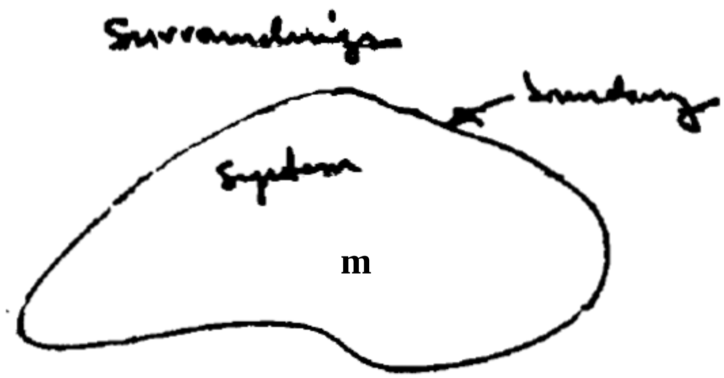
Chapters 1 Preliminary Concepts & 2 Fundamental Equations of Compressible Viscous Flow

(3) Fundamental Equations of Compressible Viscous Flow

Laws of mechanics are written for a system, i.e., a fixed amount of matter.



1. Conservation of mass: $\frac{\mathbf{dm}}{\mathbf{dt}} = \mathbf{0}$
2. Conservation of momentum: $\underline{\boldsymbol{F}} = \mathbf{m}\underline{\boldsymbol{a}} = \frac{\mathbf{d(m\underline{\boldsymbol{v}})}}{\mathbf{dt}}$
3. Conservation of energy: $\frac{dE}{dt} = \dot{Q} - \dot{W}$

ΔE=heat added – work done

Also

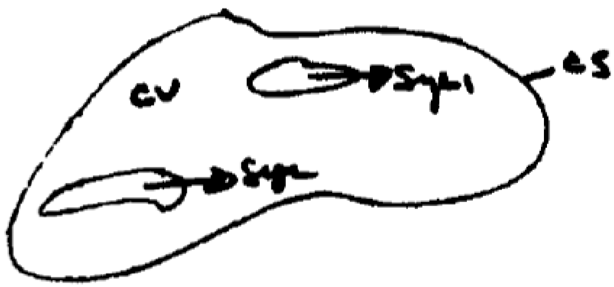
Conservation of angular momentum: $\frac{d\underline{H_G}}{dt} = \underline{M_G}$

Second Law of Thermodynamics: $\frac{dS}{dt} = \frac{\delta\dot{Q}}{T} + \dot{\sigma}$

$\dot{\sigma}$, entropy production due to system irreversibilities

$\dot{\sigma} \leq 0$

In fluid mechanics we are usually interested in a region of space, i.e, control volume and not particular systems. Therefore, we need to transform GDE's from a system to a control volume, which is accomplished through the use of RTT (actually derived in thermodynamics for CV forms of continuity and 1st and 2nd laws, but not in general form or referred to as RTT).



Note GDE's are of form:

$$\frac{\mathbf{d}}{\mathbf{dt}}(\mathbf{m}, \mathbf{m}\underline{\boldsymbol{V}}, \mathbf{E}) = \mathbf{RHS}$$

system extensive properties B_{sys} depend on mass

i.e., involve $\frac{dB_{sys}}{dt}$ which needs to be related to changes in CV. Recall, definition of corresponding system intensive properties

$\beta = (1, \underline{V}, e)$ independent of mass

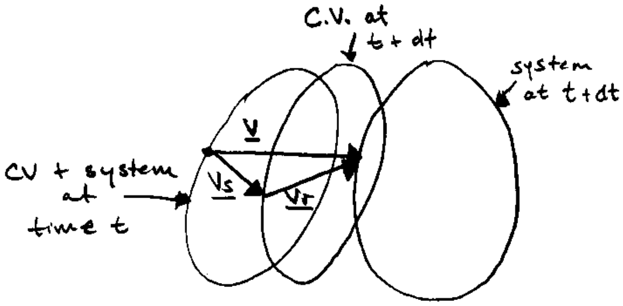
where

$$B = \int \beta dm = \int \beta \rho d\forall$$

i.e., $\beta = \frac{dB}{dm}$

Reynolds Transport Theorem (RTT)

Need relationship between $\frac{d}{dt}\left(B_{sys}\right)$ and changes in $B_{CV} = \int_{CV} \beta \, dm = \int_{CV} \beta\rho \, d\forall$.



Moving deforming CV:

$\underline{V}_r = \underline{V} - \underline{V}_s$

$$\left.\begin{aligned} &\underline{V} = \text{fluid velocity} \\ &\underline{V}_s = \text{CS defining CV velocity} \\ &\underline{V}_r = \text{relative velocity} \end{aligned}\right\} \text{all in same coordinate system}$$

$$\frac{dB_{sys}}{dt} = \lim_{\Delta t \to 0} \frac{(B_{cv} + \Delta B)_{t+\Delta t} - (B_{cv} + \Delta B)_t}{\Delta t}$$

$$= \underbrace{\lim_{\Delta t \to 0} \frac{B_{cv_{t+\Delta t}} - B_{cv_t}}{\Delta t}}_{1} + \underbrace{\lim_{\Delta t \to 0} \frac{\Delta B_{t+\Delta t} - \Delta B_t}{\Delta t}}_{2}$$

1 = time rate of change of B in CV = $\frac{dB_{CV}}{dt} = \frac{d}{dt}\int_{CV} \beta\rho \, d\forall$

2 = net outflux of B from CV across CS = $\int_{CS} \beta\rho \underline{V}_R \cdot \underline{n} \, dA$

As with Q and $\dot{m}$, $\Delta\dot{B}$ flux though A per unit time is:

$$dQ = \underline{V}_R . \underline{n} \, dA$$
$$d\dot{m} = \rho \underline{V}_R . \underline{n} \, dA$$
$$d\Delta\dot{B} = \beta\rho \underline{V}_R . \underline{n} \, dA$$

Therefore:

$$\frac{dB_{SYS}}{dt} = \frac{d}{dt}\int_{CV}\beta\rho d\forall + \int_{CS}\beta\rho \underline{V}_R \cdot \underline{n}\, dA \qquad \boxed{\underline{V_r} = \underline{V} - \underline{V_s}}$$

General form RTT for moving deforming control volume.

Specific CV cases depending on $\underline{V_s}(\underline{x}, t)$.

1) Deforming CV: $V^* = V^*(\underline{x}, t)$
   (a) $\underline{V_s} = \underline{V_s}(\underline{x}, t)$ non-uniform/accelerating velocity
   (b) $\underline{V_s} = \underline{V_s}(\underline{x})$ uniform/constant velocity (steady moving)
   (c) $\int_{CS} \underline{V_s}(\underline{x}, t) \cdot \underline{n} dA = 0$ as a whole at rest (stationary)

2) Non deforming CV: $V^* \neq V^*(\underline{x})$
   (a) $\underline{V_s} = \underline{V_s}(t)$ accelerating velocity
   (b) $\underline{V_s}$ = constant velocity, i.e., relative inertial coordinates (steady moving)
   (c) $\underline{V_s} = 0$ at rest (stationary)

3) Material volume: $\underline{V_s} = \underline{V}$, $\underline{V_r} = 0$ and RTT takes the form:

$$\frac{dB_{SYS}}{st} = \frac{d}{dt}\int_{CV}\beta(\underline{x}, t)\rho(\underline{x}, t) dV$$

Which can be written as:

$$\frac{d}{dt}\int_{MV}\beta(\underline{x},t)\rho(\underline{x},t)dV = \int_{MV}\frac{\partial(\beta\rho)}{\partial t}dV + \int_{MS}\beta\rho\underline{V}\cdot\underline{n}dA$$

Using Green's theorem: $\int_V \nabla\cdot\underline{b}dV = \int_S \underline{b}\cdot\underline{n}dA$

$$\frac{d}{dt}\int_{MV}\beta(\underline{x},t)\rho(\underline{x},t)dV = \int_{MV}\left[\frac{\partial(\beta\rho)}{\partial t} + \nabla\cdot(\beta\rho)\right]dV$$

And taking the limit for $dV \to 0$ provides GDE:

$$\frac{d}{dt}\int_{V(t)}\beta\rho dV = \frac{\partial(\beta\rho)}{\partial t} + \nabla\cdot(\beta\rho\underline{u})$$

Continuity Equation:

B = m = mass of system

β = 1

$\frac{dm}{dt} = 0$ by definition, system = fixed amount of mass

1) Most general integral form for deforming accelerating/steady moving/stationary CV depending on definition $\underline{V_s}(\underline{x}, t)$ (a) – (c) page 4:

$$\frac{dm}{dt} = 0 = \frac{d}{dt}\int_{CV} \rho(\underline{x}, t)dV + \int_{CS} \rho(\underline{x}, t)\underbrace{\left(\underline{V}(\underline{x}, t) - \underline{V_s}(\underline{x}, t)\right)}_{\boxed{\underline{V_r}}} \cdot \underline{n}dA$$

$$-\frac{d}{dt}\int_{CV} \rho dV = \int_{CS} \rho \underline{V_r} \cdot \underline{n}dA$$

Rate of decrease of mass in CV = net rate of mass outflow across CS

2) Most general integral form for non-deforming $\underline{V_s} \neq \underline{V_s}(x)$ accelerating/steady moving/stationary CV, (a)-(c) page 4:

$$\int_{CV} \frac{\partial \rho(\underline{x}, t)}{\partial t} dV + \int_{CS} \rho(\underline{x}, t)\underbrace{\left(\underline{V}(\underline{x}, t) - \underline{V_s}(t)\right)}_{\boxed{\underline{V_r}}} \cdot \underline{n}dA = 0$$

3) Incompressible flow→ $\rho(\underline{x}, t) = \text{constant}$.

(a) Deforming CV accelerating/steady moving/stationary, i.e., conservation of volume:

$$-\frac{d}{dt}\int_{CV} dV = \int_{CS} \underbrace{\left(\underline{V}(\underline{x}, t) - \underline{V_s}(\underline{x}, t)\right)}_{\boxed{\underline{V_r}}} \cdot \underline{n} dA$$

(b) Non-deforming CV accelerating/steady moving/stationary:

$$\int_{CS} \underbrace{\left(\underline{V}(\underline{x}, t) - \underline{V_s}(t)\right)}_{\boxed{\underline{V_r}}} \cdot \underline{n} dA = 0$$

(c) Steady flow, i.e., $\frac{\partial}{\partial t} = 0$. Two possibilities for $\underline{V_s}$: $\underline{V_s} = 0$, $\underline{V_s}$ =constant. The RTT takes the form:

$$\int_{CS} \underbrace{\left(\underline{V}(\underline{x}) - \underline{V_s}\right)}_{\boxed{\underline{V_r}}} \cdot \underline{n} dA = 0$$

(d) Flow over discrete inlet/outlet → the flux term can be expressed as summation:

$$\sum Q_{CS_i} = 0 \ or \ \sum (Q)_{CS\,in} = \sum (Q)_{CS\,out}$$

For inlets:
$\underline{V_r} \cdot \underline{n} < 0$
For outlets:
$\underline{V_r} \cdot \underline{n} > 0$

Non-uniform flow:

$$Q_{CS_i} = \int_{CS} \underbrace{\left(\underline{V}(\underline{x}) - \underline{V_s}\right)}_{\boxed{\underline{V_r}}} \cdot \underline{n} dA = (V_{av}A)_{CS_i}$$

$$V_{av} = \frac{1}{A}\int_{CS} \underbrace{\left(\underline{V}(\underline{x}) - \underline{V_s}\right)}_{\boxed{\underline{V_r}}} \cdot \underline{n} dA$$

Uniform flow:

$$Q_{CS_i} = \left(\underline{V}(\underline{x}) - \underline{V_s}\right) \cdot \underline{n} A$$

For fixed CV, $\underline{V_s} = 0$:
$Q_{CS_i} = \underline{V}(\underline{x}) \cdot \underline{n} A$

Differential Form:

$$\frac{dm}{dt} = 0 = \int_{CV} \left[\frac{\partial \rho}{\partial t} + \nabla.\left(\rho \underline{V}\right)\right] d\forall$$

$$\beta = 1$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \underline{V}\right) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \underline{V} + \underline{V} \cdot \nabla \rho = 0$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \underline{V} = 0$$

$$m = \rho\forall \quad \Rightarrow \qquad dm = \rho\, d\forall + \forall\, d\rho = 0 \Rightarrow \; -\frac{d\forall}{\forall} = \frac{d\rho}{\rho}$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\frac{1}{\forall}\frac{D\forall}{Dt}$$

$$\underbrace{\frac{1}{\rho}\frac{D\rho}{Dt}}_{\substack{\text{rate of change } \rho \\ \text{per unit } \rho}} + \underbrace{\nabla \cdot \underline{V}}_{} = 0$$

$$\underbrace{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{1}{\forall}\frac{D\forall}{Dt}}_{\substack{\text{rate of change } \forall \\ \text{per unit } \forall}}$$

Called the continuity equation since the implication is that ρ and $\underline{V}$ are continuous functions of $\underline{x}$.

