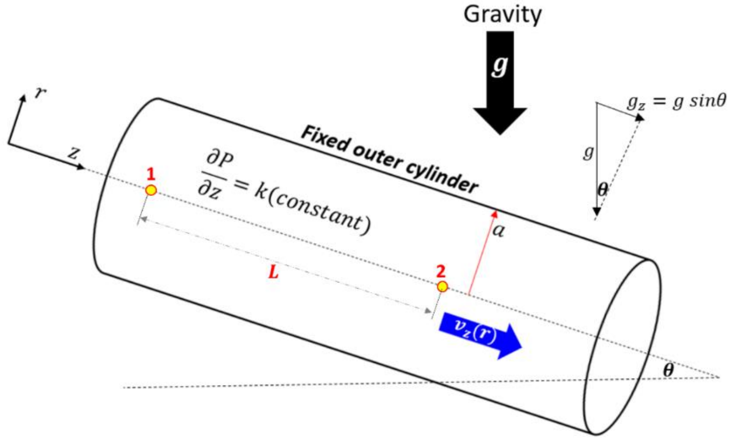
Name: ------------------ **Exam 2** **Time: 50 minutes**

ME:5160 **Fall 2024**

The exam is closed book and closed notes.

1. The viscous oil in below Figure is set into steady motion by a constant pressure gradient $\frac{\partial P}{\partial z}$ and gravity. The radius of pipe is a. Assuming fully developed flow, constant density, circumferentially symmetric flow, and a purely axial fluid motion. (a) Simplify the governing equation with these given conditions. (b) Apply appropriate boundary condition and derive the fluid velocity distribution of $v_z(r)$. (c) Calculate wall shear stress at pipe wall.



The equations of motion of an incompressible Newtonian fluid with constant density and viscosity in cylindrical coordinates (*r, θ, z*) with velocity components (*vr, vθ, vz*):

Continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0$$

r-momentum:

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = \rho g_r - \frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right]$$

θ-momentum:

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = \rho g_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right]$$

z-momentum:

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right]$$

Boundary condition Hint

- At the pipe wall, the velocity is zero
- At the pipe center, the velocity gradient should be zero

Wall shear stress Hint

- $\tau_{wall} = \mu\frac{\partial v_z}{\partial y}\Big|_{y=0} = -\mu\frac{\partial v_z}{\partial r}\Big|_{r=a}$, $y = a - r$ where $a{:}\,Radius\ of\ pipe$

