Reynolds Transport Theorem

Leibniz theorem: derivative single variable integral having $f(x,t)$ integral and limits $a(t)$ & $b(t)$

$$\frac{d}{dt}\int_{x=a(t)}^{x=b(t)} f(x,t)\,dx = \underbrace{\int_a^b \frac{\partial f}{\partial t}\,dx}_{①} + \underbrace{\frac{db}{dt} f(b,t)}_{②} - \underbrace{\frac{da}{dt} f(a,t)}_{③}$$

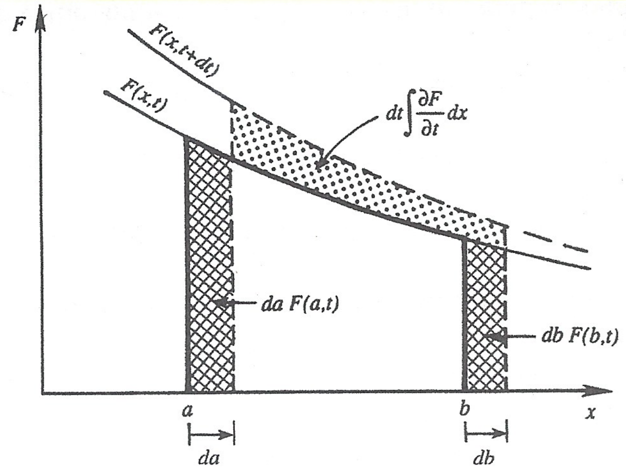

FIGURE 3.19 Graphical illustration of the Liebniz theorem. The three marked areas correspond to the three contributions shown on the right in (3.30). Here da, db, and $\partial F/\partial t$ are all shown as positive.

① integral $\frac{\partial f}{\partial t}\Big|_a^b$

② gain f at upper limit moving at db/dt

③ loss f at lower limit moving at da/dt

Total derivative LHS = integral partial derivative $\int_a^b$ + terms that account for time dependence of a and b

The one-dimensional version of Leibnitz's theorem is also very useful:

$$\frac{d}{dt}\int_{x=a(t)}^{x=b(t)} f(x,t)\,dx = \int_a^b \frac{\partial f}{\partial t}\,dx + \frac{db}{dt}\,f(x=b,t) - \frac{da}{dt}\,f(x=a,t) \qquad (3.13.4)$$

In this form the left-hand side is an integral where the integrand and the limits of integration are a function of the parameter t. The rate of change of this integral with respect to t is equal to the sum of three terms. The first term is the contribution due to the increase $\partial f/dt$ between a and b. The second term is the contribution because the right-hand limit is moving. The integral changes because f at $x = b$ is brought into the integral with the velocity *db/dt*. The third term is similarly the result of the motion of the left-hand limit, *da/dt*. Figure 3.6 depicts the terms in this equation (after the equation has been multiplied by a time increment *dt*).

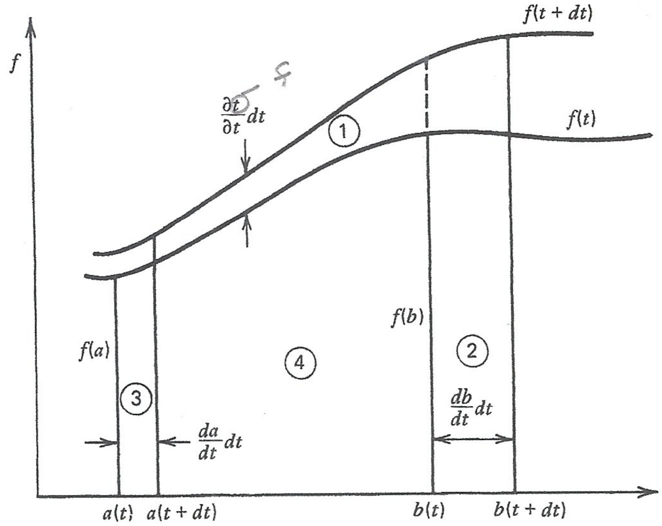

Figure 3.6 Leibnitz's theorem in one dimension.

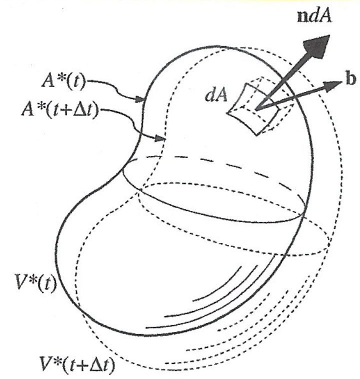

FIGURE 3.20 Geometrical depiction of a control volume $V^*(t)$ having a surface $A^*(t)$ that moves at a nonuniform velocity **b** during a small time increment Δt. When Δt is small enough, the volume increment $\Delta V = V^*(t+\Delta t) - V^*(t)$ will lie very near $A^*(t)$, so the volume-increment element adjacent to dA will be $(\mathbf{b}\Delta t)\cdot\mathbf{n}dA$ where $\mathbf{n}$ is the outward normal on $A^*(t)$.