Incompressible Fluid: ρ = constant

$$\nabla \cdot \underline{V} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Momentum Equation:
B = m$\underline{V}$ = momentum, β = $\underline{V}$

Integral Form:

$$\frac{d(m\underline{V})}{dt} = \underbrace{\frac{d}{dt}\int_{CV} \underline{V}\rho \, d\forall}_{1} + \underbrace{\int_{CS} \underline{V}\rho\underline{V}_R \cdot \underline{n} \, dA}_{2} = \underbrace{\sum \underline{F}}_{3}$$

$\sum \underline{F}$ = vector sum of all forces acting on CV

= $\underline{F}_B + \underline{F}_s$

$\underline{F}_B$ = Body forces, which act on entire CV of fluid due to external force field such as gravity or electrostatic or magnetic forces. Force per unit volume.

$\underline{F}_s$ = Surface forces, which act on entire CS due to normal (pressure and viscous stress) and tangential (viscous stresses) stresses. Force per unit area.

When CS cuts through solids $\underline{F}_s$ may also include $\underline{F}_R$ = reaction forces, e.g., reaction force required to hold nozzle or bend when CS cuts through bolts holding nozzle/bend in place.

1 = rate of change of momentum in CV

2 = rate of outflux of momentum across CS

3 = vector sum of all body forces acting on entire CV and surface forces acting on entire CS.

Many interesting applications of CV form of momentum equation: vanes, nozzles, bends, rockets, forces on bodies, water hammer, etc.

Differential Form:

$$\int_{CV}\left[\frac{\partial}{\partial t}\left(\underline{V}\rho\right)+\nabla\cdot\left(\underline{V}\rho\underline{V}\right)\right]d\forall=\sum\underline{F}$$

Where $\frac{\partial}{\partial t}\left(\underline{V}\rho\right)=\underline{V}\frac{\partial\rho}{\partial t}+\rho\frac{\partial\underline{V}}{\partial t}$

and $\underline{V}\rho\underline{V}=\rho\underline{V}\underline{V}=\rho u\hat{i}\,\underline{V}+\rho v\hat{j}\,\underline{V}+\rho w\hat{k}\,\underline{V}$ is a tensor.

$$\nabla\cdot(\underline{V}\rho\underline{V})=\nabla\cdot(\rho\underline{V}\underline{V})=\frac{\partial}{\partial x}(\rho u\underline{V})+\frac{\partial}{\partial y}(\rho v\underline{V})+\frac{\partial}{\partial z}(\rho w\underline{V})$$

$$=\underline{V}\nabla\cdot(\rho\underline{V})+\rho\underline{V}\cdot\nabla\underline{V}$$

$$\int_{CV}\left[\underline{V}\underbrace{\left(\frac{\partial\rho}{\partial t}+\nabla\cdot\left(\rho\underline{V}\right)\right)}_{=\,0,\ \text{continuity}}+\rho\left(\frac{\partial\underline{V}}{\partial t}+\underline{V}\cdot\nabla\underline{V}\right)\right]d\forall=\sum\underline{F}$$

Since $\frac{\partial\underline{V}}{\partial t}+\underline{V}\cdot\nabla\underline{V}=\frac{D\underline{V}}{Dt}$

$$\int_{CV}\rho\frac{D\underline{V}}{Dt}d\forall=\sum\underline{F}$$

$$\rho\frac{D\underline{V}}{Dt}=\sum\underline{f}$$ *per elemental fluid volume*

$$\rho\underline{a}=\underline{f}_b+\underline{f}_s$$

$\underline{f}_b$ = body force per unit volume

$\underline{f}_s$ = surface force per unit volume

Body forces are due to external fields such as gravity or magnetic fields. Here we only consider a gravitational field; that is,

$$\sum \underline{F}_{body} = d\underline{F}_{grav} = \rho \underline{g}\, dxdydz$$

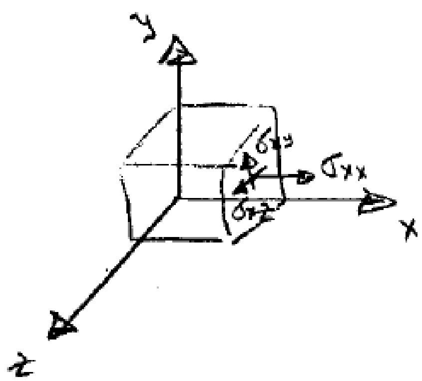
and $\underline{g} = -g\hat{k}$ for $\downarrow g$ $\uparrow z$

i.e. $\underline{f}_{body} = -\rho g\hat{k}$

Surface Forces are due to the stresses that act on the sides of the control surfaces.

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

Normal pressure Viscous stress



$$= \begin{bmatrix} -p+\tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & -p+\tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & -p+\tau_{zz} \end{bmatrix}$$

Symmetric $\sigma_{ij} = \sigma_{ji}$

2nd order tensor

Symmetry condition from requirement that for elemental fluid volume, stresses themselves cause no rotation.

As shown before, for p alone it is not the stresses themselves that cause a net force but their gradients.

$$\underline{f_s} = \underline{f_p} + \underline{f_\tau}$$

Recall $\underline{f}_p = -\nabla p$ based on 1st order TS. $\underline{f}_\tau$ is more complex since τ_{ij} is a 2nd order tensor, but similarly as for p, the force is due to stress gradients and are derived based on 1st order TS.

$$\underline{\sigma_x} = \sigma_{xx}\,\hat{i} + \sigma_{xy}\,\hat{j} + \sigma_{xz}\,\hat{k}$$

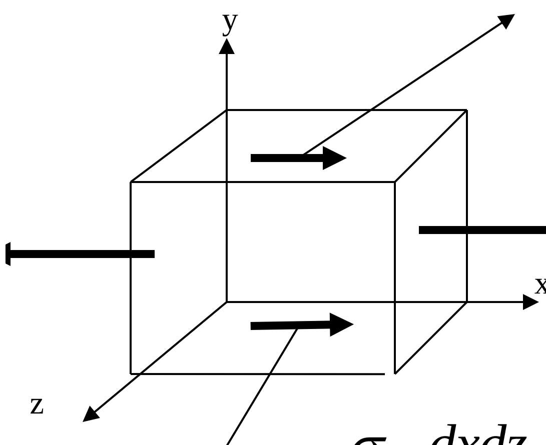
$$\underline{\sigma_y} = \sigma_{yx}\,\hat{i} + \sigma_{yy}\,\hat{j} + \sigma_{yz}\,\hat{k}$$

$$\underline{\sigma_z} = \sigma_{zx}\,\hat{i} + \sigma_{zy}\,\hat{j} + \sigma_{zz}\,\hat{k}$$

Resultant stress on each face

$$\left(\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y} dy\right) dxdz$$

$\sigma_{xx}\, dydz$



$$\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx\right) dydz$$

$\sigma_{yx}\, dxdz$

and similarly, for z face

$$\left(\sigma_{zx} + \frac{\partial \sigma_{zx}}{\partial z} dz\right) dydz - \sigma_{zx}$$

and $\hat{j}$ and $\hat{k}$ directions

$$\underline{F_s} = \left[\frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{yx}) + \frac{\partial}{\partial z}(\sigma_{zx})\right] dxdydz\ \hat{\imath}$$

$$+ \left[\frac{\partial}{\partial x}(\sigma_{xy}) + \frac{\partial}{\partial y}(\sigma_{yy}) + \frac{\partial}{\partial z}(\sigma_{zy})\right] dxdydz\ \hat{\jmath}$$

$$+ \left[\frac{\partial}{\partial x}(\sigma_{xz}) + \frac{\partial}{\partial y}(\sigma_{yz}) + \frac{\partial}{\partial z}(\sigma_{zz})\right] dxdydz\ \hat{k}$$

$$\underline{F_s} = \left[\frac{\partial}{\partial x}(\underline{\sigma_x}) + \frac{\partial}{\partial y}(\underline{\sigma_y}) + \frac{\partial}{\partial z}(\underline{\sigma_z}) \right] dxdydz$$

Divided by the volume:

$$\underline{f_s} = \frac{\partial}{\partial x}(\underline{\sigma_x}) + \frac{\partial}{\partial y}(\underline{\sigma_y}) + \frac{\partial}{\partial z}(\underline{\sigma_z})$$

$$\underline{f_s} = \left(f_{s_1}, f_{s_2}, f_{s_3}\right) = f_{s_i} = \nabla \cdot \sigma_{ij} = \frac{\partial}{\partial x_j}\sigma_{ij}$$

Since $\sigma_{ij} = \sigma_{ji}$

According to Einstein summation notation, repeated indices are implicitly summed over:
$\sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33}$

Putting together the above results,

$$\rho \underline{a} = \rho \frac{D\underline{V}}{Dt} = -\rho g \hat{k} + \nabla \cdot \sigma_{ij}$$

Inertial force | body force due to gravity | surface force = p + viscous terms (Due to stress gradients)

Note:

Δ = delta

∇ = nabla (Hebrew “nebel” means lyre or ancient harp used by David to entertain King Saul in praise of God)

∇f = vector

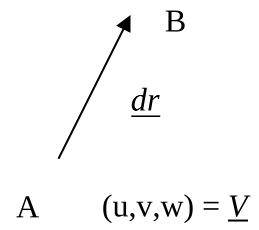
$\nabla \cdot \underline{f}$ = scalar

$\nabla \cdot \sigma_{ij}$ = vector (decreases 2nd order tensor by one)

$\nabla \underline{f}$ = tensor

$\nabla \times \underline{V}$ = vector

Next, we need to relate the stresses σ_{ij} to the fluid motion, i.e., the velocity field. To this end, we examine the relative motion between two neighboring fluid particles.



@ B: $\underline{V} + \underline{dV} = \underline{V} + \nabla \underline{V} \cdot \underline{dr}$ 1st order Taylor Series

$$u_B = u_A + u_x dx + u_y dy + u_z dz + u_{xx} \frac{dx^2}{2} + \cdots$$

$$v_B = v_A + v_x dx + v_y dy + v_z dz + v_{xx} \frac{dx^2}{2} + \cdots$$

$$w_B = w_A + w_x dx + w_y dy + w_z dz + w_{xx} \frac{dx^2}{2} + \cdots$$

$\underline{dV}$ = (u_B-u_A, v_B-v_A, w_B-w_A)

$$\underline{dV} = \nabla \underline{V} \cdot \underline{dr} = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = e_{ij}\, dx_j$$

relative motion

deformation rate tensor = e_{ij}

$$\boxed{\underline{dV} = dV_i = (dV_1, dV_2, dV_3)}$$

$$e_{ij} = \frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\substack{symmetric\ part \\ \varepsilon_{ij}=\varepsilon_{ji}}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\substack{anti-symmetric\ part \\ \omega_{ij}=-\omega_{ji}}} = \varepsilon_{ij} + \omega_{ij}$$

$$\omega_{ij} = \begin{bmatrix} 0 & \frac{1}{2}(u_y - v_x) & \overbrace{\frac{1}{2}(u_z - w_x)}^{\eta} \\ \underbrace{\frac{1}{2}(v_x - u_y)}_{\zeta} & 0 & \frac{1}{2}(v_z - w_y) \\ \frac{1}{2}(w_x - u_z) & \underbrace{\frac{1}{2}(w_y - v_z)}_{\xi} & 0 \end{bmatrix} = \begin{array}{l} rigid\ body\ rotation \\ \quad of\ fluid\ element \end{array}$$

where ξ = *rotation about x axis*
η = *rotation about y axis*
ς = *rotation about z axis*

Note that the components of ω_{ij} are related to the <u>vorticity vector defined by</u>:

$$\underline{\omega} = \nabla \times \underline{V} = \underbrace{(w_y - v_z)}_{2\xi}\,\hat{i} + \underbrace{(u_z - w_x)}_{2\eta}\,\hat{j} + \underbrace{(v_x - u_y)}_{2\zeta}\,\hat{k} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

$$= 2 \times \text{angular velocity of fluid element}$$

$$\varepsilon_{ij} = rate\ of\ strain\ tensor$$

$$= \begin{bmatrix} u_x & \frac{1}{2}(u_y + v_x) & \frac{1}{2}(u_z + w_x) \\ \frac{1}{2}(v_x + u_y) & v_y & \frac{1}{2}(v_z + w_y) \\ \frac{1}{2}(w_x + u_z) & \frac{1}{2}(w_y + v_z) & w_z \end{bmatrix}$$

$$u_x + v_y + w_z = \nabla \cdot \underline{V} = \textit{elongation (or volumetric dilatation) of fluid element} = \frac{1}{\forall}\frac{D\forall}{Dt}$$

$\frac{1}{2}(u_y + v_x)$ = distortion wrt (x,y) plane

$\frac{1}{2}(u_z + w_x)$ = distortion wrt (x,z) plane

$\frac{1}{2}(v_z + w_y)$ = distortion wrt (y,z) plane

Thus, general motion consists of:

1) pure translation described by $\underline{V}$
2) rigid-body rotation described by $\underline{\omega}$
3) volumetric dilatation described by $\nabla \cdot \underline{V}$
4) distortion in shape described by ε_{ij} $\quad i \neq j$

It is now necessary to make certain postulates concerning the relationship between the fluid stress tensor (σ_{ij}) and rate-of-deformation tensor (e_{ij}). These postulates are based on physical reasoning and experimental observations and have been verified experimentally even for extreme conditions. For a Newtonian fluid:

1) When the fluid is at rest the stress is hydrostatic and the pressure is the thermodynamic pressure

2) Since there is no shearing action in rigid body rotation, it causes no shear stress.

3) τ_{ij} is linearly related to ε_{ij} and only depends on ε_{ij}.

4) There is no preferred direction in the fluid, so that the fluid properties are point functions (condition of isotropy).