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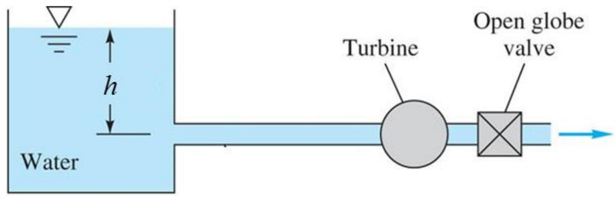
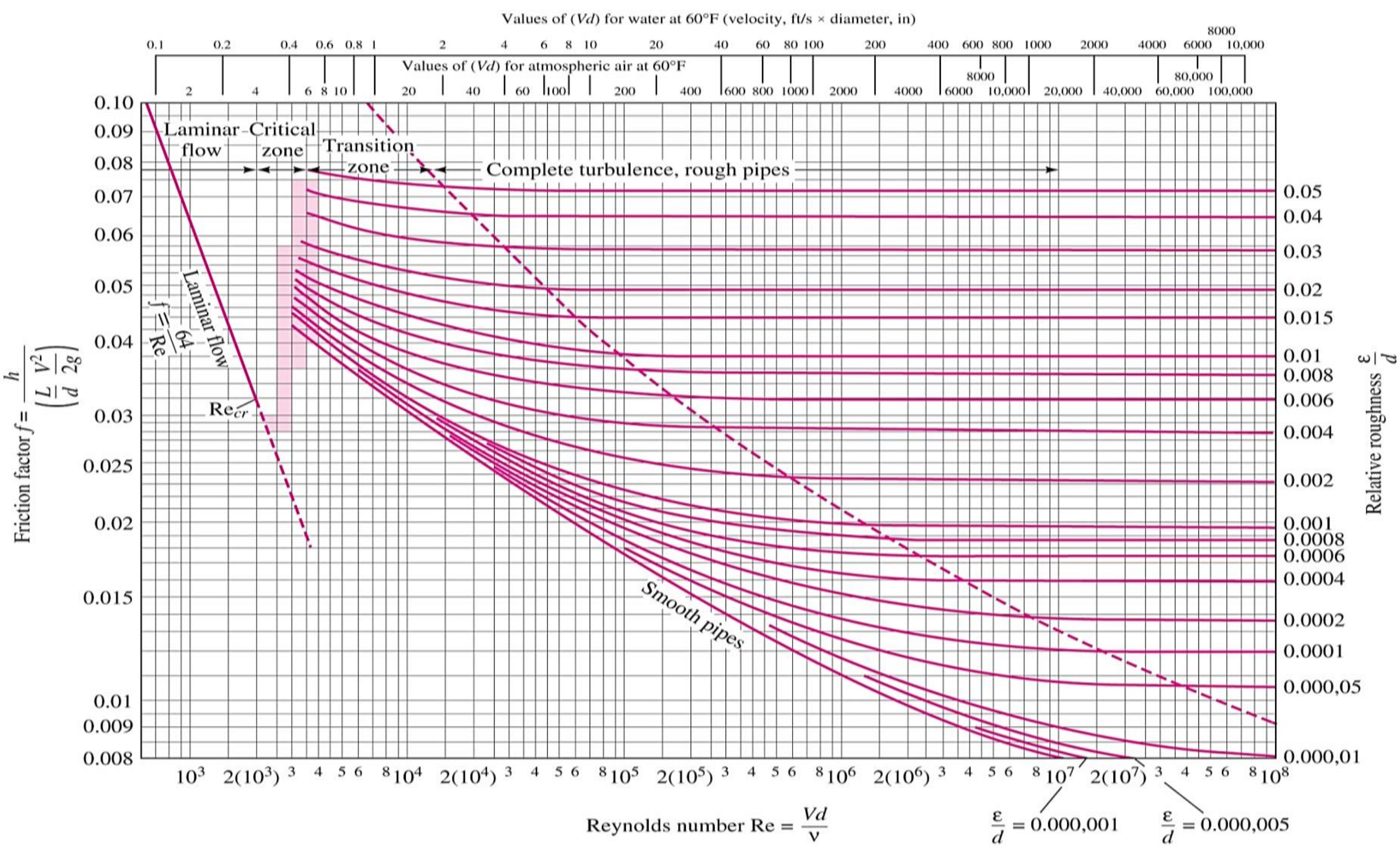
2. A tank of water with depth h is to be drained by a 5-cm-diameter exit pipe. Water density is 998 kg/m^3, water viscosity is 0.001 kg/ms. The pipe extends out for 15 m and a turbine and an open globe valve are located on the pipe. The head provided by the turbine is h_t = 10 m. (a) If the exit flow rate is Q = 0.04 m^3/s, calculate h assuming there are no minor losses, the turbine is 100% efficient, and the pipe is smooth. (b) Calculate Q if h is same as part (a) but there are minor losses (K = 0.5 for the sharp entrance and K = 6.9 for the open globe valve), the turbine has an efficiency of 80%, and the pipe is rough with ε = 0.3 mm. Use the value of f from part (a) as initial guess and stop at the end of the second iteration.





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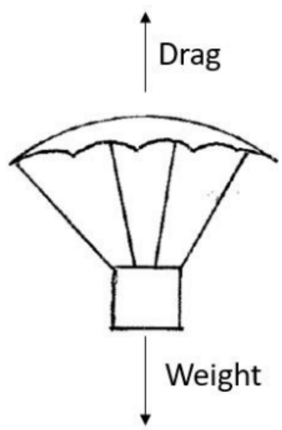
3. A parachute of a new design is tested in standard air (ρ=1.2 kg/m^3 and μ=1.8E-5 kg/m-s) with a total weight of the load and parachute of 200 N. The diameter for the tested prototype is 5 m. The results showed that the parachute reaches a constant velocity of 3 m/s. (a) Use the prototype data and find the drag coefficient for the parachute. (b) If you are to repeat the experiment for a 2.5 times smaller model using Reynolds similarity, and the weight of the model parachute comes out as 40 N, how much load do you need to add?