Generalization for $f(\underline{x},t)$ & moving $V^*(t)$ bounded $A^*(t)$ with local velocity $\underline{b}$ & normal $\underline{n}$

$\overset{CV}{V^*(t)}$ bounded by $\overset{CS}{A^*(t)}$ with local outward normal $\underline{n}$ & velocity $\underline{b}$. No coordinate system needed

$f(\underline{x},t)$ single valued continuous function

$$\frac{d}{dt}\int_{V^*(t)} f(\underline{x},t)\,dV = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{V^*(t+\Delta t)} f(\underline{x},t+\Delta t)\,dV - \cancel{\int_{V^*(t)} f(\underline{x},t)\,dV}\right]$$

$$\int_{V^*(t+\Delta t)} f(\underline{x},t+\Delta t)\,dV = \cancel{\int_{V^*(t)} f(\underline{x},t)\,dV} + \int_{V^*(t)} \Delta t\,\frac{\partial f(\underline{x},t)}{\partial t}\,dV$$

higher order

$$+\int_{\Delta V} f(\underline{x},t)\,dV + \cancel{\int_{\Delta V} \Delta t\,\frac{\partial f(\underline{x},t)}{\partial t}\,dV}$$

where $\Delta V = V^*(t+\Delta t) - V^*(t)$ $\quad f(\underline{x},t+\Delta t) = f(\underline{x},t) + \frac{\partial f(\underline{x},t)}{\partial t}\Delta t$

1st order TS

$$\frac{d}{dt}\int_{V^*(t)} f(\underline{x},t)\,dV = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{V^*(t)} \Delta t\,\frac{\partial f(\underline{x},t)}{\partial t}\,dV + \int_{\Delta V} f(\underline{x},t)\,dV\right]$$

need relationship ΔV & Δt:

$$\Delta V = (\underline{b}\,\Delta t)\cdot\underline{n}\,dA \Rightarrow \int_{\Delta V} f(\underline{x},t)\,dV = \int_{A^*(t)} f(\underline{x},t)(\underline{b}\,\Delta t\cdot\underline{n})\,dA \quad \Delta t\to 0$$

all ΔV summed via surface integral

Thus,

$$\frac{d}{dt}\int_{V^*(t)} f(\underline{x},t)\,dV = \int_{V^*(t)} \frac{\partial f(\underline{x},t)}{\partial t}\,dV + \int_{A^*(t)} f(\underline{x},t)\,\underline{b}\cdot\underline{n}\,dA$$

Specific cases depend on $f(\underline{x},t)$ & $\underline{b}(\underline{x},t)$. In particular CV behavior depends on $\underline{b}(\underline{x},t)$.

1. Deforming CV: (a) $\underline{b} = \underline{b}(\underline{x},t)$ nonuniform/accelerating velocity
   $V^* = V^*(t)$ (b) $\underline{b} = \underline{b}(\underline{x})$ uniform/constant velocity
   (c) $\int_{A^*} \underline{b}(\underline{x},t)\cdot\underline{n}\,dA = 0$ as a whole at rest

accelerating 2. Non deforming CV: (a) $\underline{b} = \underline{b}(t)$ accelerating velocity
steady moving $V^* \neq V^*(t)$ (b) $\underline{b}$ = constant velocity i.e. relative inertial coordinates
stationary (c) $\underline{b} = 0$ at rest

3. Steady flow:
   $V^* \neq V^*(t)$ (a) $\underline{b}$ = constant
   $f(\underline{x},t) = f(\underline{x})$ (b) $\underline{b} = 0$

$$0 = \int_{A^*} f(\underline{x})\,\underline{b}\cdot\underline{n}\,dA = \sum_{A_i^*} \underbrace{f(\underline{x})\,\underline{b}\cdot\underline{n}}_{\text{constant over discrete flow areas}} A \quad (a)$$

$$\text{or } \sum_{A_i} \underbrace{\int_{A_i} f(\underline{x})\,\underline{b}\cdot\underline{n}\,dA}_{\text{if nonuniform over discrete flow areas}} \quad (b)$$

4. MV: $\underline{b} = \underline{u}$ & $V^* = V$, $A^* = A$

Physical interpretations:

1. $f = 1$: conservation of volume

Exercise 3.33. Starting from (3.35), set $F = 1$ and derive (3.14) when $\mathbf{b} = \mathbf{u}$ and $V^*(t) = \delta V \to 0$.

Solution 3.33. With $F = 1$, $\mathbf{b} = \mathbf{u}$, and $V^*(t) = \delta V$ with surface δA, (3.35) becomes:

$$\frac{d}{dt}\int_{\delta V} dV = 0 + \int_{\delta A} \mathbf{u}\cdot\mathbf{n}dA.$$

The first integral is merely δV. Use Gauss' divergence theorem on the second term to convert it to volume integral.

$$\frac{d}{dt}(\delta V) = \int_{\delta V} \nabla\cdot\mathbf{u}dV.$$

As $\delta V \to 0$ the integral reduces to a product of δV and the integrand evaluated at the center point of δV. Divide both sides of the last equation by δV and take the limit as $\delta V \to 0$:

$$\lim_{\delta V\to 0}\frac{1}{\delta V}\frac{d}{dt}(\delta V) = \lim_{\delta V\to 0}\frac{1}{\delta V}\int_{\delta V}\nabla\cdot\mathbf{u}dV = \lim_{\delta V\to 0}\frac{1}{\delta V}\left[(\nabla\cdot\mathbf{u})\delta V + ...\right] = \nabla\cdot\mathbf{u} = S_{ii},$$

and this is (3.14).

$$\frac{1}{\delta V}\frac{D(\delta V)}{Dt} = u_x + v_y + w_z = u_{i,i}$$

2. RTT = $\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{u}\cdot\nabla f = \frac{\partial f}{\partial t} + u_i\frac{\partial f}{\partial x_i}$

Extended CV

3.35. Show that (3.35) reduces to (3.5) when $V^*(t) = \delta V \to 0$ and the control surface velocity $\mathbf{b}$ is equal to the fluid velocity $\mathbf{u}(\mathbf{x},t)$.