Using statements 1-3

$$\sigma_{ij} = -p\delta_{ij} + k_{ijmn}\varepsilon_{mn} \qquad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

k_{ijmn} = 4th order tensor with 81 components (3x3x3x3) such that each stress is linearly related to all nine components of ε_{mn}.

However, statement (4) requires that the fluid has no directional preference, i.e., σ_{ij} is independent of rotation of coordinate system, which means k_{ijmn} is an isotropic tensor = even order tensor made up of products of δ_{ij}.

$$k_{ijmn} = \lambda\delta_{ij}\delta_{mn} + \mu\delta_{im}\delta_{jn} + \gamma\delta_{in}\delta_{jm}$$

$$(\lambda, \mu, \gamma) = scalars$$

Lastly, the symmetry condition $\sigma_{ij} = \sigma_{ji}$ requires:

$$k_{ijmn} = k_{jimn} \quad \rightarrow \quad \gamma = \mu = \text{viscosity}$$

$$\sigma_{ij} = -p\delta_{ij} + \mu\delta_{im}\delta_{jn}\varepsilon_{ij} + \mu\delta_{in}\delta_{jm}\varepsilon_{ij} + \lambda\delta_{ij}\delta_{mn}\varepsilon_{ij}$$

Take $\mu\delta_{im}\delta_{jn}\varepsilon_{ij} \rightarrow \delta_{im} \neq 0$ if $i = m$ and $\delta_{jn} \neq 0$ if $j = n \rightarrow$ equivalent to $\mu\varepsilon_{mn}$. Similar reasoning for other terms:

$$\sigma_{ij} = -p\delta_{ij} + 2\mu\varepsilon_{ij} + \lambda\,\underbrace{\varepsilon_{mm}}_{\nabla\cdot\underline{V}}\,\delta_{ij}$$

λ and μ can be further related if one considers mean normal stress vs. thermodynamic p.

$$\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \sigma_{ii} = -3p + (2\mu + 3\lambda)\nabla \cdot \underline{V}$$

$$p = \underbrace{-\frac{1}{3}\sigma_{ii}}_{\substack{\bar{p} = mean \\ normal\ stress}} + \left(\frac{2}{3}\mu + \lambda\right)\nabla \cdot \underline{V}$$

$$p - \bar{p} = \left(\frac{2}{3}\mu + \lambda\right)\nabla \cdot \underline{V}$$

Incompressible flow: $p = \bar{p}$ and absolute pressure is indeterminant since there is no equation of state for p. Equations of motion determine ∇p.

Compressible flow: $p \neq \bar{p}$ and λ = bulk viscosity must be determined; however, it is a very difficult measurement requiring large $\nabla \cdot \underline{V} = -\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{1}{\forall}\frac{D\forall}{Dt}$, e.g., within shock waves.

Stokes Hypothesis also supported kinetic theory monotonic gas.

$$\lambda = -\tfrac{2}{3}\mu$$

$$p = \bar{p}$$

$$\sigma_{ij} = -\left(p + \frac{2}{3}\mu\nabla\cdot\underline{V} \right)\delta_{ij} + 2\mu\varepsilon_{ij}$$

Generalization $\tau = \mu\dfrac{du}{dy}$ for 3D flow.

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad i \neq j \qquad \textit{relates shear stress to strain rate}$$

$$\sigma_{ii} = -p - \frac{2}{3}\mu\nabla\cdot\underline{V} + 2\mu\left(\frac{\partial u_i}{\partial x_i} \right) = -p + \underbrace{2\mu\left[-\frac{1}{3}\nabla\cdot\underline{V} + \frac{\partial u_i}{\partial x_i} \right]}_{\textit{normal viscous stress}}$$

Where the normal viscous stress is the difference between the extension rate in the x_i direction and average expansion at a point. Only differences from the average = $\frac{1}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)$ generate normal viscous stresses. For incompressible fluids, average = 0 i.e., $\nabla\cdot\underline{V} = 0$.

Non-Newtonian fluids:

$\tau_{ij} \propto \varepsilon_{ij}$ for small strain rates $\dot{\theta}$, which works well for air, water, etc. Newtonian fluids

$$\tau_{ij} \propto \underset{\textit{non–linear}}{\varepsilon_{ij}^{n}} + \underset{\textit{history effect}}{\frac{\partial}{\partial t}\varepsilon_{ij}}$$

Non-Newtonian

Viscoelastic materials

Non-Newtonian fluids include:

(1) Polymer molecules with large molecular weights and form long chains coiled together in spongy ball shapes that deform under shear.

(2) Emulsions and slurries containing suspended particles such as blood and water/clay.

Navier Stokes Equations:

$$\rho \underline{a} = \rho \frac{D\underline{V}}{Dt} = -\rho g \hat{k} + \nabla \cdot \sigma_{ij}$$

$$\rho \frac{D\underline{V}}{Dt} = -\rho g \hat{k} - \nabla p + \frac{\partial}{\partial x_j}\left[2\mu\varepsilon_{ij} - \frac{2}{3}\mu\nabla \cdot \underline{V}\delta_{ij}\right]$$

Recall $\mu = \mu(T)$ μ increases with T for gases, decreases with T for liquids, but if it is assumed that μ = constant:

$$\rho \frac{D\underline{V}}{Dt} = -\rho g \hat{k} - \nabla p + 2\mu \frac{\partial}{\partial x_j}\varepsilon_{ij} - \frac{2}{3}\mu \frac{\partial}{\partial x_j}\nabla \cdot \underline{V}$$

$$2\frac{\partial}{\partial x_j}\varepsilon_{ij} = \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{\partial^2 u_i}{\partial x_j \partial x_j} = \nabla^2 u_i = \nabla^2 \underline{V}$$

$$\rho \frac{D\underline{V}}{Dt} = -\rho g\,\hat{k} - \nabla p + \mu\left[\nabla^2\underline{V} - \frac{2}{3}\frac{\partial}{\partial x_j}\nabla\cdot\underline{V}\right]$$

For incompressible flow $\nabla\cdot\underline{V} = 0$

$$\rho \frac{D\underline{V}}{Dt} = \underbrace{-\rho g\hat{k} - \nabla p}_{\substack{-\nabla\hat{p}\ \text{where}\ \hat{p}=p+\gamma z \\ \text{piezometric pressure}}} + \mu\nabla^2\underline{V}$$

For μ = 0

$$\rho \frac{D\underline{V}}{Dt} = -\rho g\,\hat{k} - \nabla p \qquad \text{Euler Equation}$$

NS equations for ρ, μ constant

$$\rho \frac{D\underline{V}}{Dt} = -\nabla\hat{p} + \mu\nabla^2\underline{V}$$

$$\rho\left[\frac{\partial\underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right] = -\nabla\hat{p} + \mu\nabla^2\underline{V}$$

$$\left[\frac{\partial\underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right] = -\frac{1}{\rho}\nabla\hat{p} + \nu\nabla^2\underline{V} \qquad \nu = \frac{\mu}{\rho}$$ *kinematic viscosity/ diffusion coefficient*

Non-linear 2nd order PDE, as is the case for ρ, μ not constant.

Combine with $\nabla\cdot\underline{V}$ for 4 equations for 4 unknowns $\underline{V}$, p and can be, albeit difficult, solved subject to initial and boundary conditions for $\underline{V}$, p at t = t_0 and on all boundaries i.e. “well posed” IBVP.

Application of differential momentum equation:

1. NS valid both laminar and turbulent flow; however, many orders of magnitude difference in temporal and spatial resolution, i.e., turbulent flow requires very small time and spatial scales.

2. Laminar flow $Re_{crit} = \frac{U\delta}{\nu} \leq$ about 2000

   $Re > Re_{crit}$ instability

3. Turbulent flow $Re_{transition} \geq 10$ or $20\ Re_{crit}$

   Random motion superimposed on mean coherent structures.

   Cascade: energy from large scale dissipates at smallest scales due to viscosity.
   Kolmogorov hypothesis for smallest scales

4. No exact solutions for turbulent flow: RANS, DES, LES, DNS (all CFD)

5. 80 exact solutions for simple laminar flows are mostly linear $\underline{V} \cdot \nabla \underline{V} = 0$. Topics of exact analytical solutions:

I. Couette (wall/shear-driven) steady flows
   a. Channel flows
   b. Cylindrical flows.
II. Poiseuille (pressure-driven) steady flows
   a. Channel flows
   b. Duct flows
III. Combined Couette and Poiseuille steady flows
IV. Gravity and free-surface steady flows
V. Unsteady flows
VI. Suction and injection flows
VII. Wind-driven (Ekman) flows
VIII. Similarity solutions

6. Also, many exact solutions for low Re linearized creeping motion Stokes flows and high Re nonlinear BL approximations.

7. Can also use CFD for non-simple laminar flows.

8. AFD or CFD requires well posed IBVP; therefore, exact solutions are useful for setup of IBVP, physics, and verification CFD since modeling errors yield $U_{SM} = 0$ and only errors are numerical errors U_{SN}, i.e., assume analytical solution = truth, called analytical benchmark.

<u>**The Stream Function**</u>

Powerful tool for 2-D flow in which $\underline{V}$ is obtained by differentiation of a scalar ψ which automatically satisfies the continuity equation.

Note for 2D flow

$$\nabla \times V = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = (0, 0, \omega_z)$$

Continuity: $u_x + v_y = 0$

say: $u = \psi_y$ and $\mathrm{v} = -\psi_x$

then: $\frac{\partial}{\partial x}(\psi_y) + \frac{\partial}{\partial y}(-\psi_x) = \psi_{yx} - \psi_{xy} = 0$ by definition!

$\underline{\mathrm{V}} = \psi_y \hat{i} - \psi_x \hat{j}$

$curl\underline{\mathrm{V}} = \hat{k}\omega_z = -\hat{k}\nabla^2\psi \quad \left(\omega_z = v_x - u_y = -\psi_{xx} - \psi_{yy} = -\nabla^2\psi\right)$

NS equation for unsteady constant property flow:

$$\rho\frac{\partial \underline{V}}{\partial t} + \rho(\underline{V} \cdot \nabla)\underline{V} = -\nabla(p + \gamma z) + \mu\nabla^2\underline{V}$$

Taking the curl gives:

$$\rho\left(\nabla \times \frac{\partial \underline{V}}{\partial t}\right) + \rho\nabla \times (\underline{V} \cdot \nabla)\underline{V} = \mu\nabla^2(\nabla \times \underline{V}) \qquad (1)$$

For the unsteady term:

$$\rho\left(\nabla \times \frac{\partial \underline{V}}{\partial t}\right) = \rho\frac{\partial}{\partial t}(\nabla \times \underline{V}) = \rho\frac{\partial \underline{\omega}}{\partial t}$$

Recall vector identity:

$$\underline{V} \times (\nabla \times \underline{V}) = \frac{1}{2}\nabla(\underline{V}^2) - (\underline{V} \cdot \nabla)\underline{V}$$

Such that:

$$(\underline{V} \cdot \nabla)\underline{V} = \frac{1}{2}\nabla(\underline{V}^2) - \underline{V} \times (\nabla \times \underline{V}) \quad (2)$$

Taking the curl of (2), recalling that the curl of the gradient of a scalar equal zero and using $\nabla \times \underline{V} = \underline{\omega}$, gives:

$$\nabla \times \{(\underline{V} \cdot \nabla)\underline{V}\} = -\nabla \times (\underline{V} \times \underline{\omega}) = \nabla \times (\underline{\omega} \times \underline{V}) \quad (3)$$

And using Eq. (3) into Eq. (1) gives:

$$\rho \frac{\partial \underline{\omega}}{\partial t} + \rho \nabla \times (\underline{\omega} \times \underline{V}) = \mu \nabla^2 \underline{\omega} \quad (4)$$

Recall vector identity:

$$\nabla \times (\underline{a} \times \underline{b}) = \underline{a}(\nabla \cdot \underline{b}) + (\underline{b} \cdot \nabla)\underline{a} - \underline{b}(\nabla \cdot \underline{a}) - (\underline{a} \cdot \nabla)\underline{b}$$

Such that:

$$\nabla \times (\underline{\omega} \times \underline{V}) = \cancel{\underline{\omega}(\nabla \cdot \underline{V})} + (\underline{V} \cdot \nabla)\underline{\omega} - \cancel{\underline{V}(\nabla \cdot \underline{\omega})} - (\underline{\omega} \cdot \nabla)\underline{V}$$

And Eq. (4) becomes (vorticity transport equation):

$$\rho \frac{\partial \underline{\omega}}{\partial t} + \rho[(\underline{V} \cdot \nabla)\underline{\omega} - (\underline{\omega} \cdot \nabla)\underline{V}] = \mu \nabla^2 \underline{\omega} \quad (4)$$

The second term in brackets in Eq. (4) represents vortex stretching and it is exactly zero for 2D flow, since the velocity and vorticity vector are orthogonal, i.e., $\underline{\omega} \cdot \nabla = \omega_z \frac{\partial}{\partial z} = 0$.

