Name: ----------------- **Exam 1** **Time: 50 minutes**

ME:5160 **Fall 2023**

The exam is closed book and closed notes.

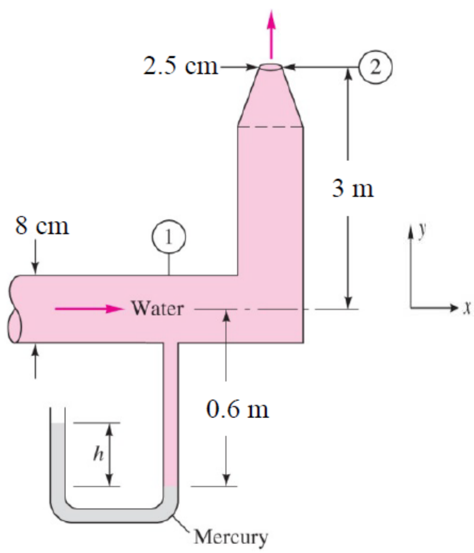
1. In the Figure below the bend is flanged at section 1 (the flange is not shown) and the flow exits to atmosphere at section 2. If V_1 = 0.5 m/s, h = 40 cm, ρ_{water} = 998 kg/m3, and $\rho_{mercury}$ = 13,550 kg/m^3, neglecting the gravity forces and assuming uniform flows at 1 and 2, find: a) V_2 using continuity equation; b) p_1 using manometry equation; c) the force components on the flange bolts in x and y directions using linear momentum equations; d) the friction head loss between 1 and 2 using energy equation.

Continuity equation: $-\frac{d}{dt}\int_{CV}\rho d\forall = \int_{CS}\rho \underline{V_R}\cdot \underline{n}dA$

Momentum equation: $\sum F = \frac{d}{dt}\int_{CV}\rho \underline{V}d\forall + \int_{CS}\rho \underline{V}\underline{V_R}\cdot \underline{n}dA$

Energy Equation (for incompressible steady flow):

$$\left(\frac{p}{\rho g}+\frac{\alpha V^2}{2g}+z\right)_1 = \left(\frac{p}{\rho g}+\frac{\alpha V^2}{2g}+z\right)_2 + h_{turbine} - h_{pump} + h_{friction}$$



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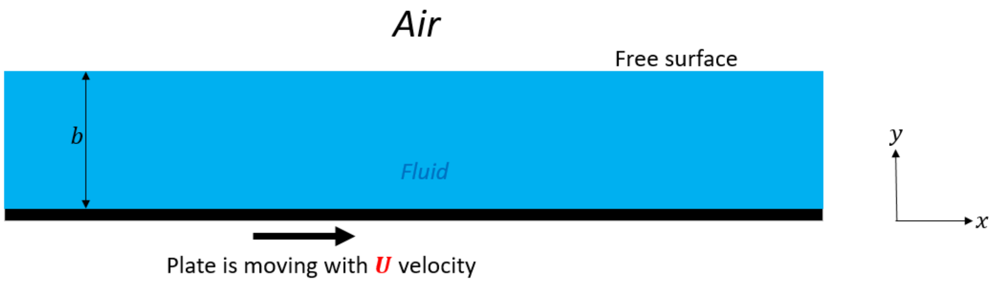
2. The viscous, incompressible flow between the parallel plates shown in Figure is caused by the motion of the bottom plate. Assuming pressure gradient along the x direction is constant $\left(\frac{\partial P}{\partial x} = C\right)$, steady, 2D, and parallel flow and using differential analysis: (a) Show that the flow is fully developed using continuity equation; (b) Find the velocity profile $u(y)$ using Navier-Stokes equations with appropriate boundary conditions; (c) Find wall shear stress at bottom wall; and (d) Find the flow rate (hint: $Q = \int \underline{V} \cdot \underline{n}\, dA$ and assume constant width w). **Explicitly state all assumptions**.



Incompressible Continuity Equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Incompressible Navier-Stokes Equations in Cartesian Coordinates:

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Shear stress (At free surface, shear stress is zero)

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

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3. The power P developed by a wind turbine is a function of turbine's diameter D, air density ρ, wind speed V, and rotation rate ω. Namely it can be written as $P = f(D, \rho, V, \omega)$. Viscosity effects are negligible. Rewrite this relationship in dimensionless form.