Solution 3.35. When $V^*(t) = \delta V$ with surface δA, δV is small, and $\mathbf{b} = \mathbf{u}$, δV represents a fluid particle. Under these conditions (3.35) becomes:

$$\frac{d}{dt}\int_{\delta V} F(\mathbf{x},t)dV = \int_{\delta V}\frac{\partial F(\mathbf{x},t)}{\partial t}dV + \int_{\delta A} F(\mathbf{x},t)\mathbf{u}\cdot\mathbf{n}dA,$$

and the time derivative is evaluated following δV. Use Gauss' divergence theorem on the final term to convert it to a volume integral,

$$\int_{\delta A} F(\mathbf{x},t)\mathbf{u}\cdot\mathbf{n}dA = \int_{\delta V}\nabla\cdot(F(\mathbf{x},t)\mathbf{u})dV,$$

so that (3.35) becomes:

$$\frac{d}{dt}\int_{\delta V} F(\mathbf{x},t)dV = \int_{\delta V}\left[\frac{\partial F(\mathbf{x},t)}{\partial t} + \nabla\cdot(F(\mathbf{x},t)\mathbf{u})\right]dV = \int_{\delta V}\left[\frac{\partial F(\mathbf{x},t)}{\partial t} + F(\mathbf{x},t)\nabla\cdot\mathbf{u} + (\mathbf{u}\cdot\nabla)F(\mathbf{x},t)\right]dV,$$

where the second equality follows from expanding the divergence of the product $F\mathbf{u}$.

As $\delta V \to 0$ the various integrals reduce to a product of δV and the integrand evaluated at the center point of δV. Divide both sides of the prior equation by δV and take the limit as $\delta V \to 0$ to find:

$$\lim_{\delta V\to 0}\frac{1}{\delta V}\frac{d}{dt}\int_{\delta V} F(\mathbf{x},t)dV = \lim_{\delta V\to 0}\frac{1}{\delta V}\int_{\delta V}\left[\frac{\partial F(\mathbf{x},t)}{\partial t} + F(\mathbf{x},t)\nabla\cdot\mathbf{u} + (\mathbf{u}\cdot\nabla)F(\mathbf{x},t)\right]dV,$$

$$\lim_{\delta V\to 0}\frac{1}{\delta V}\frac{d}{dt}\left[F(\mathbf{x},t)\delta V + ...\right] = \lim_{\delta V\to 0}\frac{1}{\delta V}\left[\left(\frac{\partial F(\mathbf{x},t)}{\partial t} + F(\mathbf{x},t)\nabla\cdot\mathbf{u} + (\mathbf{u}\cdot\nabla)F(\mathbf{x},t)\right)\delta V + ...\right], \text{ or}$$

$$\frac{d}{dt}F(\mathbf{x},t) + F(\mathbf{x},t)\lim_{\delta V\to 0}\frac{1}{\delta V}\frac{d}{dt}(\delta V) = \frac{\partial F(\mathbf{x},t)}{\partial t} + F(\mathbf{x},t)\nabla\cdot\mathbf{u} + (\mathbf{u}\cdot\nabla)F(\mathbf{x},t),$$

where the product rule for derivative has been used on product $F\delta V$ in [,]-braces on the left. From (3.14) or Exercise 3.33: $\lim_{\delta V\to 0}\frac{1}{\delta V}\frac{d}{dt}(\delta V) = \nabla\cdot\mathbf{u}$, so the second terms on both sides of the last equation are equal and may be subtracted out leaving:

$$\frac{d}{dt}F(\mathbf{x},t) = \frac{\partial F(\mathbf{x},t)}{\partial t} + (\mathbf{u}\cdot\nabla)F(\mathbf{x},t),$$

and this is (3.5) when the identification $D/Dt \equiv d/dt$ is made.

Relationship MV & CV

$$MV: \frac{d}{dt}\int_{\forall(t)} f(\underline{x},t)\, d\forall = \int_{\forall(t)} \frac{\partial f}{\partial t}\, d\forall + \int_{A(t)} f\, \underline{u}\cdot\underline{n}\, dA$$

$\forall$ = MV

A = boundary with local normal $\underline{n}$

moving at nonuniform $\underline{u}(\underline{x},t)$

Greens theorem:

$$\int_{\forall} \nabla\cdot\underline{b}\, d\forall = \int_{S} \underline{b}\cdot\underline{n}\, dA$$

$$\frac{d}{dt}\int_{\forall} f\, d\forall = \int_{\forall} \left[\frac{\partial f}{\partial t} + \nabla\cdot(f\underline{u})\right] d\forall$$

lim $d\forall \to 0$ provides GDE $\frac{d}{dt}\int_{\forall} f\, d\forall = \frac{\partial f}{\partial t} + \nabla\cdot(f\underline{u})$

Assume at time t MV & CV coincide

$$\frac{d}{dt}\int_{\forall} f\, d\forall = \int_{\forall^*} \frac{\partial f}{\partial t}\, d\forall + \int_{A^*} f\, \underline{u}\cdot\underline{n}\, dA$$

$$\int_{\forall^*} \frac{\partial f}{\partial t}\, d\forall = \frac{d}{dt}\int_{\forall^*} f\, d\forall - \int_{A^*} f\, \underline{u_s}\cdot\underline{n}\, dA$$ $\underline{u_s} = \underline{b}$ = velocity of CS A^*

$$\therefore \frac{d}{dt}\int_{\forall} f\, d\forall = \frac{d}{dt}\int_{\forall^*} f\, d\forall + \int_{A^*} f\, \underbrace{(\underline{u}-\underline{u_s})}_{\underline{u_R}}\cdot\underline{n}\, dA$$ $\underline{u_R}$ = relative velocity

GDE : $\frac{d}{dt}(m, m\underline{u}, E) = RHS = (0, m\underline{a}, \dot{Q} - \dot{W})$

$\frac{dB_{sys}}{dt} = RHS$

$B_{sys} = f(m) =$ extensive property

$\beta = \frac{dB}{dm} = (1, \underline{u}, e) =$ intensive property
independent m

$B = \int \beta \, dm = \int \beta \rho \, d\forall$

$f = \beta \rho \quad \underline{u}_R = \underline{u} - \underline{u}_s$

$$\frac{d}{dt}\int_{\forall} \beta \rho \, d\forall = \frac{d}{dt}\int_{\forall^*} \beta \rho \, d\forall + \int_{A^*} \beta \rho \, \underline{u}_R \cdot \underline{n} \, dA = \frac{dB_{sys}}{dt}$$

