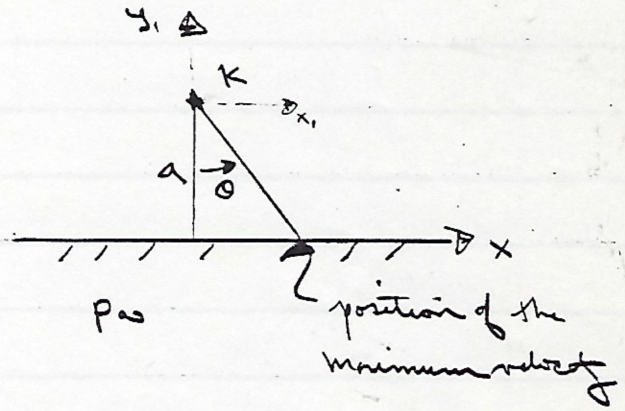
A source of strength K is located at a distance a from a wall as shown in the figure below. Calculate the total force on the wall. Is the wall drawn towards or away from the source. At what angle θ is the velocity along the wall a maximum.



$$Q_1 = \frac{K}{2\pi} \ln r$$

$$r = \sqrt{x_1^2 + y_1^2} \qquad x = x_1$$

$$y = y_1 + a$$

$$y_1 = y - a$$

$$Q_1 = \frac{K}{2\pi} \ln \sqrt{x^2 + (y-a)^2}$$

$\Phi = Q_1 + Q_2$ where Q_2 is the image

$$= \frac{K}{2\pi}\left[\ln\sqrt{x^2+(y-a)^2} + \ln\sqrt{x^2+(y+a)^2}\right]$$

$$u = \frac{\partial \Phi}{\partial x} = \frac{K}{2\pi}\left[\frac{x}{x^2+(y-a)^2} + \frac{x}{x^2+(y+a)^2}\right]$$

$$u(y=0) = \frac{K}{\pi}\frac{x}{x^2+a^2}$$

$$p + \tfrac{1}{2}\rho u^2 = p_\infty$$

$$p = p_\infty - \tfrac{1}{2}\rho u^2$$

$$F = -\int_{-\infty}^{\infty} (p - p_\infty)\,dx = \frac{1}{2}\rho\frac{K^2}{\pi^2}\int_{-\infty}^{\infty}\frac{x^2}{(x^2+a^2)^2}\,dx$$

$$F = \frac{1}{2}\rho\left(\frac{k}{\pi}\right)^2 \int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^2}\,dx = \frac{1}{2}\rho\left(\frac{k}{\pi}\right)^2 \left[\frac{-x}{2(x^2+a^2)} + \underbrace{\frac{1}{2}\int \frac{dx}{a^2+x^2}}_{\frac{1}{2a}\tan^{-1}x} \right]_{-\infty}^{\infty}$$

$$= \frac{1}{2}\rho\left(\frac{k}{\pi}\right)^2 \cdot \frac{\pi}{2a}$$

$$= \frac{\rho k^2}{4\pi a} \qquad \text{for } m = \frac{k}{2\pi}, \quad k^2 = 4\pi^2 m^2$$

$$= \frac{\rho \pi m^2}{a}$$

$F > 0$, thus the wall is urged towards the source with a force (per unit width) given as above!

$$u = \frac{k}{\pi}\frac{x}{x^2+a^2} \qquad \tan\theta = \frac{x}{a}$$

$$= \frac{k}{\pi}\frac{a\tan\theta}{a^2\tan^2\theta + a^2} = \frac{k}{\pi a}\cdot\frac{\tan\theta}{\tan^2\theta+1}$$

$$u = \frac{k}{2\pi a}\sin 2\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{k}{2\pi a}\cos 2\theta \times 2 = \frac{k}{\pi a}\cos 2\theta = 0$$

ie $\theta = \pm\frac{\pi}{4}$

$$\int \frac{x^2}{(x^2+a^2)^2}\,dx = \frac{-x}{2(x^2+a^2)} + \frac{1}{2a}\tan^{-1}x$$

$$\frac{\tan\theta}{\tan^2\theta+1} = \frac{1}{2}\sin 2\theta$$

$$\phi_1 = \frac{k}{2\pi} \ln \sqrt{x_1^2 + y_1^2} \qquad x_1 = x_2 = x = r_1 \cos\theta_1 = r_2 \cos\theta_2$$

$$\phi_2 = \frac{k}{2\pi} \ln \sqrt{x_2^2 + y_2^2} \qquad y_1 = y - a = r_1 \sin\theta_1$$

$$\underline{\Phi} = \phi_1 + \phi_2 \qquad y_2 = y + a = r_2 \sin\theta_2$$

$$\underline{V}_1 = v_{r_1} \hat{e}_{r_1} \qquad v_{r_1} = \frac{k}{2\pi r_1} \qquad \hat{e}_{r_1} = \cos\theta_1 \hat{i} + \sin\theta_1 \hat{j}$$

$$\underline{V_2} = v_{r_2} \hat{e}_{r_2} \qquad v_{r_2} = \frac{k}{2\pi r_2} \qquad \hat{e}_{r_2} = \cos\theta_2 \hat{i} + \sin\theta_2 \hat{j}$$

$$\underline{V} = \underline{V_1} + \underline{V_2}$$

$$U^2 = \underline{V} \cdot \underline{V} = \left(v_{r_1} \hat{e}_{r_1} + v_{r_2} \hat{e}_{r_2}\right) \cdot \left(v_{r_1} \hat{e}_{r_1} + v_{r_2} \hat{e}_{r_2}\right)$$

$$= v_{r_1}^2 + v_{r_2}^2 + 2 v_{r_1} v_{r_2} \hat{e}_{r_1} \cdot \hat{e}_{r_2}$$

$$= v_{r_1}^2 + v_{r_2}^2 + 2 v_{r_1} v_{r_2} \left[\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2\right]$$

$$= \left(\frac{k}{2\pi r_1}\right)^2 + \left(\frac{k}{2\pi r_2}\right)^2 + 2\left(\frac{k}{2\pi r_1}\right)\left(\frac{k}{2\pi r_2}\right)\left[\frac{x_1 x_2}{r_1 r_2} + \frac{y_1 y_2}{r_1 r_2}\right]$$

$$= \frac{k^2}{4\pi^2}\left(\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{2}{r_1^2 r_2^2}\left(x_1 x_2 + y_1 y_2\right)\right)$$

on $r_1 = \sqrt{x^2 + a^2} = r_2 \qquad x_1 = x_2 = x \qquad y_1 = -a$

$y_2 = a$

$$= \frac{k^2}{4\pi^2}\left(\frac{1}{x^2+a^2} + \frac{1}{x^2+a^2} + \frac{2}{(x^2+a^2)^2}\left(x^2 - a^2\right)\right)$$

$$= \frac{k^2}{\pi^2} \cdot \frac{x^2}{(x^2+a^2)^2} \qquad \frac{2}{x^2+a^2} + \frac{2(x^2-a^2)}{(x^2+a^2)^2} = \frac{2(x^2+a^2) + 2(x^2-a^2)}{(x^2+a^2)^2}$$