The resulting equation is (2D vorticity transport equation):

$$\rho \frac{\partial \underline{\omega}}{\partial t} + \rho\left[\left(\underline{V} \cdot \nabla\right)\underline{\omega}\right] = \mu \nabla^2 \underline{\omega} \quad (5)$$

Recall:

$$u = \ \psi_y \quad v = \psi_x$$

$$\underline{\omega} = \nabla \times \underline{V} = \hat{k}\omega_z = -\hat{k}\nabla^2\psi$$

Such that Eq. (5) becomes:

$$\rho \frac{\partial(-\hat{k}\nabla^2\psi)}{\partial t} + \rho\left[\left(\underline{V} \cdot \nabla\right)\left(-\hat{k}\nabla^2\psi\right)\right] = \mu\nabla^2\left(-\hat{k}\nabla^2\psi\right)$$

And writing $\left(\underline{V} \cdot \nabla\right)$ by components gives:

$$\rho \frac{\partial(-\hat{k}\nabla^2\psi)}{\partial t} + \rho\left[\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\left(-\hat{k}\nabla^2\psi\right)\right] = \mu\nabla^2\left(-\hat{k}\nabla^2\psi\right) \quad (6)$$

Substituting the definition of stream function in Eq. (6) for u and v gives:

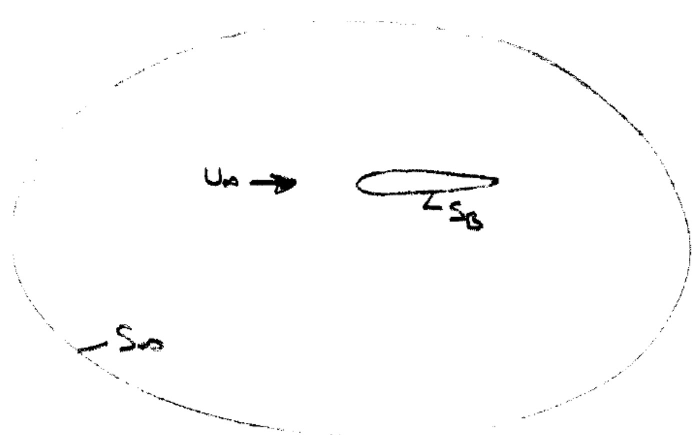
$$\frac{\partial\nabla^2\psi}{\partial t} + \left[\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\psi) - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\psi)\right] = \nu\nabla^4\psi$$

This represents a single scalar equation, but 4th order!

boundary conditions (4 required):

$$\text{at infinity:} \quad u = \psi_y = U_\infty \qquad v = -\psi_x = 0$$

$$\text{on body:} \quad u = v = 0 = \psi_y = -\psi_x$$



Irrotational Flow

$\nabla^2\psi = 0$ 2nd order linear Laplace equation

on S_∞ : $\psi = U_\infty y + const.$

on S_B : $\psi = const.$

$$u = \psi_y = \phi_x$$

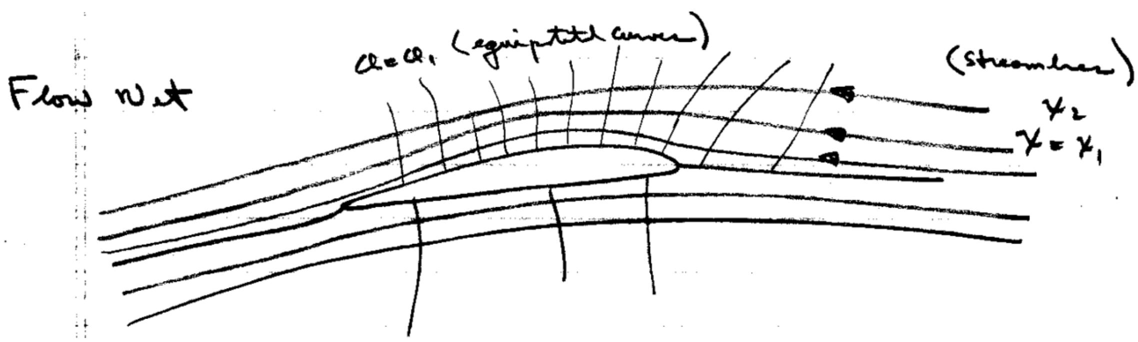
$$v = -\psi_x = \phi_y$$

Ψ and φ are orthogonal.

$$d\phi = \phi_x dx + \phi_y dy = udx + vdy$$

$$d\psi = \psi_x dx + \psi_y dy = -vdx + udy$$

i.e. $\left.\dfrac{dy}{dx}\right|_{\phi = const} = -\dfrac{u}{v} = \dfrac{-1}{\left.\dfrac{dy}{dx}\right|_{\psi = const}}$



Geometric Interpretation of ψ

Besides its importance mathematically ψ also has important geometric significance.

ψ = constant = streamline
Recall definition of a streamline:

$$\underline{V} \times \underline{dr} = 0 \qquad dr = dx\hat{i} + dy\hat{j}$$

$$\frac{dx}{u} = \frac{dy}{v}$$

$$udy - vdx = 0$$

$$\text{compare with} \quad d\psi = \psi_x dx + \psi_y dy = -vdx + udy$$

$$\text{i.e.} \quad d\psi = 0 \quad \text{along a streamline}$$

Or ψ =constant along a streamline and curves of constant ψ are the flow streamlines. If we know ψ(x, y) then we can plot ψ = constant curves to show streamlines.

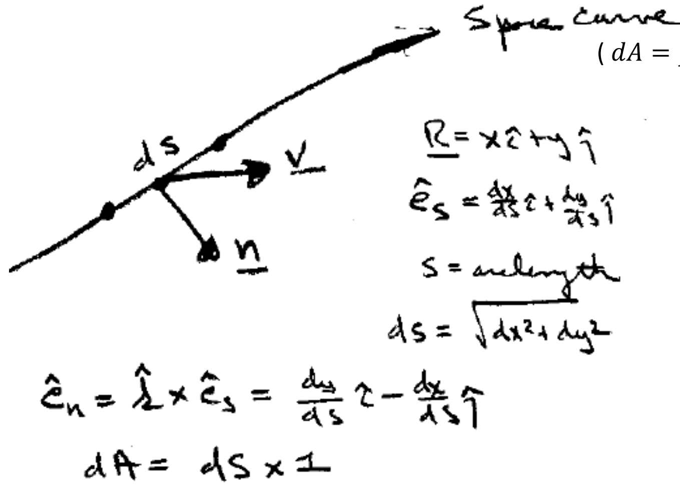
Physical Interpretation

$$dQ = \underline{V}.\underline{n}dA$$

$$= (\hat{i}\,\frac{\partial\psi}{\partial y} - \hat{j}\,\frac{\partial\psi}{\partial x}).(\frac{dy}{ds}\hat{i} - \frac{dx}{ds}\,\hat{j}) \times ds \times 1$$

$$= \psi_y dy + \psi_x dx$$

$$= d\psi$$

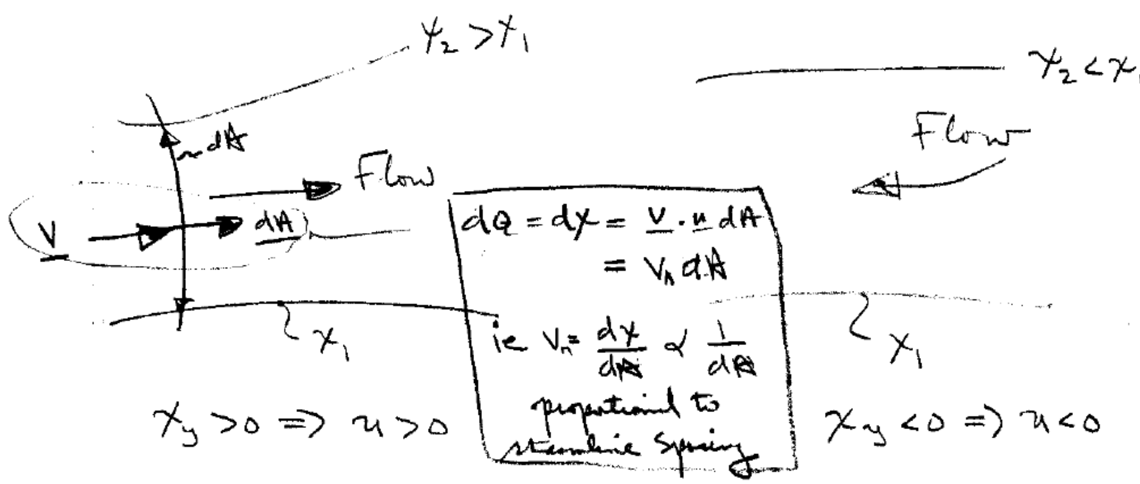


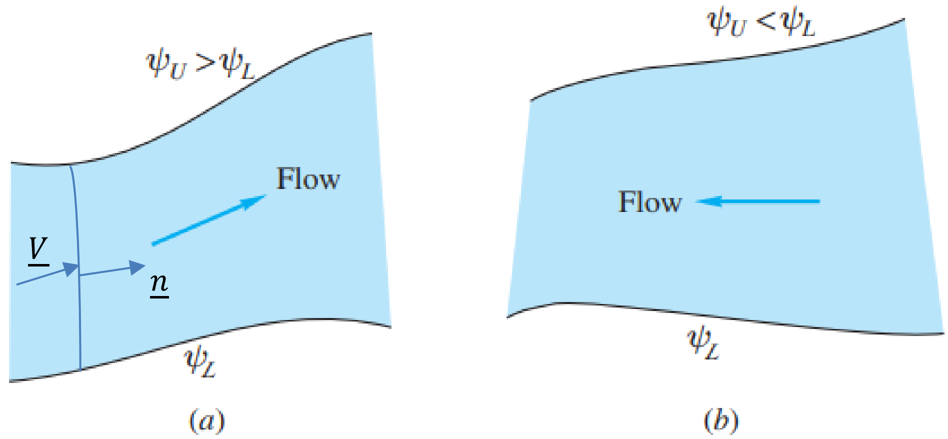
($dA = flow\ area\ \ ds \times 1$ with 2D unit tangent and normal vectors)

i.e., change in $d\psi$ is volume flux and across streamline $dQ = 0$.

$$Q_{1\to 2} = \int_1^2 \underline{V}.\underline{n}dA = \int_1^2 d\psi = \psi_2 - \psi_1$$

Consider flow between two streamlines: $u = \frac{\partial\psi}{\partial y},\ v = -\frac{\partial\psi}{\partial x}$





$$\boxed{\begin{array}{l} \psi_y > 0 \rightarrow u > 0 \\ \psi_y < 0 \rightarrow u < 0 \end{array}}$$

$$dQ = d\psi = \underline{V} \cdot \underline{n} dA = V_n dA$$

$$V_n = \frac{d\psi}{dA} \propto \frac{1}{dA}$$

i.e., proportional to streamline spacing.

Pressure Distribution in Irrotational Flow; Bernoulli Equation

Navier-Stokes for constant property incompressible flow:

$$\rho \underline{a} = -\nabla(p) - \rho g\hat{k} + \mu\nabla^2\underline{V} = -\nabla(p+\gamma z) + \mu\nabla^2\underline{V}$$

$$\rho\left[\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right] = -\nabla(p+\gamma z) + \mu\left[\nabla(\nabla\cdot\underline{V}) - \nabla\times(\nabla\times\underline{V})\right]$$

Viscous term=0 for ρ=constant and $\underline{\omega}$=0, i.e., potential flow solutions also solutions NS under such conditions! But cannot satisfy no slip condition and suffers from D'Alembert's paradox that drag = 0.

In fluid dynamics, d'Alembert's paradox (or the hydrodynamic paradox) is a contradiction reached in 1752 by French mathematician Jean le Rond d'Alembert. D'Alembert proved that – for incompressible and inviscid potential flow – the drag force is zero on a body moving with constant velocity relative to the fluid. Zero drag is in direct contradiction to the observation of substantial drag on bodies moving relative to fluids, such as air and water, especially at high velocities corresponding with high Reynolds numbers. It is a particular example of the reversibility paradox.