Specific cases same as CV derivation

Next step : apply $\beta = (1, \underline{u}, e)$

I. Conservation of mass $B_{sys} = m$
$\beta = 1$

$$\frac{dm}{dt} = 0 = \frac{d}{dt}\int_{\forall^*} \rho \, d\forall + \int_{A^*} \rho \, \underline{u}_R \cdot \underline{n} \, dA$$

① most general case $\rho(\underline{x}, t)$ and $\underline{u}_R$, $\underline{u}$ & $\underline{u}_s = f(\underline{x}, t)$

$$-\frac{d}{dt}\int_{\forall^*} \rho \, d\forall = \int_{A^*} \rho \, \underline{u}_R \cdot \underline{n} \, dA$$

rate of decrease = net outflow

② Other specific cases depend on $\rho \neq f(t)$ or $\rho \neq f(\underline{x})$ & form of $\underline{u}(\underline{x}, t)$ or per RTT for $\forall^*$

ie deforming or nondeforming and each case either accelerating, steady moving, or stationary

II Conservation of Momentum

$B_{sys} = m\underline{u}$

$\beta = \underline{u}$

$$\frac{dB_{sys}}{dt} = \frac{d(m\underline{u})}{dt} = \frac{d}{dt}\int_{V^*} \rho \underline{u}\, dV + \int_{A^*} \rho \underline{u}\; \underline{u}_R \cdot \underline{n}\, dA$$

RHS = $\Sigma \underline{F}$ acting on CV V^*

$$= \underbrace{\int_{V} \rho \underline{g}\, dV}_{\text{body force}} + \underbrace{\int_{A^*} \underline{\Sigma}(\underline{n}, \underline{x}, t)\, dA}_{\text{surface force}} = \text{RHS}$$

Here again, (1) most general case $\rho = \rho(\underline{x}, t)$ & $\underline{u}_R$, $\underline{u}$, & $\underline{u}_S$ all $f(\underline{x}, t)$

& (2) other specific cases depend on
and form $\underline{\Sigma}(\underline{x}, t)$ or per RTT for V^*

Body force $\rho \underline{g}\, dV$ acts on dV without physical contact; and is conservative since by definition conservative body forces can be expressed as the gradient of a potential function

$\frac{m}{s^2}$ $\quad \underline{g} = -\nabla \Phi$ or $g_i = -\frac{\partial \Phi}{\partial x_i}$ $\quad \Phi$ = force potential with units energy per unit mass

ρ per unit mass

PE gz $\frac{m^2}{s^2}$ $\quad z \uparrow \downarrow \underline{g}$ $\quad \Phi = gz$ $\frac{m^2}{s^2}$

KE $\frac{1}{2}u^2$

$\underline{g} = -g\hat{z}$

internal $\hat{u}$

Surface forces act on fluid elements via direct contact with the CS with units of stress W/m2 and normal and tangential components

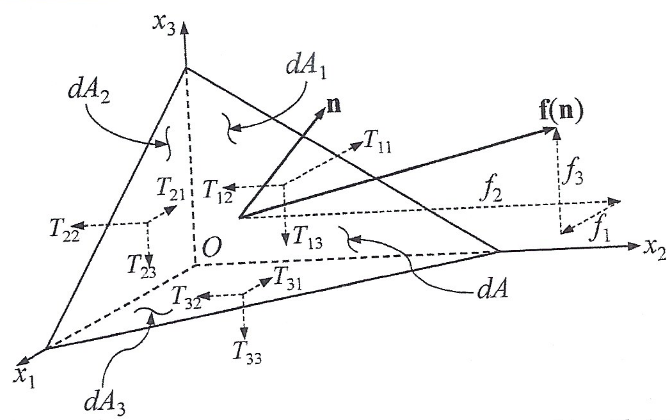

FIGURE 2.5 Force f per unit area on a surface element whose outward normal is **n**. The areas of the tetrahedron's faces that are perpendicular to the i^{th} coordinate axis are dA_i. The area of the largest tetrahedron face is dA. As in Figure 2.4, the directions of positive normal and shear stresses are shown.

Arbitrarily oriented dA with normal $\underline{n} = n_i$

Surface force $\underline{f}(\underline{n}, \underline{x}, t) = f_j = n_i T_{ij}$ per unit area

$f_1 = n_1 T_{11} + n_2 T_{21} + n_3 T_{31}$ $\quad T_{ij}$ = stress tensor

$f_2 = n_1 T_{12} + n_2 T_{22} + n_3 T_{32}$

$f_3 = n_1 T_{13} + n_2 T_{23} + n_3 T_{33}$

Normal component $= \underline{n} \cdot \underline{f} = n_i f_i$

tangential component vector $= \underline{f} - (\underline{n} \cdot \underline{f})\underline{n}$

$= f_k - (n_i f_i) n_k$

$\underline{n} \cdot \underline{f} = n_1 f_1 + n_2 f_2 + n_3 f_3$

Cubical element

x-force : $\underline{n} = (1,0,0)$ $\qquad f_1 = T_{11}$

$f_2 = T_{12}$

$\underline{f_x} = T_{11}\hat{\imath} + T_{12}\hat{\jmath} + T_{13}\hat{k}$ $\qquad f_3 = T_{13}$

$\underline{n} \cdot \underline{f_x} = T_{11}$ $\qquad$ tangential component $\underline{f_t} = \underline{f_x} - T_{11}\hat{\imath}$

$= T_{12}\hat{\jmath} + T_{13}\hat{k}$

y-force : $\underline{n} = (0,1,0)$ $\qquad f_1 = T_{21}$

$f_2 = T_{22}$

$\underline{f_y} = T_{21}\hat{\imath} + T_{22}\hat{\jmath} + T_{23}\hat{k}$ $\qquad f_3 = T_{23}$

z-force : $\underline{n} = (0,0,1)$ $\qquad f_1 = T_{31}$

$f_2 = T_{32}$

$\underline{f_z} = T_{31}\hat{\imath} + T_{32}\hat{\jmath} + T_{33}\hat{k}$ $\qquad f_3 = T_{33}$.