Quantity	Symbol	Dimensions $MLT\Theta$	Dimensions $FLT\Theta$
Length	L	L	L
Area	A	L^2	L^2
Volume	$\mathcal{V}$	L^3	L^3
Velocity	V	LT^{-1}	LT^{-1}
Acceleration	dV/dt	LT^{-2}	LT^{-2}
Speed of sound	a	LT^{-1}	LT^{-1}
Volume flow	Q	L^3T^{-1}	L^3T^{-1}
Mass flow	$\dot{m}$	MT^{-1}	FTL^{-1}
Pressure, stress	p, σ, τ	$ML^{-1}T^{-2}$	FL^{-2}
Strain rate	$\dot{\epsilon}$	T^{-1}	T^{-1}
Angle	θ	None	None
Angular velocity	ω, Ω	T^{-1}	T^{-1}
Viscosity	μ	$ML^{-1}T^{-1}$	FTL^{-2}
Kinematic viscosity	ν	L^2T^{-1}	L^2T^{-1}
Surface tension	Υ	MT^{-2}	FL^{-1}
Force	F	MLT^{-2}	F
Moment, torque	M	ML^2T^{-2}	FL
Power	P	ML^2T^{-3}	FLT^{-1}
Work, energy	W, E	ML^2T^{-2}	FL
Density	ρ	ML^{-3}	FT^2L^{-4}
Temperature	T	Θ	Θ
Specific heat	c_p, c_v	$L^2T^{-2}\Theta^{-1}$	$L^2T^{-2}\Theta^{-1}$
Specific weight	γ	$ML^{-2}T^{-2}$	FL^{-3}
Thermal conductivity	k	$MLT^{-3}\Theta^{-1}$	$FT^{-1}\Theta^{-1}$
Thermal expansion coefficient	β	Θ^{-1}	Θ^{-1}

Solution 1:

a) Continuity:

$$A_1V_1 = A_2V_2$$

$$V_2 = \frac{A_1V_1}{A_2} = \frac{(\pi/4)D_1^2V_1}{(\pi/4)D_2^2} = \frac{D_1^2V_1}{D_2^2} = \frac{(0.08)^2(0.5)}{(0.025)^2} = 5.12\ m/s$$ +3.5

b) Manometry:

$$p_1 + \rho_{water}g(0.6) - \rho_{mercury}gh = p_a = 0$$

$$p_1 = \rho_{mercury}gh - \rho_{water}g(0.6)$$

$$= (13550)(9.81)(0.4) - (998)(9.81)(0.6) = 47{,}296\ pa\ (gage)$$ +3

c) $x-$momentum:

$$\sum F_x = \dot{m}u_2 - \dot{m}u_1$$

$$-F_{x,bolts} + p_1A_1 = \rho Q(0 - V_1)$$

$$F_{x,bolts} = p_1A_1 + \rho A_1V_1(V_1) = A_1(p_1 + \rho V_1^2)$$

$$F_{x,bolts} = \left[\frac{\pi}{4}(0.08)^2\right][(47296) + (998)(0.5)^2] = 239\ N$$ +2

$y-$momentum:

$$\sum F_y = \dot{m}v_2 - \dot{m}v_1$$

$$-F_{y,bolts} = \rho Q(V_2 - 0)$$

$$F_{y,bolts} = -\rho A_2V_2^2 = -(998)\left[\frac{\pi}{4}(0.025)^2\right](5.12)^2 = -12.84\ N$$ +1

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d) Energy equation:

$$\frac{p_1}{\rho g} + \frac{\alpha_1 V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{\alpha_2 V_2^2}{2g} + z_2 + h_f$$

$$\frac{(47296)}{(998)(9.81)} + \frac{(1.0)(0.5)^2}{2(9.81)} + 0 = 0 + \frac{(1.0)(5.12)^2}{2(9.81)} + (3.0) + h_f$$

$$h_f = 0.51\ m$$

+0.5

Solution 2:

KNOWN: *Flow condition, Boundary condition*

FIND: **(2)**

- Show that the flow is fully developed
- velocity profile $u(y)$
- Find wall shear stress at bottom wall
- Find the flow rate

ASSUMPTIONS:

- steady: $\frac{\partial}{\partial t} = 0$
- 2D flow: $w = 0$; $\frac{\partial}{\partial z} = 0$
- Parallel flow: $v = w = 0$
- $\frac{\partial p}{\partial x} = \text{constant}$