Hint: $C_D = \frac{Drag}{\frac{1}{2}\rho V^2 A}$, $Re = \frac{\rho V D}{\mu}$



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1. Solution:

ASSUMPTIONS:

1. Steady flow ($\frac{\partial}{\partial t}$=0)
2. Incompressible flow (ρ=constant)
3. Purely axial flow (vr=vθ=0)
4. Circumferentially symmetric flow, so properties do not vary with θ ($\frac{\partial}{\partial \theta}$=0)
5. Constant pressure gradient ($\partial p/\partial z$ =k)

(a)

Continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0$$

$$0(3) + 0(3) + \frac{\partial v_z}{\partial z} = 0$$ +1

z-momentum:

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right]$$

$$\rho(0(1) + 0(3) + 0(3{,}4) + 0(continuity)) = \rho g sin\theta - k(5) + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + 0(4) + 0(continuity)\right]$$ +1

$$\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = k - \rho g sin\theta$$ +1

(b) Integrate

$$\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{k - \rho g sin\theta}{\mu}r$$

$$r\frac{\partial v_z}{\partial r} = \frac{k - \rho g sin\theta}{2\mu}r^2 + C_1$$

$$\frac{\partial v_z}{\partial r} = \frac{k - \rho g sin\theta}{2\mu}r + \frac{C_1}{r}$$

$$\therefore v_z(r) = \frac{k - \rho g sin\theta}{4\mu}r^2 + C_1\ln(r) + C_2$$ +1

Apply two boundary conditions

$$v_z(a) = 0 \rightarrow \frac{k-\rho g sin\theta}{4\mu} a^2 + C_1 \ln(a) + C_2 = 0 \quad \boxed{+1}$$

$$\left.\frac{\partial v_z}{\partial r}\right|_{r=0} = 0 \rightarrow \frac{k-\rho g sin\theta}{2\mu}(0) + \frac{C_1}{(0)} = 0 \quad \boxed{+1}$$

$$\therefore C_1 = 0$$

$$\therefore C_2 = -\frac{k - \rho g sin\theta}{4\mu} a^2$$

Hence,

$$\therefore v_z(r) = \frac{k - \rho g sin\theta}{4\mu} r^2 - \frac{k - \rho g sin\theta}{4\mu} a^2 = \frac{k - \rho g sin\theta}{4\mu}(r^2 - a^2) \quad \boxed{+0.5}$$

$$= \frac{1}{4\mu}\left(\frac{\partial P}{\partial z} - \rho g sin\theta\right)(r^2 - a^2)$$

(c) Wall shear stress at the pipe wall

$$\tau_{wall} = -\mu \left.\frac{\partial v_z}{\partial r}\right|_{r=a}$$

$$\mu \frac{\partial v_z}{\partial r} = -\frac{k - \rho g sin\theta}{2} r \quad \boxed{+0.5}$$

Apply

$$r = a$$

$$\therefore \tau_{wall} = -\frac{(k - \rho g sin\theta)a}{2} = -\frac{a}{2}\left(\frac{\partial P}{\partial z} - \rho g sin\theta\right) \quad \boxed{+1}$$

Solution 2:

ANALYSIS:

Energy equation

between free-surface (1) and exit (2):

$$\left(\frac{p}{\rho g}+\frac{V^2}{2g}+z\right)_1=\left(\frac{p}{\rho g}+\frac{V^2}{2g}+z\right)_2+h_f-h_p+h_t$$

$$p_1=p_2=p_{atm}$$

$$z_1-z_2=h$$

$$V_1=0;\quad h_p=0$$

$$h_f=\frac{V_2^2}{2g}\left(f\frac{L}{D}+\sum K\right)$$ **(1)**

Replace and find *h*:

$$h=\frac{V_2^2}{2g}\left(1+f\frac{L}{D}+\sum K\right)+h_t$$ **(1)**

Find velocity using the flow rate and then *Re*:

$$V_2=\frac{Q}{\frac{\pi}{4}D^2}=\frac{(0.04\ \mathrm{m^3/s})}{\frac{\pi}{4}(0.05\ \mathrm{m})^2}=20.4\ \mathrm{m/s}$$

$$Re=\frac{\rho V_2 D}{\mu}=\frac{(998\ \mathrm{kg/m^3})(20.4\ \mathrm{m/s})(0.05\ \mathrm{m})}{(0.001\ \mathrm{kg/ms})}=1.02\mathrm{E}6\quad(turb.)$$ **(1)**

(a)