Write momentum equation for MV : $\underline{u}_R = 0$
$\forall^* = \forall$, $A^* = A$

$$\frac{d}{dt}\int_{\forall} \rho \underline{u}\, d\forall = \int_{\forall} \frac{\partial}{\partial t}(\rho \underline{u})\, d\forall + \int_{A} \rho \underline{u}\,\underline{u}\cdot\underline{n}\, dA$$

$$= \int_{\forall} \rho \underline{g}\, d\forall + \int_{A} \underline{f}\, dA$$

$$\int_{A} \rho \underline{u}\,\underline{u}\cdot\underline{n}\, dA = \int_{\forall} \nabla\cdot(\rho \underline{u}\,\underline{u})\, d\forall = \int_{\forall} \frac{\partial}{\partial x_i}(\rho u_i u_j)\, d\forall$$

$$\int_{A} \underline{f}\, dA = \int_{A} n_i T_{ij}\, dA = \int_{\forall} \frac{\partial}{\partial x_i} T_{ij}\, d\forall$$

$$\int_{\forall} \left[\frac{\partial}{\partial t}(\rho u_j) + \frac{\partial}{\partial x_i}(\rho u_i u_j) - \rho g_j - \frac{\partial}{\partial x_i}(T_{ij})\right] d\forall = 0$$

$$\lim_{\forall \to 0} : \frac{\partial}{\partial t}(\rho u_j) + \frac{\partial}{\partial x_i}(\rho u_i u_j) = \rho g_j + \frac{\partial}{\partial x_i}(T_{ij})$$

$$\frac{\partial}{\partial t}(\rho u_j) + \frac{\partial}{\partial x_i}(\rho u_i u_j) = \frac{\partial \rho}{\partial t} u_j + \rho\frac{\partial u_j}{\partial t} + u_j \frac{\partial}{\partial x_i}(\rho u_i) + \rho u_i \frac{\partial u_j}{\partial x_i}$$

$$= \rho\frac{\partial u_j}{\partial t} + u_j \underbrace{\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i)\right]}_{=0 \text{ continuity}} + \rho u_i \frac{\partial u_j}{\partial x_i}$$

(1)

$$= \rho\frac{\partial u_j}{\partial t} + \rho u_i \frac{\partial u_j}{\partial x_i} = \rho\frac{D u_j}{D t}$$

(2) $$\rho\frac{D u_j}{D t} = \rho g_j + \frac{\partial}{\partial x_i}(T_{ij})$$ Cauchy equation of motion

unknowns : ρ, u_j, T_{ij} = 1 + 3 + 9 = 13 need stress-strain relationship

equations : 1 + 3 + 2 = 6

↑ thermodynamic equation (ρ, p)

$$\begin{array}{cc} \text{row} & \text{column} \\ 1 \times & 3 \end{array} \qquad V = 1^{st} \text{ order tensor (column matrix)}$$

$\underline{u} \cdot \underline{v}$ $\quad u_i v_i \quad v^T V \quad v^T = [u_1\ u_2\ u_3] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

3×1

$$A = \overset{m}{1} \times \overset{n}{3} \qquad B = \overset{n}{3} \times \overset{p}{1} \qquad C = \overset{m}{1} \times \overset{p}{1}$$

$v^T v = 1 \times 1 = 0$ order tensor

$$c_{ij} = \sum_{k=1}^{n=3} a_{ik} b_{kj} \qquad i = 1, m = 1 \quad j = 1, p = 1 \quad = a_{11} b_{11} + a_{12} b_{21} + a_{13} b_{31}$$

$$= u_1 v_1 + u_2 v_2 + u_3 v_3$$

$\underline{u}\,\underline{v}$ $\quad u_i v_j \quad v\, v^T \quad \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} [v_1\ v_2\ v_3]$

$$c_{ij} = \sum_{k=1}^{n=1} a_{i1} b_{1j} \qquad i = 1,3 \quad j = 1,3$$

$$\overset{}{3} \times 1 \qquad 1 \times 3 \qquad = \qquad 3 \times 3$$

$$\begin{array}{cccccc} m & n & n & p & m & p \end{array}$$

$a_{11} b_{11} \quad a_{11} b_{12} \quad a_{11} b_{13} \qquad u_1 v_1 \quad u_1 v_2 \quad u_1 v_3$

$a_{21} b_{11} \quad a_{21} b_{12} \quad a_{21} b_{13} \qquad u_2 v_1 \quad u_2 v_2 \quad u_2 v_3$

$a_{31} b_{11} \quad a_{31} b_{12} \quad a_{31} b_{13} \qquad u_3 v_1 \quad u_3 v_2 \quad u_3 v_3$

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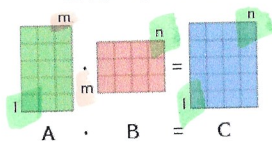
Matrix multiplication

columns A = rows B



C = rows A
x columns B

For matrix multiplication, the number of columns in the first matrix must be equal to the number of rows in the second matrix. The result matrix has the number of rows of the first and the number of columns of the second matrix.

In mathematics, particularly in linear algebra, **matrix multiplication** is a binary operation that produces a matrix from two matrices. For matrix multiplication, the number of columns in the first matrix must be equal to the number of rows in the second matrix. The resulting matrix, known as the **matrix product**, has the number of rows of the first and the number of columns of the second matrix. The product of matrices **A** and **B** is denoted as **AB**.[1]

Matrix multiplication was first described by the French mathematician Jacques Philippe Marie Binet in 1812,[2] to represent the composition of linear maps that are represented by matrices. Matrix multiplication is thus a basic tool of linear algebra, and as such has numerous applications in many areas of mathematics, as well as in applied mathematics, statistics, physics, economics, and engineering.[3][4] Computing matrix products is a central operation in all computational applications of linear algebra.