1. Additionally, assuming inviscid flow: μ=0 and using vector identity

   $$\underline{V}\cdot\nabla\underline{V} = \frac{1}{2}\nabla\underline{V}\cdot\underline{V} - \underline{V}\times(\nabla\times\underline{V})$$

$$\rho\left[\frac{\partial \underline{V}}{\partial t} + \left(\frac{1}{2}\nabla\underline{V}\cdot\underline{V} - \underline{V}\times(\nabla\times\underline{V})\right)\right] = -\nabla(p+\gamma z) \text{ Euler Equation}$$

$$\frac{\partial \underline{V}}{\partial t} + \nabla\left[\frac{V^2}{2} + \frac{p}{\rho} + gz\right] = \underline{V}\times\underline{\omega} \qquad V^2 = \underline{V}\cdot\underline{V} \qquad (\underline{\omega} \neq 0)$$

2. Additionally, assuming steady flow: $\frac{\partial}{\partial t} = 0$

$$\nabla B = \underline{V} \times \underline{\omega}$$

$$B = \frac{V^2}{2} + \frac{p}{\rho} + gz$$

Consider:

∇B perpendicular B= constant

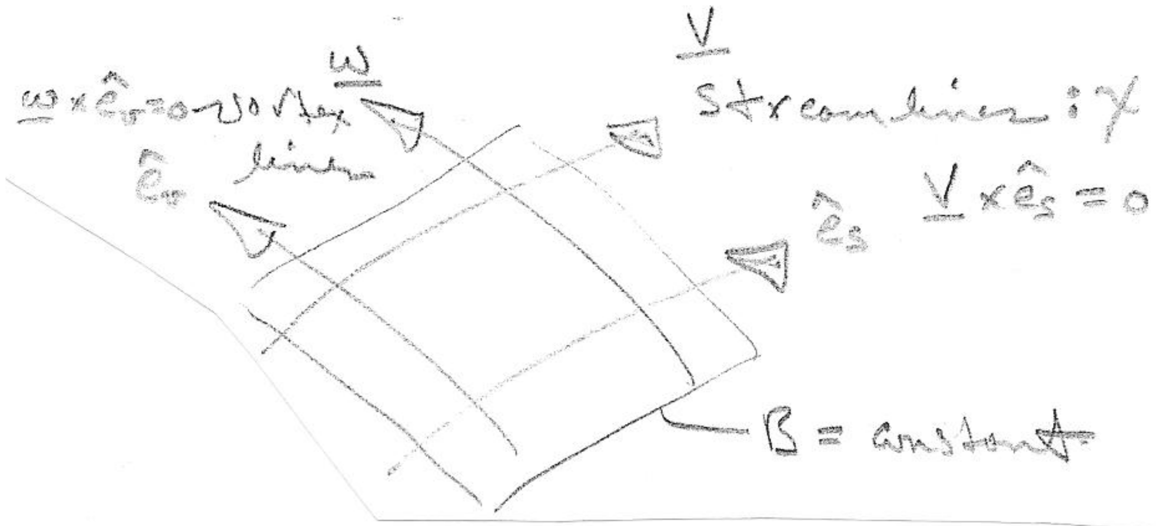
$\underline{V} \times \underline{\omega} = \nabla B$ perpendicular $\underline{\text{V}}$ and $\underline{\omega}$

Therefore, B=constant contains streamlines and vortex lines:

$$\hat{e}_s \cdot \nabla B = \frac{\partial B}{\partial s} = 0$$

$$\hat{e}_v \cdot \nabla B = 0$$

$B = \frac{V^2}{2} + \frac{p}{\rho} + gz$ =constant along streamlines and vortex lines.



3. Additionally assuming irrotational flow: $\underline{\omega}=0$

$\nabla B = 0$ B= constant (everywhere same constant)

$$\frac{V^2}{2} + \frac{p}{\rho} + gz = B$$

4. Unsteady, inviscid, incompressible, and irrotational flow:
   $\mu=0$, ρ=constant, $\underline{\omega}=0$, i.e., potential flow

$$\underline{V} = \nabla\varphi$$

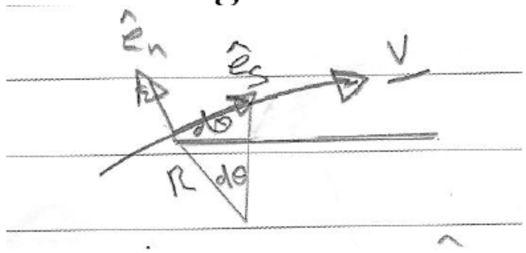
$$V^2 = \nabla\varphi \cdot \nabla\varphi$$

$$\nabla\left[\frac{\partial\varphi}{\partial t} + \frac{\nabla\varphi\cdot\nabla\varphi}{2} + \frac{p}{\rho} + gz\right] = 0$$

$$\frac{\partial\varphi}{\partial t} + \frac{\nabla\varphi\cdot\nabla\varphi}{2} + \frac{p}{\rho} + gz = B(t)$$

B(t)= time dependent constant

Alternate derivation using stream line coordinates:

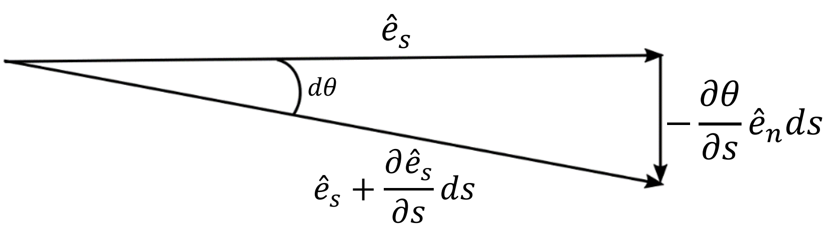


R = local radius of curvature along streamline

$$\underline{V} = v_s(s,t)\hat{e}_s + v_n\hat{e}_n = v_s(s,t)\hat{e}_s$$

$$\nabla = \hat{e}_s\frac{\partial}{\partial s} + \hat{e}_n\frac{\partial}{\partial n}$$

$$\underline{a} = \frac{D\underline{V}}{Dt} = \frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V} = \frac{\partial \underline{V}}{\partial t} + v_s\frac{\partial \underline{V}}{\partial s} = [\frac{\partial v_s}{\partial t}\hat{e}_s + v_s\frac{\partial \hat{e}_s}{\partial t}] + v_s[\frac{\partial v_s}{\partial s}\hat{e}_s + v_s\frac{\partial \hat{e}_s}{\partial s}]$$



To 1st order $\hat{e}_s$ changes by $\frac{\partial \hat{e}_s}{\partial s}$ along ψ for increments $ds = Rd\theta$

In a space increment ds, the tangent unit vector $\hat{e}_s$ is transformed into $\hat{e}_s + \frac{\partial \hat{e}_s}{\partial s}ds$ and its direction changes by $d\theta$. The vector connecting the two can be obtained using the triangle rule, and its magnitude is equal to $d\theta$, pointing in the $-\hat{e}_n$ direction. Alternatively, this can be written as: $-\frac{\partial\theta}{\partial s}\hat{e}_n ds$.

Therefore:

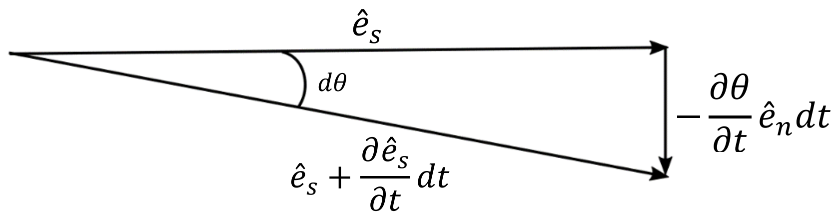
$$\hat{e}_s + \frac{\partial \hat{e}_s}{\partial s}ds = \hat{e}_s - \frac{\partial\theta}{\partial s}\hat{e}_n ds$$

i.e.,

$$\frac{\partial \hat{e}_s}{\partial s} = -\frac{\partial\theta}{\partial s}\hat{e}_n = -\frac{1}{R}\hat{e}_n$$

$$\boxed{\frac{\partial\theta}{\partial s} = \frac{1}{R}}$$

Where $\frac{\partial\theta}{\partial s}$ represents the curvature k of the trajectory, or equivalently $1/R$.



Similarly, in a time increment dt, the tangent unit vector $\hat{e}_s$ is transformed into $\hat{e}_s + \frac{\partial \hat{e}_s}{\partial t} dt$ and its direction changes by $d\theta$. The vector connecting the two can be obtained using the triangle rule, and its magnitude is equal to $d\theta$, pointing in the $-\hat{e}_n$ direction. Alternatively, this can be written as: $-\frac{\partial \theta}{\partial t}\hat{e}_n dt$.

Therefore:

$$\hat{e}_s + \frac{\partial \hat{e}_s}{\partial t} dt = \hat{e}_s - \frac{\partial \theta}{\partial t}\hat{e}_n dt$$

i.e.,

$$\frac{\partial \hat{e}_s}{\partial t} = -\frac{\partial \theta}{\partial t}\hat{e}_n$$

Consequently, the acceleration vector can be expressed as:

$$\underline{a} = [\frac{\partial v_s}{\partial t} + v_s \frac{\partial v_s}{\partial s}]\hat{e}_s + [-v_s \frac{\partial \theta}{\partial t} - \frac{v_s^2}{R}]\hat{e}_n$$

$\frac{\partial v_s}{\partial t}$ =local a_s in direction of flow

$\frac{\partial v_n}{\partial t} = -v_s \frac{\partial \theta}{\partial t}$ =local a_n normal to flow

$v_s \frac{\partial v_s}{\partial s}$ =convective a_s due to convergence/divergence of streamlines

$-\frac{v_s^2}{R}$ =normal a_n due to streamline curvature

Euler Equation

$$\rho \underline{a} = -\nabla(p + \gamma z)$$

Steady flow s equation:

$$\rho v_s \frac{\partial v_s}{\partial s} = -\frac{\partial}{\partial s}(p + \gamma z)$$
$$\frac{\partial}{\partial s}(\frac{v_s^2}{2} + \frac{p}{\rho} + gz) = 0$$

i.e., B=constant along streamline

Steady flow n equation:

$$-\rho \frac{\partial v_s^2}{R} = -\frac{\partial}{\partial n}(p + \gamma z)$$

$$-\int \frac{v_s^2}{R} dn + \frac{p}{\rho} + gz = \text{constant across streamline}$$

Larger speed/density or smaller R require larger pressure gradient or elevation gradient normal to streamline.

Highlights that the Bernoulli equation can also be obtained by integration of the Euler equation along a streamline.

Energy Equation:

B = E = energy

β = e = dE/dm = energy per unit mass

Integral Form (fixed CV):

$$\frac{dE}{dt} = \underbrace{\int_{CV} \frac{\partial}{\partial t}(e\rho)\,d\forall}_{\substack{rate\,of\;change\\ E\,in\;CV}} + \underbrace{\int_{CS} e\rho\,\underline{V}\cdot\underline{n}\,dA}_{\substack{rate\,of\;outflux\\ E\,across\,CS}} = \dot{Q} - \dot{W}$$

Rate of change E

Rate of heat added CV

Rate work done by CV

$$e = \hat{u} + \frac{1}{2}v^2 + gz = \text{internal} + KE + PE$$

$$\dot{Q} = \textit{conduction} + \textit{convection} + \textit{radiation}$$

$$\dot{W} = \underset{pump/turbine}{\dot{W}_{shaft}} + \underset{pressure}{\dot{W}_p} + \underset{viscous}{\dot{W}_v}$$

$$d\dot{W}_p = \left(p\,\underline{n}dA\right)\cdot\underline{V} \qquad \text{- pressure force} \times \text{velocity}$$

$$\dot{W}_p = \int_{CS} p\left(\underline{V}\cdot\underline{n}\right)dA$$

$$d\dot{W}_v = -\underline{\tau}\,dA\cdot\underline{V} \qquad \text{- viscous force} \times \text{velocity}$$

$$\dot{W}_v = -\int_{CS} \underline{\tau} \cdot \underline{V}\, dA$$

$$\dot{Q} - \dot{W}_s - \dot{W}_v = \int_{CV} \frac{\partial}{\partial t}(e\rho)\, d\forall + \int_{CS} \left(e + p/\rho\right) \rho \underline{V} \cdot \underline{n}\, dA$$

For our purposes, we are interested in steady flow with one inlet and outlet. Also $\dot{W}_v \approx 0$ in most cases; since, $\underline{V}$ = 0 at solid surface; on inlet and outlet τ_n ~ 0 since its perpendicular to flow; or for $\underline{V} \neq 0$ and $\tau_{streamline}$ ~ 0 if outside BL.

$$\dot{Q} - \dot{W}_S = \int_{inlet\&outlet} \left(\hat{u} + \frac{1}{2}V^2 + gz + p/\rho \right) \rho \underline{V} \cdot \underline{n}\, dA$$

Assume parallel flow with $\underbrace{p/\rho + gz}$ and $\hat{u}$ constant over inlet and outlet.

= constant i.e., hydrostatic pressure variation

$$\dot{Q} - \dot{W}_S = \left(\hat{u} + p/\rho + gz\right) \int_{inlet\&outlet} \rho \underline{V} \cdot \underline{n}\, dA + \frac{\rho}{2} \int_{inlet\&outlet} V^2 (\underline{V} \cdot \underline{n})\, dA$$

$$\dot{Q} - \dot{W}_S = \left(\hat{u} + p/\rho + gz\right)_{in} (-\dot{m}_{in}) - \frac{\rho}{2} \int_{in} V_{in}^{\ 3}\, dA_{in}$$

$$+ \left(\hat{u} + p/\rho + gz\right)_{out} (\dot{m}_{out}) + \frac{\rho}{2} \int_{out} V_{out}^{\ 3}\, dA_{out}$$

Define kinetic energy correction factor.

$$\alpha = \frac{1}{A}\int_A \left(\frac{V}{V_{ave}}\right)^3 dA \quad \rightarrow \quad \frac{\rho}{2}\int_A V^2(\underline{V}\cdot\underline{n})\,dA = \alpha\frac{V_{ave}^2}{2}\dot{m}$$

Laminar flow: $u = U_0\left(1-\left(\frac{r}{R}\right)^2\right)$

V_{ave}=0.5 *β = 4/3* *α=2*

Turbulent flow: $u = U_0\left(1-\frac{r}{R}\right)^m$

$$\alpha = \frac{(1+m)^3(2+m)^3}{4(1+3m)(2+3m)}$$

m=1/7 *α=1.058* *as with β, α~1 for turbulent flow*

$$\frac{\dot{Q}}{\dot{m}} - \frac{\dot{W}_s}{\dot{m}} = (\hat{u} + p/\rho + gz + \alpha\frac{V_{ave}^2}{2})_{out} - (\hat{u} + p/\rho + gz + \alpha\frac{V_{ave}^2}{2})_{in}$$

Let in = 1, out = 2, $V = V_{\text{ave}}$, and divide by g

$$\frac{p_1}{\rho g} + \frac{\alpha_1}{2g}V_1^2 + z_1 + h_p = \frac{p_2}{\rho g} + \frac{\alpha_2}{2g}V_2^2 + z_2 + h_t + h_L$$

$$\frac{\dot{W}_s}{g\dot{m}} = \frac{\dot{W}_t}{g\dot{m}} - \frac{\dot{W}_p}{g\dot{m}} = h_t - h_p$$

Where h_t extracts and h_p adds energy

$$h_L = \frac{1}{g}(u_2 - u_1) - \frac{\dot{Q}}{\dot{m}g} \text{ = head loss}$$

h_L = thermal energy (other terms represent mechanical energy

$$\dot{m} = \rho A_1 V_1 = \rho A_2 V_2$$

Assuming no heat transfer mechanical energy converted to thermal energy through viscosity and cannot be recovered; therefore, it is referred to as head loss ≥ 0, which can be shown from 2nd law of thermodynamics.