Find the friction factor from the moody diagram using *Re*:

$$f_{smooth}\sim 0.011$$

$$h=\frac{V_2^2}{2g}\left(1+f_{smooth}\frac{L}{D}\right)+h_t$$ **(1)**

$$h=\frac{\left(20.4\frac{\mathrm{m}}{\mathrm{s}}\right)^2}{(2)\left(9.81\frac{\mathrm{m}}{\mathrm{s}^2}\right)}\left[1+(0.011)\frac{(15\ \mathrm{m})}{(0.05\ \mathrm{m})}\right]+(10\ \mathrm{m})=101\ \mathrm{m}$$ **(1)**

(b)

Consider minor losses, turbine efficiency, and roughness of the pipe with h from part (a).

$$h = \frac{V_2^2}{2g}\left(1 + f_{rough}\frac{L}{D} + K_{ent} + K_{valve}\right) + \frac{h_t}{\eta} \qquad \textbf{(1)}$$

Find an expression of V_2 as a function of f_{rough}:

$$V_2 = \sqrt{\frac{2g\left(h - \frac{h_t}{\eta}\right)}{1 + f_{rough}\frac{L}{D} + K_{ent} + K_{valve}}} = \sqrt{\frac{(2)(9.81)\left(101 - \frac{10}{0.8}\right)}{1 + f_{rough}\frac{(15)}{(0.05)} + 0.5 + 6.9}}$$

$$V_2 = \sqrt{\frac{1736.37}{300f_{rough} + 8.4}} \qquad \textbf{(1)}$$

Use f_{smooth} as initial guess to compute new velocity and *Re*:

$$V_2 = \sqrt{\frac{1736.37}{300(0.011) + 8.4}} = 12.18\,\frac{\text{m}}{\text{s}} \quad \rightarrow \quad Re = 6.08\text{E}5$$

Find the friction factor from the Moody diagram using *Re* and relative roughness and iterate twice.

$$\frac{\varepsilon}{D} = \frac{(0.0003\ \text{m})}{(0.05\ \text{m})} = 0.006 \qquad \textbf{(1)}$$

<u>Iteration 1:</u>

$$f_{rough} \sim 0.032$$

$$V_2 = \sqrt{\frac{1736.37}{300(0.032) + 8.4}} = 9.82\,\frac{\text{m}}{\text{s}} \quad \rightarrow \quad Re = 4.90\text{E}5 \qquad \textbf{(1)}$$

<u>Iteration 2:</u>

$$f_{rough} \sim 0.032 \text{ (converged)}$$

Compute flow rate:

$$Q = V_2\left(\frac{\pi}{4}D^2\right) = \left(9.82\,\frac{\text{m}}{\text{s}}\right)\frac{\pi}{4}(0.05\ \text{m})^2 = 0.019\ \text{m}^3/\text{s} \qquad \textbf{(1)}$$

Compared to part (a), the flow rate is reduced by 52.5% due to additional losses.

3. Solution

(a)

$$C_D = \frac{D}{\frac{1}{2}\rho V^2 A} \qquad \textbf{(1)}$$

$$A = \frac{\pi D^2}{4} \qquad \textbf{(1)}$$

If in equilibrium at constant velocity, then:

$$D = W \qquad \textbf{(1)}$$

$$C_D = \frac{W}{\frac{1}{2}\rho V^2 A} = \frac{(200)}{\frac{1}{2}(1.2)(3)^2\frac{\pi}{4}(5)^2} = 1.89 \qquad \textbf{(1)}$$

(b)

To repeat the experiment, it has to be designed to reach same Reynolds number as prototype:

$$Re_m = Re_p \qquad \textbf{(2)}$$

$$\frac{V_m D_m}{\nu_m} = \frac{V_p D_p}{\nu_p} \quad \rightarrow \quad V_m = V_p \frac{D_p}{D_m} = V_p \lambda = (3)(2.5) = 7.5\, m/s \qquad \textbf{(1)}$$

At this speed the drag coefficient should be the same as prototype since $C_D = f(Re)$. Therefore:

$$C_{D_m} = C_{D_p} = 1.89 \qquad \textbf{(2)}$$

$$W_{total} = C_D \frac{1}{2}\rho V^2 A = (1.89)(0.5)(1.2)(7.5)^2 \frac{\pi}{4}(5/2.5)^2 = 200\, N$$

$$W_{load} = W_{total} - W_{Parachute} = 200 - 40 = 160 \qquad \textbf{(1)}$$