ANALYSIS:

(a) Continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} + 0 + 0 = 0 \quad \rightarrow \quad \frac{\partial u}{\partial x} = 0 \quad \rightarrow \quad fully\ developed$$ **(2)**

(b) *x*-momentum equation

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$0 = -\frac{\partial P}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$$ **(1.5)**

Integrate twice

$$u(y) = \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 + c_1 y + c_2$$ **(1.5)**

Apply boundary conditions

$$y = 0 \quad \rightarrow \quad u = U \quad \rightarrow \quad c_2 = U$$

$$y = b \quad \rightarrow \quad \tau = 0 \quad \rightarrow \mu\left(\frac{\partial u}{\partial y}\right) = 0 \;\rightarrow\; \mu\left(\frac{1}{\mu}\left(\frac{\partial p}{\partial x}\right)b + c_1\right) = 0$$ **(2)**

$$c_1 = -\frac{1}{\mu}\left(\frac{\partial p}{\partial x}\right)b$$

Therefore,

$$u(y) = \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 - \frac{b}{\mu}\left(\frac{\partial p}{\partial x}\right)y + U$$

$$u(y) = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(y^2 - 2by) + U$$

(c) Shear stress at bottom wall

$$\tau = \mu\frac{du}{dy} = \mu\left\{\frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(2y - 2b)\right\} = \left(\frac{\partial p}{\partial x}\right)(y - b)$$

$$\tau_{wall} = \tau(0) = -b\left(\frac{\partial p}{\partial x}\right)$$ **(0.5)**

(d) Flow rate

$$Q = \int \underline{V}\cdot\underline{n}\,dA = w\int_0^b u(y)dy = \frac{w}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left(\frac{1}{3}y^3 - by^2\right)_0^b + wU(y)_0^b$$

$$Q = \frac{w}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left(\frac{1}{3}b^3 - b^3\right) + wUb = -\frac{wb^3}{3\mu}\left(\frac{\partial p}{\partial x}\right) + wUb$$ **(0.5)**

Solution 3:

ASSUMPTIONS: the problem is only a function of the above dimensional variables
ANALYSIS:

$$P = f(D, \rho, V, \omega).$$

$$n = 5$$

$$P = \left\{\frac{ML^2}{T^3}\right\},\ D = \{L\},\ \rho = \left\{\frac{M}{L^3}\right\},\ V = \left\{\frac{L}{T}\right\},\ \omega = \left\{\frac{1}{T}\right\}$$

(4)

$$j = 3 \ \rightarrow \ k = n - j = 5 - 3 = 2$$

The repeating variables are D, ρ, V

Calculate first non-dimensional group

$$\Pi_0 = PD^a\rho^bV^c$$

$$unit\ [\Pi_0] \rightarrow \left\{\frac{ML^2}{T^3}\right\}\{L^a\}\left\{\frac{M^b}{L^{3b}}\right\}\left\{\frac{L^c}{T^c}\right\}$$ **(2)**

Above one should be non-dimensional

$$[M] \rightarrow 1 + b = 0$$
$$[L] \rightarrow 2 + a - 3b + c = 0$$
$$[T] \rightarrow -3 - c = 0$$
$$\therefore b = -1, c = -3, a = -2$$

$$\therefore \Pi_0 = PD^{-2}\rho^{-1}V^{-3} = \frac{P}{\rho D^2V^3}$$ **(1)**

Calculate second non-dimensional group

$$\Pi_1 = \omega D^a\rho^bV^c$$
$$unit\ [\Pi_1] \rightarrow \left\{\frac{1}{T}\right\}\{L^a\}\left\{\frac{M^b}{L^{3b}}\right\}\left\{\frac{L^c}{T^c}\right\}$$ **(2)**
$$[M] \rightarrow b = 0$$
$$[L] \rightarrow a - 3b + c = 0$$
$$[T] \rightarrow -1 - c = 0$$

$$\therefore b = 0, c = -1, a = 1$$
$$\therefore \Pi_1 = \omega D^1\rho^0V^{-1} = \frac{\omega D}{V}$$

As a result,

$$\Pi_0 = fnc(\Pi_1) \rightarrow \frac{P}{\rho D^2V^3} = fnc\left(\frac{\omega D}{V}\right)$$ **(1)**