1D energy equation can be considered as modified Bernoulli equation for h_p, h_t, and h_L.

Application of 1D Energy equation fully developed pipe flow without h_p or h_t.

Recall for horizontal pipe flow using continuity and momentum: $\tau_w = \frac{R}{2}\left(-\frac{dp}{dx}\right)$, i.e., $-\frac{dp}{dx} = \frac{2\tau_w}{R}$

Similarly, for non-horizontal pipe: $-\frac{d}{dx}(p+\gamma z) = \frac{2\tau_w}{R}$

Using energy equation, $L = dx$ and $\hat{p} = p + \gamma z$:

$$h_L = \frac{p_1 - p_2}{\rho g} + (z_1 - z_2) = \frac{L}{\rho g}\left[-\frac{d}{dx}(p+\gamma z)\right] \qquad \frac{\alpha_1}{2g}V_1^2 = \frac{\alpha_2}{2g}V_2^2$$

$$h_L = \frac{L}{\rho g}\left(-\frac{d\hat{p}}{dx}\right) = \frac{L}{\rho g}\left(\frac{2\tau_w}{R}\right)$$ (If $\frac{d\hat{p}}{dx} < 0$ flow moves from left to right)

Where $\tau_w = \frac{1}{8} f \rho V_{ave}^2$

$$h_L = h_f = f\,\frac{L}{D}\frac{V_{ave}^2}{2g}$$ Darcy-Weisbach Equation (valid for laminar or turbulent flow)

Where h_f is the friction loss

Also recall for laminar flow that $\tau_w = \frac{4\mu V_{ave}}{R}$

$$f = \frac{8\tau_w}{\rho V_{ave}^2} = \frac{32\mu}{\rho R V_{ave}} = 64/\mathrm{Re}_D$$

$$\mathrm{Re}_D = V_{ave} D/\nu$$

$$h_L = \frac{32\mu L V_{ave}}{\gamma D^2} \propto V_{\mathrm{ave}}$$ exact solution friction loss for laminar pipe flow!

Note:

Po = Poiseuille number = fRe = 64 = pure constant, which is the case for all laminar flows regardless duct cross section but with different constant depending on cross section; since, $\tau_w \propto V_{ave}$

For turbulent flow, $Re_{crit} \sim 2000$ ($2x10^3$), $Re_{trans} \sim 3000$

f=f (Re, k/D) Re = $\underline{V}_{ave}D/\nu$, k = roughness

τ_w and $h_L \propto V_{ave}^2$

Pipe with minor losses,

$$h_L = h_f + \Sigma h_m \quad \text{where} \quad \begin{aligned} h_m &= K\frac{V^2}{2g} \\ K &= loss\,coefficient \end{aligned}$$

h_m = "so called" minor losses, e.g., entrance/exit, expansion/contraction, bends, elbows, tees, other fitting, and valves.

Differential Form of Energy Equation:

$$\frac{dE}{dt} = \int_{CV} \left[\underbrace{\frac{\partial}{\partial t}(e\rho) + \nabla \cdot (e\rho \underline{V})} \right] d\forall = \dot{Q} - \dot{W}$$

$$\rho \frac{\partial e}{\partial t} + \underbrace{e\frac{\partial \rho}{\partial t} + e\nabla.\left(\rho \underline{V}\right)}_{=0} + \rho \underline{V}.\nabla e = \rho \frac{De}{Dt} = \rho\left(\frac{\partial e}{\partial t} + \underline{V}.\nabla e\right)$$

The RHS can be expressed through surface integrals:

$$\dot{Q} = \int_{CS} \underline{q} \cdot \underline{n} dA$$

$$\dot{W} = \int_{CS} \underline{f} \cdot \underline{V} dA$$

$\underline{q} = -k\nabla T$ heat flux
$\underline{f} = f_j$ = surface forces per unit area acting on CS.

And the surface integrals can be converted into volume integrals using Gauss' theorem:

$$\int_{CS} \underline{q} \cdot \underline{n} dA = \int_{CS} q_i n_i \, dA = \int_{CV} \nabla \cdot \underline{q} \, dA = \int_{CV} \frac{\partial}{\partial x_i} q_i \, dV$$

$$\int_{CS} \underline{f} \cdot \underline{V} dA = \int_{CS} n_i \sigma_{ij} u_j \, dA = \int_{CV} \frac{\partial}{\partial x_i}\left(\sigma_{ij} u_j\right) dV$$

Where:

$$\nabla \cdot \left(\sigma_{ij} u_j\right) = \frac{\partial}{\partial x_i}\left(\sigma_{ij} u_j\right) = \frac{\partial}{\partial x_j}\left(u_i \sigma_{ij}\right)$$

Which enables expressing the energy equation as:

$$\frac{dE}{dt} = \int_{CV} \left[\frac{\partial}{\partial t}(e\rho) + \nabla \cdot (e\rho \underline{V})\right] d\forall$$

$$= \int_{CV} \frac{\partial}{\partial \mathrm{x_i}} q_i \, d\forall - \int_{CV} \frac{\partial}{\partial x_i}\left(\sigma_{ij} u_j\right) d\forall$$

And in the limit as the CV goes to 0, i.e., for a material volume the differential form becomes:

$$\frac{\partial}{\partial t}(e\rho) + \nabla \cdot \left(e\rho \underline{V}\right) = \nabla \cdot \underline{q} - \nabla \cdot (\sigma_{ij} u_j)$$

For the LHS:

$$e = \hat{u} + \frac{1}{2}V^2 + gz = \hat{u} + \frac{1}{2}V^2 - \underline{g} \cdot \underline{r}$$

$$\frac{D\left(-\underline{g} \cdot \underline{r}\right)}{Dt} = -\underline{g} \cdot \frac{D\underline{r}}{Dt} = -\underline{g} \cdot \underline{V} \qquad \boxed{\underline{g} = -g\hat{k}}$$

$$\rho \frac{De}{Dt} = (\dot{Q} - \dot{W})/\forall = \nabla \cdot \underline{q} - \nabla \cdot (\sigma_{ij} u_j)$$
$$= \rho \underbrace{\left(\frac{D\hat{u}}{Dt} + V\frac{DV}{Dt} - \underline{g} \cdot \underline{V}\right)}_{\boxed{\frac{De}{dt}}}$$

All the terms in this equation have dimensions $\left[\frac{N}{m^2 s}\right]$ or equivalently $\left[\frac{kg}{ms^3}\right]$

$$\dot{q} = -\nabla \cdot \underline{q} = \nabla \cdot (k\nabla T) \qquad \text{Fourier's Law Heat Conduction}$$

$$\dot{w} = -\nabla \cdot \left(u_i \sigma_{ij}\right) = -\frac{\partial}{\partial x_j}\left(u_i \sigma_{ij}\right) = -\underline{V} \cdot \underbrace{\left(\nabla \cdot \sigma_{ij}\right)}_{\substack{\rho\left(\frac{D\underline{V}}{Dt} - \underline{g}\right) \\ using\ NS}} - \sigma_{ij}\frac{\partial u_i}{\partial x_j}$$

$\nabla \cdot \underline{f} =$ scalar

$\nabla \cdot \sigma_{ij}$ = vector (decreases 2nd order tensor by one)

$$\frac{\partial}{\partial x_j}\left(u_i\sigma_{ij}\right) = u_i\frac{\partial \sigma_{ij}}{\partial x_j} + \sigma_{ij}\frac{\partial u_i}{\partial x_j}$$

Total work of surface force	Deformation work w/o $\underline{a}$ lost to internal energy.	Increase of KE since contributes fluid $\underline{a}$

First term for $\dot{w}$

$$-\underline{V}\cdot\left(\nabla\cdot\sigma_{ij}\right) = -\underline{V}\cdot\rho\left(\frac{D\underline{V}}{Dt} - \underline{g}\right) = -\rho\left(\underline{V}\cdot\frac{D\underline{V}}{Dt} - \underline{V}\cdot\underline{g}\right)$$

Where:

$$\underline{V}\cdot\frac{D\underline{V}}{Dt} = \underline{V}\cdot\left(\frac{\partial\underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right) = \frac{\partial V^2}{\partial t} + \underline{V}\cdot\nabla V^2 = \frac{DV^2}{Dt} = V\frac{DV}{Dt}$$

Therefore

$$-\underline{V}\cdot\left(\nabla\cdot\sigma_{ij}\right) = -\rho\left(V\frac{DV}{Dt} - \underline{V}\cdot\underline{g}\right)$$

And

$$\dot{w} = -\rho\left(V\frac{DV}{Dt} - \underline{V}\cdot\underline{g}\right) - \sigma_{ij}\frac{\partial u_i}{\partial x_j}$$

Substitute equation for $\dot{q}$ and $\dot{w}$

$$\dot{q} - \dot{w} = -\nabla \cdot (k\nabla \mathrm{T}) + \rho\left(\cancel{V\frac{DV}{Dt} - \underline{V} \cdot \underline{g}}\right) + \sigma_{ij}\frac{\partial u_i}{\partial x_j}$$

$$= \rho\left(\frac{D\hat{u}}{Dt} + \cancel{V\frac{DV}{Dt} - \underline{V} \cdot \underline{g}}\right)$$

$$\rho\frac{D\hat{u}}{Dt} = -\nabla \cdot (k\nabla \mathrm{T}) + \sigma_{ij}\frac{\partial u_i}{\partial x_j}$$

$$\boxed{\begin{gathered}\sigma_{ij} = -p\delta_{ij} + \tau_{ij}\\ \tau_{ij} = 2\mu\varepsilon_{ij}\\ \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})\end{gathered}}$$

Second term on right hand side

$$\sigma_{ij}\frac{\partial u_i}{\partial x_j} = (\tau_{ij} - p\delta_{ij})\frac{\partial u_i}{\partial x_j} = \tau_{ij}\frac{\partial u_i}{\partial x_j} - p\nabla \cdot \mathrm{V}$$

From continuity

$$\frac{D\rho}{Dt} + \rho\nabla.\underline{V} = 0 \rightarrow \nabla.\underline{V} = -\frac{1}{\rho}\frac{D\rho}{Dt}$$

$$-p\nabla.\underline{V} = \frac{p}{\rho}\frac{D\rho}{Dt} = -\rho\frac{D}{Dt}\left(\frac{p}{\rho}\right) + \frac{Dp}{Dt}$$

$$\boxed{\begin{gathered}-\rho\left[\frac{D}{Dt}\left(\frac{p}{\rho}\right)\right]\\ = -\rho\frac{1}{\rho}\frac{Dp}{Dt} - \rho p\frac{D}{Dt}\left(\frac{1}{\rho}\right)\\ = -\frac{Dp}{Dt} + \frac{p}{\rho}\frac{D\rho}{Dt}\\ \\ \frac{D}{Dt}\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}\frac{d\rho}{dt}\end{gathered}}$$

Therefore

$$\sigma_{ij}\frac{\partial u_i}{\partial x_j} = \tau_{ij}\frac{\partial u_i}{\partial x_j} - \rho\frac{D}{Dt}\left(\frac{p}{\rho}\right) + \frac{Dp}{Dt}$$

Such that

$$\rho\frac{D\hat{u}}{Dt} = -\nabla \cdot (k\nabla \mathrm{T}) + \tau_{ij}\frac{\partial u_i}{\partial x_j} - \rho\frac{D}{Dt}\left(\frac{p}{\rho}\right) + \frac{Dp}{Dt}$$

Rearranging equation and substituting dissipation function $\Phi = \tau_{ij}\frac{\partial u_i}{\partial x_j} \geq 0$

$$\rho \frac{D}{Dt}\underbrace{\left(\hat{u} + \frac{p}{\rho}\right)}_{\boxed{h=entalpy}} = -\nabla \cdot (k\nabla \mathrm{T}) + \frac{Dp}{Dt} + \Phi$$

Consider energy equation in form:

$$\rho \frac{D\hat{u}}{Dt} = -\nabla \cdot (k\nabla \mathrm{T}) - p\nabla . \underline{V} + \Phi$$

And compare with mechanical energy equation derived by multiplying u_i x NS:

$$\rho \frac{D\left(\frac{1}{2}\mathrm{u}_i{}^2\right)}{Dt} = \rho \underline{g} \cdot \underline{V} + \frac{\partial\left(\mathrm{u}_i \sigma_{ij}\right)}{\partial x_j} + p\nabla \cdot \underline{V} - \Phi$$

$\rho \underline{g} \cdot \underline{V}$	$\frac{\partial\left(\mathrm{u}_i \sigma_{ij}\right)}{\partial x_j}$	$p\nabla \cdot \underline{V}$	Φ
Rate of work done by body force $\underline{g}$	Total rate of work done σ_{ij}	Rate of work due to volume expansion; converts mechanical energy to internal energy and viceversa	Rate of viscous dissipation

$\Phi \geq 0$ loss mechanical energy = gain internal energy due to deformation of the fluid element

Summary GDE for compressible non-constant property fluid flow

Continuity: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{V}) = 0$

Momentum: $\rho \frac{D\underline{V}}{Dt} = \rho \underline{g} - \nabla p + \nabla . \sigma_{ij}$

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda \nabla . \underline{V} \delta_{ij}$$

$$\underline{g} = -g\hat{k}$$

Energy $\rho \frac{Dh}{Dt} = \frac{Dp}{Dt} + \nabla \cdot (k \nabla T) + \Phi$

Primary variables: p, $\underline{V}$, T

Auxiliary relations: ρ = ρ (p,T) μ = μ (p,T)
(equations of state) h = h (p,T) k = k (p,T)

Restrictive Assumptions:

1) Continuum
2) Newtonian fluids
3) Thermodynamic equilibrium
4) $\underline{g} = -g\hat{k}$
5) heat conduction follows Fourier's law.
6) no internal heat sources.