Notation

This article will use the following notational conventions: matrices are represented by capital letters in bold, e.g. **A**; vectors in lowercase bold, e.g. **a**; and entries of vectors and matrices are italic (they are numbers from a field), e.g. A and a. Index notation is often the clearest way to express definitions, and is used as standard in the literature. The entry in row i, column j of matrix **A** is indicated by $(\mathbf{A})_{ij}$, A_{ij} or a_{ij}. In contrast, a single subscript, e.g. $\mathbf{A}_1$, $\mathbf{A}_2$, is used to select a matrix (not a matrix entry) from a collection of matrices.

Definition

If **A** is an $m \times n$ matrix and **B** is an $n \times p$ matrix,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}$$

the *matrix product* **C** = **AB** (denoted without multiplication signs or dots) is defined to be the $m \times p$ matrix[5][6][7][8]

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{pmatrix}$$

such that

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj},$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, p$.

That is, the entry c_{ij} of the product is obtained by multiplying term-by-term the entries of the ith row of **A** and the jth column of **B**, and summing these n products. In other words, c_{ij} is the dot product of the ith row of **A** and the jth column of **B**.

Therefore, **AB** can also be written as

$$\mathbf{C} = \begin{pmatrix} a_{11}b_{11} + \cdots + a_{1n}b_{n1} & a_{11}b_{12} + \cdots + a_{1n}b_{n2} & \cdots & a_{11}b_{1p} + \cdots + a_{1n}b_{np} \\ a_{21}b_{11} + \cdots + a_{2n}b_{n1} & a_{21}b_{12} + \cdots + a_{2n}b_{n2} & \cdots & a_{21}b_{1p} + \cdots + a_{2n}b_{np} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{11} + \cdots + a_{mn}b_{n1} & a_{m1}b_{12} + \cdots + a_{mn}b_{n2} & \cdots & a_{m1}b_{1p} + \cdots + a_{mn}b_{np} \end{pmatrix}$$

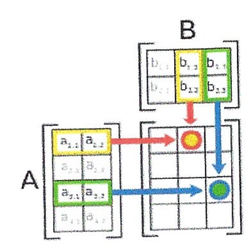
Thus the product **AB** is defined if and only if the number of columns in **A** equals the number of rows in **B**,[1] in this case n.

In most scenarios, the entries are numbers, but they may be any kind of mathematical objects for which an addition and a multiplication are defined, that are associative, and such that the addition is commutative, and the multiplication is distributive with respect to the addition. In particular, the entries may be matrices themselves (see block matrix).

Illustration

The figure to the right illustrates diagrammatically the product of two matrices **A** and **B**, showing how each intersection in the product matrix corresponds to a row of **A** and a column of **B**.

$$\underset{4\times2\text{ matrix}}{\begin{bmatrix} a_{11} & a_{12} \\ \cdot & \cdot \\ a_{31} & a_{32} \\ \cdot & \cdot \end{bmatrix}} \underset{2\times3\text{ matrix}}{\begin{bmatrix} \cdot & b_{12} & b_{13} \\ \cdot & b_{22} & b_{23} \end{bmatrix}} = \underset{4\times3\text{ matrix}}{\begin{bmatrix} \cdot & c_{12} & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & c_{33} \\ \cdot & \cdot & \cdot \end{bmatrix}}$$



The values at the intersections, marked with circles in figure to the right, are:

$c_{12} = a_{11} b_{12} + a_{12} b_{22}$

$c_{33} = a_{31} b_{13} + a_{32} b_{23}$

Constitutive equation for Newtonian fluid

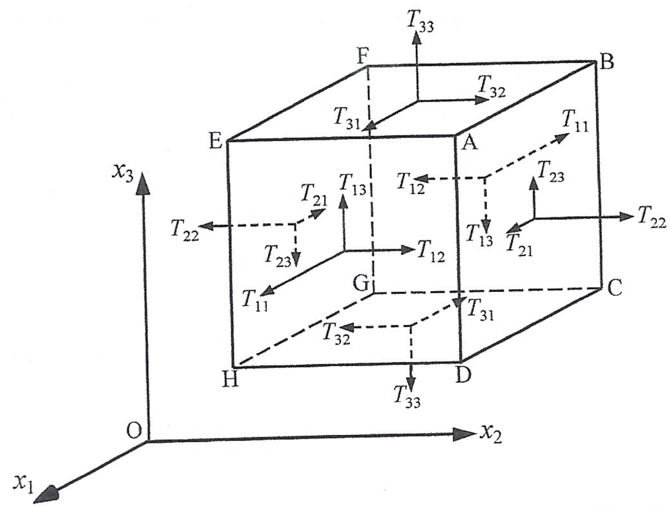


FIGURE 2.4 Illustration of the stress field at a point via stress components on a cubic volume element. Here each surface may experience one normal and two shear components of stress. The directions of positive normal and shear stresses are shown.

Stress at a point fully described by T_{ij}: 9 components
However,

$$T_{ij} = T_{ji}$$

is symmetric such that only six independent components; since, the stresses themselves cause no rotation, which is shown by considering the differential equation of angular momentum for limit $dV = dx_1 dx_2 dx_3 \to 0$ assuming no external body force moments $\propto \rho$ such as electric field or polarized fluid molecules.

$$T_{ij} = f(u_{i,j}) = \text{constitutive equation}$$

$$= \alpha\, u_{i,j} \quad \text{Newtonian fluid}$$

For fluid at rest, only normal stresses which do not depend on the orientation of the surface, ie, isotropic. The only 2nd order isotropic tensor is the Kronecker delta, δ_{ij}, $\overline{\overline{o}}$

(1) $T_{ij} = -p\delta_{ij}$ — sign as must be compressive
p = thermodynamic p related ρ, T via equation of state

