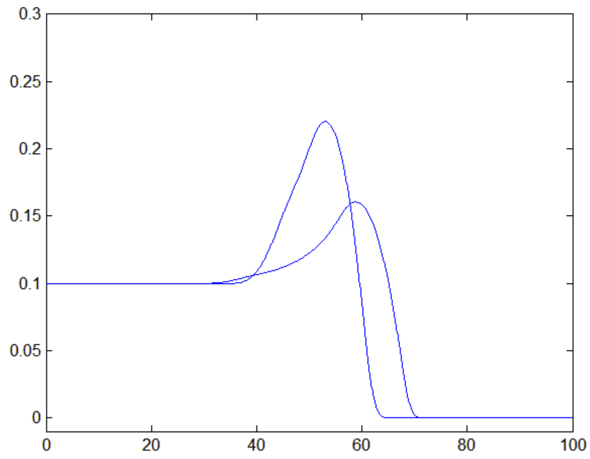

Fig. 3. Plot $\eta(x,53.6)$ for different Froude number $F=3,5$ smaller front wave and $F=6.5$.

5 Conclusion

A model of flow on an inclined wall has been derived based on mass and momentum conservations by including the bottom friction. The analytical kinematic wave is obtained, and it is then used to construct the discretization of the numerical procedure of the model. The simulation confirms that the solution is obtained for Froude number greater than 2, and when Froude number tends to 2 the solution tends to the kinematic wave.

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