For incompressible constant property fluid flow

$$d\hat{u} = c_v \, dT \qquad c_v, \mu, k, \rho \sim constant$$

$$\rho c_v \frac{DT}{Dt} = k\nabla^2 T + \Phi$$

For static fluid or $\underline{V}$ small

$$\rho c_p \frac{\partial T}{\partial t} = k\nabla^2 T \qquad \textit{heat conduction equation (also valid for solids)}$$

Summary GDE for incompressible constant property fluid flow ($c_v \sim c_p$)

$$\nabla \cdot \underline{V} = 0$$

$$\rho \frac{D\underline{V}}{Dt} = -\rho g\hat{k} - \nabla p + \mu \nabla^2 \underline{V} \qquad \text{“elliptic”}$$

$$\rho c_p \frac{DT}{Dt} = k\nabla^2 T + \Phi \qquad \text{where } \Phi = \tau_{ij} \frac{\partial u_i}{\partial x_j}$$

Continuity and momentum uncoupled from energy; therefore, solve separately and use solution post facto to get T.

For compressible flow, ρ solved from continuity equation, T from energy equation, and p = (ρ, T) from equation of state (e.g., ideal gas law). For incompressible flow, ρ = constant and T uncoupled from continuity and momentum equations, the latter of which contains ∇p such that reference p is arbitrary and specified post facto (i.e., for incompressible flow, there is no connection between p and ρ). The connection is between ∇p and $\nabla\cdot\underline{V}=0$, i.e., a solution for p requires $\nabla\cdot\underline{V}=0$.

NS:

$$\frac{\partial u_i}{\partial x_i}=0$$

$$\rho\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)=-\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 u_i}{\partial x_j^2}$$

$\nabla\cdot(NS)$:

$$\nabla\cdot\left[\frac{\partial \underline{V}}{\partial t}+\underline{V}\cdot\nabla\underline{V}=-\nabla\left(\frac{p}{\rho}\right)+\nu\nabla^2\underline{V}\right]$$

$$\nabla\cdot\left(\frac{\partial \underline{V}}{\partial t}-\nu\nabla^2\underline{V}\right)+\nabla\cdot\left(\underline{V}\cdot\nabla\underline{V}\right)=-\nabla^2\left(\frac{p}{\rho}\right)$$

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\nabla\cdot\underline{V}+\nabla\cdot\left(\underline{V}\cdot\nabla\underline{V}\right)=-\nabla^2\left(\frac{p}{\rho}\right)$$

$$\underline{V}\cdot\nabla\underline{V}=u_j\frac{\partial u_i}{\partial x_j}$$

$$\nabla\cdot\left(\underline{V}\cdot\nabla\underline{V}\right)=\frac{\partial}{\partial x_i}\left(u_j\frac{\partial u_i}{\partial x_j}\right)=\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}+u_j\cancel{\frac{\partial}{\partial x_i}\frac{\partial u_i}{\partial x_j}}$$

$$\nabla \cdot \left(\underline{V} \cdot \nabla \underline{V}\right) = \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}$$

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2\right)\nabla \cdot \underline{V} = -\frac{1}{\rho}\nabla^2 p - \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$$

For $\nabla \cdot \underline{V} = 0$:

$$\nabla^2 p = -\rho \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$$

Poisson equation determines pressure up to additive constant.

<u>Approximate Models</u>:

1) Stokes Flow

For low $\mathrm{Re} = \dfrac{UL}{\nu} << 1, \quad \underline{V} \cdot \nabla \underline{V} \sim 0$

$$\left.\begin{aligned} &\nabla \cdot \underline{V} = 0 \\ &\frac{\partial \underline{V}}{\partial t} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \underline{V} \end{aligned}\right] \rightarrow$$

Linear, "elliptic"
Most exact solutions NS; and for steady flow superposition, elemental solutions, and separation of variables

$$\nabla \cdot (NS) \Rightarrow \nabla^2 p = 0$$

2) Boundary Layer Equations

For high Re >> 1 and attached boundary layers or fully developed free shear flows (wakes, jets, mixing layers), v<<U, $\dfrac{\partial}{\partial x} << \dfrac{\partial}{\partial y}$, $p_y = 0$, and for free shear flow $p_x = 0$.

$$u_x + v_y = 0$$

$$u_t + uu_x + vu_y = -\hat{p}_x + \nu u_{yy} \qquad \textit{non-linear, "parabolic"}$$

$$p_y = 0$$

$$-\hat{p}_x = U_t + UU_x$$

Many exact solutions; similarity methods

3) Inviscid Flow

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \underline{V} \right) = 0$$

$$\rho \frac{D\underline{V}}{Dt} = \rho \underline{g} - \nabla p \qquad \textit{Euler Equation, nonlinear, "hyperbolic"}$$

$$\rho \frac{Dh}{Dt} = \frac{Dp}{Dt} + \nabla \cdot (k \nabla T) \quad p, \underline{V}, T \textit{ unknowns} \quad \textit{and } \rho, h, k = f(p, T)$$

4) Inviscid, Incompressible, Irrotational

$$\nabla \times \underline{V} = 0 \rightarrow \underline{V} = \nabla \varphi$$
$$\nabla . \underline{V} = 0 \rightarrow \nabla^2 \varphi = 0 \quad \textit{linear} \text{ elliptic}$$

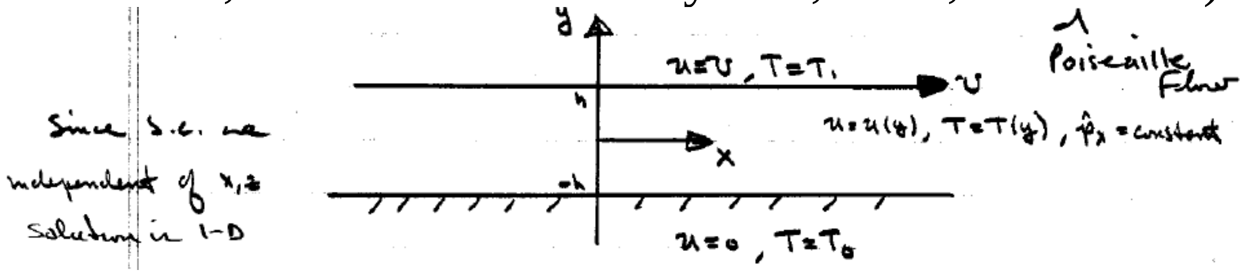
$\int$ Euler Equation → Bernoulli Equation:

$$p + \frac{\rho}{2} V^2 + \rho g z = const$$

Many elegant solutions: Laplace equation using superposition elementary solutions, separation of variables, complex variables for 2D, and Boundary Element methods.

<u>Couette Shear Flows</u>: 1-D shear flow between surfaces of like geometry (parallel plates or rotating cylinders).

<u>Steady Incompressible Flow Between Parallel Plates</u>: *Combined Couette and Poiseuille Flow.* IBVP: geometry, conditions, domain/coordinate system, GDE, and IC/BC)



$\nabla \cdot \underline{V} = 0$

$u_x + v_y + w_z = 0$

$u_x = 0$ i.e., fully developed flow

$$\rho \frac{D\underline{V}}{Dt} = -\nabla \hat{p} + \mu \nabla^2 \underline{V} \qquad \frac{\partial u}{\partial t} + u u_x + v u_y + w u_z = 0$$

$$0 = -\hat{p}_x + \mu u_{yy}$$

$$\rho c_p \frac{DT}{Dt} = k\nabla^2 T + \Phi \qquad \frac{\partial T}{\partial t} + uT_x + vT_y + wT_z = 0$$

$$\Phi = \tau_{ij}\frac{\partial u_i}{\partial x_j} = \mu(u_{i,j} + u_{j,i})\frac{\partial u_i}{\partial x_j}$$

$$\mu\big[2u_x^2 + 2v_y^2 + 2w_z^2$$
$$+(v_x + u_y)^2 + (w_y + v_z)^2 + (u_z + w_x)^2\big]$$
$$= \mu u_y^2$$

$$0 = kT_{yy} + \mu u_y^2$$

(Note inertia terms vanish identically and ρ is absent from equations)

*Non-dimensional equations, but drop **

$$u^* = u/U \qquad T^* = \frac{T - T_0}{T_1 - T_0} \qquad y^* = y/h$$

$$u_x = 0 \tag{1}$$

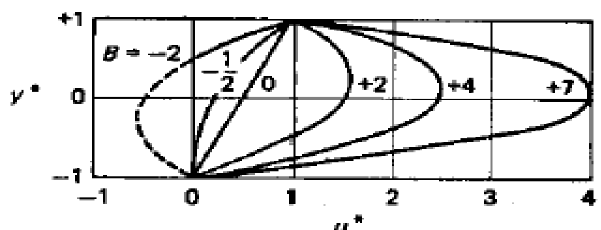
$$u_{yy} = \frac{h^2}{\mu U}\hat{p}_x = -B = constant \tag{2}$$

$$T_{yy} = \underbrace{\frac{\mu U^2}{k(T_1 - T_0)}}_{\mathrm{Pr}Ec}\left[-u_y^2\right] \tag{3}$$

B.C. y = 1 u = 1 T = 1
y = -1 u = 0 T = 0

(1) is consistent with 1-D flow assumption. Simple form of (2) and (3) allow for solution to be obtained by double integration.

$$\Rightarrow \quad u = \underbrace{\frac{1}{2}(1+y)}_{\text{Linear flow due to U}} + \underbrace{\frac{1}{2}B(1-y^2)}_{\text{Parabolic flow due to } p_x} \qquad y{=}y/h$$



Linear flow due to U

Parabolic flow due to p_x

Note: linear superposition since $\underline{V}\cdot\nabla\underline{V} = 0$

Solution depends on $B = -\dfrac{h^2}{\mu U}\hat{p}_x$ ($\hat{p}_x = \partial p/\partial x + \gamma\partial z/\partial x$)

B < 0 (favorable) $\hat{p}_x$ is opposite to U
B < -0.5 backflow occurs near lower wall
|B| >> 1 flow approaches parabolic profile.

$$T = \frac{1}{2}(1+y) + \frac{\Pr E_c}{8}(1-y^2) \overbrace{- \frac{\Pr E_c B}{6}(y-y^3) + \frac{\Pr E_c B^2}{12}(1-y^4)}^{\text{Pressure gradient effect}}$$

Pure conduction

T rises due to viscous dissipation

Dominant term for B→ ∞

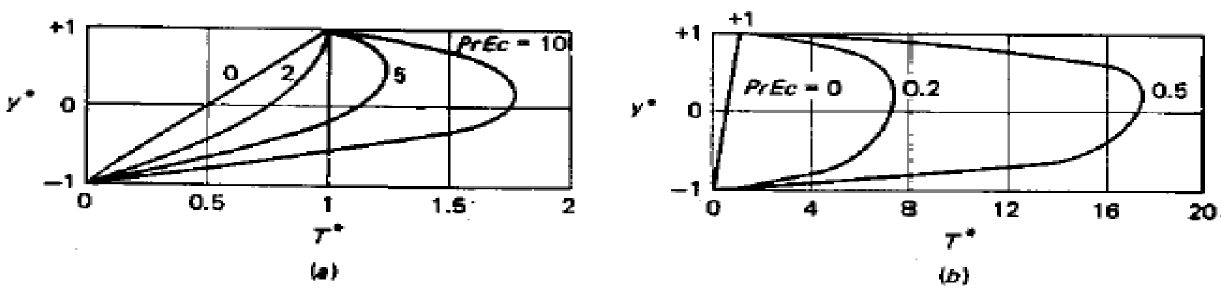


FIGURE 3-3
Temperature distributions for flow between parallel plates, Eq. (3-12): (*a*) pure Couette flow: *B* = 0; (*b*) mostly Poiseuille flow: *B* = 20.