For a moving fluid, additional stresses due μ, ie,

(2) $T_{ij} = -p\delta_{ij} + \tau_{ij}$ reduces (1) $\tau_{ij} = 0$

assumes thermodynamic equilibrium as per continuum hypothesis such that p still thermodynamic pressure

τ_{ij} = deviatoric stress tensor
$\neq f(u_i)$ to be invariant Galilean transformation (two inertial reference frames moving at constant velocity relative each other; also called Newtonian transformation)

$\tau_{ij} = f(u_{i,j})$ however stresses due deformation not rotation

$= f(S_{ij})$ $u_{i,j} = S_{ij} + \frac{1}{2}R_{ij}$ deformation

$S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ strain

$\frac{1}{2}R_{ij} = (u_{i,j} - u_{j,i})$ rotation

$\tau_{ij} = K_{ijmn} S_{mn}$ Most general linear relationship including $\tau_{ij} = 0$ for $S_{ij} = 0$, ie static fluid

K_{ijmn} = 4th order tensor with 81 components ie 9x9, ie, allows each of the 9 τ_{ij} to be linearly related all nine components of S_{ij}

However, $T_{ij} = T_{ji}$, ie, symmetric & fluid is isotropic. In isotropic medium stress-strain relationship must be independent orientation coordinate system, which is only possible if K_{ijmn} is an isotropic tensor. The only possible 4th order isotropic tensor must be of the form

$$K_{ijmn} = \lambda \delta_{ij}\delta_{mn} + \mu \delta_{im}\delta_{jn} + \gamma \delta_{in}\delta_{jm}$$

with λ, μ, γ scalars = f(thermodynamic state)

$$\tau_{ij} = \tau_{ji} \Rightarrow K_{ijmn} = K_{jimn}$$

$$K_{ijmn} = \lambda \delta_{ij}\delta_{mn} + \mu \delta_{im}\delta_{jn} + \gamma \delta_{in}\delta_{jm} =$$
$$K_{jimn} = \lambda \delta_{ji}\delta_{mn} + \mu \delta_{jm}\delta_{in} + \gamma \delta_{jn}\delta_{im}$$

$\therefore \quad \gamma = \mu$ & 81 constants reduces to 2 i.e. λ & μ

$$K_{ijmn} = \lambda \delta_{ij}\delta_{mn} + \mu(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})$$

$$\tau_{ij} = [\lambda \delta_{ij}\delta_{mn} + \mu(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})] S_{mn}$$

$$= 2\mu S_{ij} + \lambda S_{mm}\delta_{ij} \qquad S_{mm} = \nabla \cdot \underline{u}$$

$$T_{ij} = -p\delta_{ij} + \tau_{ij} = -p\delta_{ij} + 2\mu S_{ij} + \lambda S_{mm}\delta_{ij}$$

$$T_{ii} = -3p + (2\mu + 3\lambda) S_{mm}$$

$S_{11} = u_x$, $S_{22} = v_y$, $S_{33} = w_z$ — not equal in all; use avg to define $\bar{p}$

$$p = -\tfrac{1}{3}T_{ii} + \left(\tfrac{2}{3}\mu + \lambda\right)\nabla \cdot \underline{u}$$

define $\bar{p} = -\frac{1}{3}T_{ii}$ mean pressure

$$p - \bar{p} = \left(\tfrac{2}{3}\mu + \lambda\right)\nabla \cdot \underline{u}$$

$\neq p =$ thermodynamic pressure

(1) ρ = constant ie incompressible

$p = \bar{p}$ ie can only define mean = mechanical pressure

which means absolute / thermodynamic

p indeterminant for ρ = constant

since no equation of state for liquids.

Only $\partial p / \partial x_i$ can be determined via NS

$T_{ij} = -p\delta_{ij} + 2\mu S_{ij}$ λ not needed

(2) $\rho = \rho(p, T)$

$\Delta p \propto$ expansion / contraction fluid particle

$$p - \bar{p} = \underbrace{\left(\tfrac{2}{3}\mu + \lambda\right)}_{\mu_v \,=\, \text{bulk viscosity}} \nabla \cdot \underline{u}$$

important sound absorption & shock-waves

Stokes assumption

$\lambda = -2/3\,\mu$ often

reasonable since often μ_v small and/or $\nabla \cdot \underline{u}$

EXAMPLE 4.8

Write out all the components of the stress tensor **T** in (x, y, z)-coordinates in terms of $\mathbf{u} = (u, v, w)$, and its derivatives.

Solution

Evaluate each component of (4.36) and abbreviate $S_{mm} = \partial u/\partial x + \partial v/\partial y + \partial w/\partial z = \nabla \cdot \mathbf{u}$ to find:

$$\mathbf{T} = \begin{bmatrix} -p + 2\mu\dfrac{\partial u}{\partial x} + \left(\mu_v - \dfrac{2}{3}\mu\right)\nabla\cdot\mathbf{u} & \mu\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \mu\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) \\ \mu\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & -p + 2\mu\dfrac{\partial v}{\partial y} + \left(\mu_v - \dfrac{2}{3}\mu\right)\nabla\cdot\mathbf{u} & \mu\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) \\ \mu\left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\right) & \mu\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right) & -p + 2\mu\dfrac{\partial w}{\partial z} + \left(\mu_v - \dfrac{2}{3}\mu\right)\nabla\cdot\mathbf{u} \end{bmatrix}$$

Without Stokes assumption,

$$T_{ij} = -p\delta_{ij} + \tau_{ij} = -p\delta_{ij} + \underbrace{2\mu\left(S_{ij} - \tfrac{1}{3}S_{mm}\delta_{ij}\right) + \mu_v S_{mm}\delta_{ij}}_{2\mu S_{ij} + \lambda S_{mm}\delta_{ij}}$$

$$\mu_v = \lambda + \tfrac{2}{3}\mu$$

$\tau \propto S_{ij}$ = Newtonian fluid with μ and μ_v thermodynamic properties

Presumed valid small S_{ij}, but found accurate many common fluids: air, water, gasoline, and oils.

Other liquids display non Newtonian behavior at moderate strain rates: long chain polymer molecules; concentrated soaps; melted plastics; emulsions; slurries with suspended particles; and many biological liquids.

(1) Non linear Shear Stress

For example $\underline{u} = (u_1(x_2), 0, 0)$

$\eta = m\dot{\gamma}^{n-1}$

$$\tau_{12} = \eta\, u_{1,2} = (m\dot{\gamma}^{n-1})\, u_{1,2} = m\dot{\gamma}^{n} \qquad \dot{\gamma} = u_{1,2}$$

η non Newtonian viscosity m = power law coeff

liquid plastic
polymeric solutions $n<1$ shear thinning / pseudoplastic n = power law

water and starch
concentrated suspensions $n>1$ shear thickening / dilatant

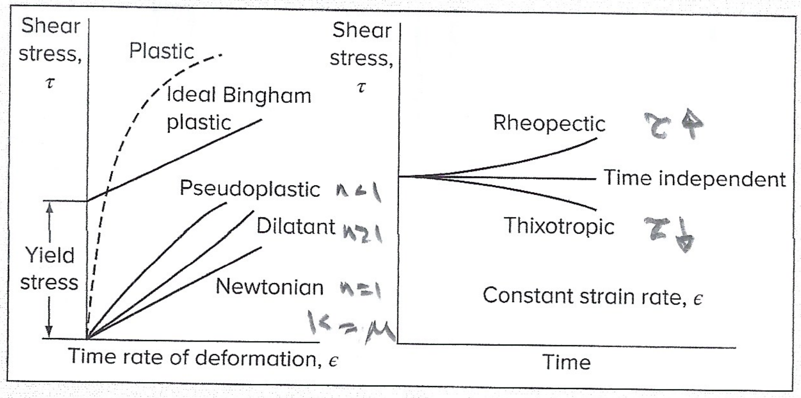
② $T_{ij} = f(\text{history } S_{ij})$, eg, linear viscoelastic materials, which have mixed solid/fluid behavior

$$\tau_{ij} = \int_{-\infty}^{t} K_{ijmn}(t-t')\, S_{mn}(t')\, dt'$$

↑ tensorial relaxation modulus

③ Normal stress differences such as 1D shear flow $\underline{u} = (u_1(x_2), 0, 0)$ with $T_{11} - T_{22} \neq 0$ & $T_{22} - T_{33} \neq 0$

polymeric fluid
$T_{11} - T_{22} < 0$
$T_{22} - T_{33} > 0$



$\tau_{xy} = f(\epsilon_{xy})$

$\epsilon_{xy} = \frac{1}{2} u_y$

FIGURE 1-14
Viscous behavior of various materials.

True fluids $\tau(0) = 0$; yielding fluids are part fluid part solid
'ideal' = clay suspension, putty, toothpaste, drilling muds, and cement slurries

Power law: $\tau_{xy} \simeq 2K\epsilon_{xy}^{n}$
K and n material properties
= f(p, t, composition)

Not realistic $\epsilon = 0$: pseudoplastic $\tau \to \infty$
∴ other formulas dilatant $\tau \to 0$

Navier-Stokes equations

$$\rho\left[\frac{\partial u_j}{\partial t} + u_i\frac{\partial u_j}{\partial x_i}\right] = -\frac{\partial p}{\partial x_j} + \rho g_j + \frac{\partial}{\partial x_i}\Big[\mu\left(u_{j,i}+u_{i,j}\right) + \left(\mu_v - \tfrac{2}{3}\mu\right)u_{m,m}\delta_{ij}\Big]$$

μ & μ_v thermodynamic properties

$$+ \quad \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underline{u}) = \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0$$

or $\dfrac{D\rho}{Dt} + \rho\nabla\cdot\underline{u} = 0 \qquad \dfrac{D\rho}{Dt} = \dfrac{\partial \rho}{\partial t} + \underline{u}\cdot\nabla\rho = 0$

$\rho=\rho(p)$ barotropic fluid

$1+3=4$ equations in $\rho, p,$ & $u_j = 5$ unknowns
if ρ known constant or $\rho=\rho(p)$ can be solved
otherwise if $\rho=\rho(p,T)$ energy equation (e)
also needed such that T & e also unknown.
Additionally, equations of state needed
for thermodynamic variables.

For μ & $\mu_v \neq f(T)$ = constants

$$\rho\frac{Du_j}{Dt} = -\frac{\partial p}{\partial x_j} + \rho g_j + \mu\frac{\partial^2 u_j}{\partial x_i^2} + \left(\mu_v + \tfrac{1}{3}\mu\right)\frac{\partial}{\partial x_j}\frac{\partial u_m}{\partial x_m}$$

For ρ = constant ie incompressible fluid

Euler equation: $\rho\dfrac{D\underline{u}}{Dt} = -\nabla p + \rho \underline{g}$

$\nabla\cdot\underline{u} = 0 \qquad \rho\dfrac{D\underline{u}}{Dt} = -\nabla p + \rho\underline{g} + \mu\nabla^2\underline{u}$

since $\mu\dfrac{\partial^2 u_j}{\partial x_i^2} = 2\mu\dfrac{\partial S_{ij}}{\partial x_i} = \mu\dfrac{\partial}{\partial x_i}\left(u_{j,i}+u_{i,j}\right) = -\mu\varepsilon_{ijk}\dfrac{\partial \omega_k}{\partial x_i}$

or $\mu\nabla^2\underline{u} = -\mu\nabla\times\underline{\omega}$ paradox net viscous force $f(\underline{\omega})$
even though $\tau_{ij} \neq f(\underline{\omega})$ however $\mu\nabla^2\underline{u} = f(\nabla\times\underline{\omega})$
ie spatial derivative S_{ij} or ω_k & if ω_k = const net viscous force $= 0$