Note: usually PrE_c is quite small

Substance	*PrE_c*	*dissipation*
Air	0.001	very small
Water	0.02	
Crude oil	20	large

$$Br = \Pr E_c$$
$$= Brinkman\ \#$$

Prandtl number Pr = $\mu C_p/k$ = momentum diffusivity/thermal diffusivity

Eckert number Ec = $U^2/C_p(T_1-T_0)$ = advection transport/heat dissipation potential

Br# = heat produced viscous dissipation/heat transported molecular conduction

Shear Stress

1) $\hat{p}_x = 0$ i.e., pure Couette Flow

$$B = -\frac{h^2}{\mu U}\hat{p}_x = 0$$

Using solution shown previously

$$u^* = \frac{1}{2}(1 + y^*) + \frac{1}{2}B\left(1 - {y^*}^2\right) = \frac{1}{2}(1 + y^*)$$

Calculating wall shear stress

$$\frac{u}{U} = \frac{1}{2}\left(1 + \frac{y}{h}\right)$$

$$\frac{\partial\left(\frac{u}{U}\right)}{\partial\left(\frac{y}{h}\right)} = \frac{1}{2}$$

$$\tau_w = \mu\frac{du}{dy}\bigg]_{y=-1} = \frac{\mu U}{2h}$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{\frac{\mu U}{2h}}{\frac{1}{2}\rho U^2} = \frac{\mu}{\rho U h}$$

Since $Re_h = \rho U h/\mu$

$$C_f = \frac{1}{Re_h}$$

$P_0 = C_f Re = 1$: Better for non-accelerating flows since ρ is not in equations and P_0 = pure constant

2) U = 0   i.e. pure Poiseuille Flow

$$u^* = \frac{1}{2}B(1-y^{*2}) \qquad u^*_{y^*} = -By^* \qquad u_y = -\frac{BU}{h^2}y \qquad V_{ave} = \bar{u}$$

Where $B = \frac{-h}{\mu U}\hat{p}_x = \frac{2u_{\max}}{U}$

Dimensional form $u = \underbrace{-\frac{1}{2}\frac{h^2}{\mu}\hat{p}_x}_{u_{\max}}\left(1-\left(y/h\right)^2\right) \quad Q = \int_{-h}^{h} u\,dy = \frac{4}{3}hu_{\max}$

$$\bar{u} = \frac{Q}{2h} = \frac{2}{3}u_{\max} = V_{ave}$$

Remember that for laminar pipe flow, $V_{ave} = \frac{1}{2}u_{max}$

$$\tau_w = \mu u_y\Big|_{y=\pm h} = -\mu\frac{BU}{h} \qquad \textit{upper}$$

$$= +\mu\frac{BU}{h} \qquad \textit{lower}$$

$$|\tau_w| = \mu\frac{BU}{h} = \mu\frac{2u_{\max}}{h} = \mu 3\bar{u}/h \qquad \begin{array}{l} \propto \bar{u} \quad \textit{lam.} \\ \propto \rho\bar{u}^2 \quad \textit{turb.} \end{array}$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{6\mu}{\rho\bar{u}h} = \frac{6}{\mathrm{Re}_h} \quad \textit{or} \quad P_0 = C_f\,\mathrm{Re}_h = 6$$

Remember that for laminar pipe flow, $C_f = \frac{16}{Re_D}$ and $\tau_w = \frac{\mu 8 V_{ave}}{D}$, i.e., except for numerical constants same functionality as for circular pipe.

Rate of heat transfer at the walls:

$$q_w = \left| k \frac{\partial T}{\partial y} \right|_{y \pm h} = \frac{k}{2h}(T_1 - T_0) \pm \mu \frac{U^2}{4h}$$

+ = upper, - = lower

Heat transfer coefficient:

$$\varsigma = q_w \big/ \left(T_1 - T_0\right)$$

$$Nu = \frac{2h\varsigma}{k} = 1 \pm Br \big/ 2$$

For Br > 2, both upper & lower walls must be cooled to maintain T_1 and T_0

Conservation of Angular Momentum: moment form of momentum equation (not new conservation law!)

$B = \underline{H}_0 = \int_{sys} \underline{r} \times \underline{V}\, dm =$ *angular momentum of system about inertial coordinate system 0 (extensive property)*

$\beta = \frac{dB}{dM} = \underline{r} \times \underline{V}$ *(Intensive property)*

$$\underbrace{\frac{d\underline{H}_0}{dt}}_{\substack{\text{Rate of}\\ \text{change of}\\ \text{angular}\\ \text{momentum}}} = \frac{d}{dt}\int_{CV} (\underline{r} \times \underline{V})\rho\, d\forall + \int_{CS} (\underline{r} \times \underline{V})\rho\, \underline{V}_R . \underline{n}\, dA$$

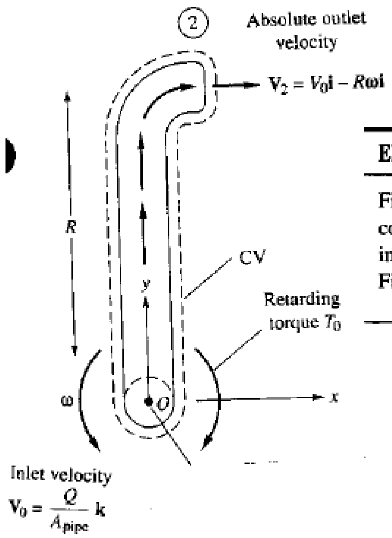
$= \sum \underline{M}_0 =$ *vector sum all external moments applied on CV due to both* $\underline{F}_B$ *and* $\underline{F}_S$*, including reaction forces.*

For uniform flow across discrete inlet/outlet:

$$\int_{CS} (\underline{r} \times \underline{V})\rho\, \underline{V}_R . \underline{n}\, dA = \sum (\underline{r} \times \underline{V})_{out} \dot{m}_{out} - \sum (\underline{r} \times \underline{V})_{in} \dot{m}_{in}$$

$$\underline{M}_0 = \underbrace{\int_{CS} \underline{\tau} \cdot dA \times \underline{r}}_{surface\ force\ moment} + \underbrace{\int_{CV} (\rho \underline{g}\, d\forall) \times \underline{r}}_{body\ force\ moment} + \underline{M}_R$$

$\underline{M}_R =$ *moment of reaction forces*



EXAMPLE 3.15

Figure 3.14 shows a lawn sprinkler arm viewed from above. The arm rotates about O at constant angular velocity ω. The volume flux entering the arm at O is Q, and the fluid is incompressible. There is a retarding torque at O, due to bearing friction, of amount $-T_o\mathbf{k}$. Find an expression for the rotation ω in terms of the arm and flow properties.

Fig. 3.14 View from above of a single arm of a rotating lawn sprinkler.

Take inertial frame 0 as fixed to earth such that CS moving at $\underline{V}_S$= -Rω $\hat{\imath}$

$$\underline{V} = \underline{V}_R + \underline{V}_S$$

$$\underline{V}_2 = V_0\hat{\imath} - R\omega\hat{\imath} = (V_0 - R\omega)\hat{\imath} \qquad \underline{r}_2 = R\,\hat{\jmath}$$

$$\underline{V}_1 = V_0\hat{k} \qquad \underline{r}_1 = 0\,\hat{\jmath}$$

$$V_0 = Q/A_{pipe}$$

Retarding torque due to bearing friction

$$\sum \underline{M}_z = 0 = -T_0\hat{k} = (\underline{r}_2 \times \underline{V}_2)\dot{m}_{out} - (\underline{r}_1 \times \underline{V}_1)\dot{m}_{in}$$

$$\dot{m}_{out} = \dot{m}_{in} = \rho Q \qquad -T_o\hat{k} = R(V_0 - R\omega)(-\hat{k})\rho Q$$

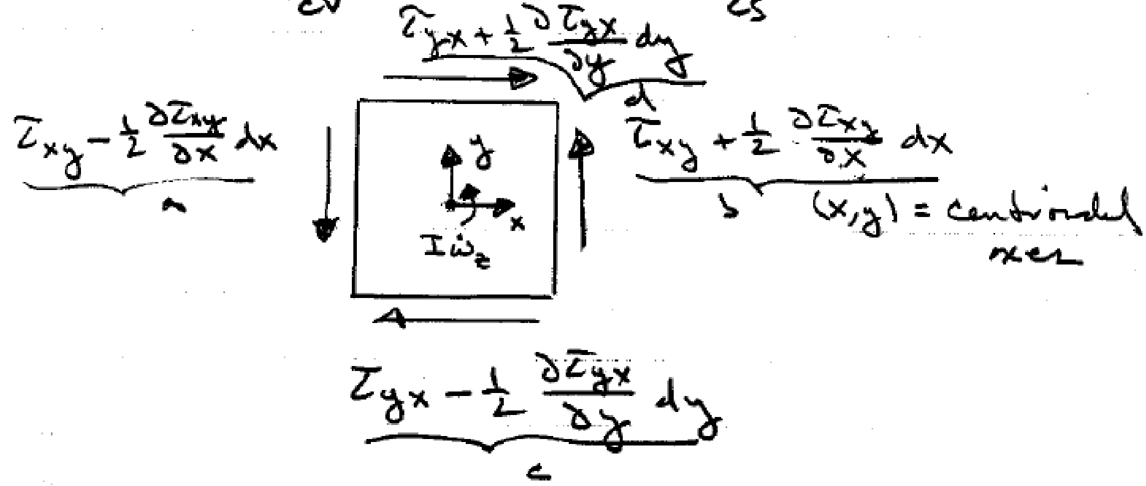
$$\omega = \frac{V_0}{R} - \frac{T_0}{\rho Q R^2}$$ → *interestingly, even for T_0=0, ω_{max}=V_0/R*

(limited by ratio such that large R small ω; large V_0 large ω)

<u>Differential Equation of Conservation of Angular Momentum</u>:

Apply CV form for fixed CV:

$$\Sigma \underline{M}_o = \frac{d}{dt}\int_{CV} (\underline{r} \times \underline{V})\, \rho\, d\forall + \int_{CS} (\underline{r} \times \underline{V})\, \rho\, \underline{V}\cdot\underline{n}\, dA$$



$\dot{\omega}_z$ = angular acceleration

I = moment of inertia

$$I\dot{\omega}_z = a\, dy\frac{dx}{2} + b\, dy\frac{dx}{2} - c\, dx\frac{dy}{2} - d\, dx\frac{dy}{2}$$

$$I\dot{\omega}_z = \left(\tau_{xy} - \tau_{yx}\right) dxdy$$

Since $I = \frac{\rho}{12}\left[dxdy^3 + dydx^3\right] = \frac{\rho}{12}dxdy\left[dx^2 + dy^2\right]$

$$\frac{\rho}{12}\left[dx^2 + dy^2\right]\dot{\omega}_z = \tau_{xy} - \tau_{yx}$$

$$\lim_{dx\to 0, dy\to 0} \tau_{xy} = \tau_{yx}, \text{ similarly, } \quad \tau_{xz} = \tau_{zx}, \quad \tau_{yz} = \tau_{zy}$$

i.e. $\tau_{ij} = \tau_{ji}$ *stress tensor is symmetric (stresses themselves cause no rotation)*

2-8 Mathematical Character of the Basic Equations

Solution of this system of equations

Clearly, the ^ represents a formidable challenge. In fact, even with modern supercomputers solutions for arbitrary geometries are difficult, especially for large Reynolds numbers (ie turbulent flow). The difficulties arise due to:

1) strong coupling between p, $\underline{V}$, and T
2) each equation contains one or more nonlinearities
3) equations display elliptic, parabolic, and hyperbolic characteristics depending on the specific flow and geometry
4) numerous length + time scales required for turbulent flow

The general form of the momentum and energy eq is:

$$\underbrace{A Q_{xx} + B Q_{yy} + C Q_{zz}}_{\text{viscous diffusion}} = \underset{\text{unsteady}}{D Q_t} + \underbrace{E Q_x + F Q_y + G Q_z}_{\text{convection}} + \underset{\text{source/sink}}{H(p, Q, \text{etc.})}$$

or for steady 2-D ⇒ $A Q_{xx} + B Q_{xy} + C Q_{yy} = D$

+ in standard form

$B^2 - 4AC$	< 0	elliptic	BVP
	$= 0$	parabolic	mixed
	> 0	hyperbolic	initial value problem

elliptic: b.c. required on complete contour enclosing region
parabolic: b.c. closed in one direction and open at one end
hyperbolic: b.c. only req. on one portion of boundary others open

Examples: $\phi_{xx} + \phi_{yy} = 0$ Laplace (elliptic)

$\phi_{xx} - \phi_{y} = 0$ Heat conduction (parabolic)

$\phi_{xx} - \phi_{yy} = 0$ Wave Eq. (hyperbolic)

The full N-S eq. contain all these possibilities depending on the particular flow situation at hand. As a very simple example consider incompressible flow with zero convective derivatives:

$$\underline{V}\cdot\nabla\underline{V} = \underline{V}\cdot\nabla T = 0 \quad \& \quad \rho, \mu, k \text{ constant}$$

$$\left.\begin{array}{ll} (1) & \nabla\cdot\underline{V} = 0 \\ (2) & \rho\underline{V}_t = \underline{f} - \nabla p + \mu\nabla^2\underline{V} \\ (3) & \rho c_v T_t = k\nabla^2 T \end{array}\right\} \text{linear system}$$

uncoupled ie solve for $\underline{V}, p$, then solve for T

(3) is $T_t = \alpha\nabla^2 T$ $\quad \alpha = \frac{k}{\rho c_v}$ Heat conduction: elliptic in space + parabolic in t

$\nabla\cdot(2) \Rightarrow \nabla^2 p = 0$ Laplace eq.

$\nabla\times(2) \Rightarrow \underline{\omega}_t = \nu\nabla^2\underline{\omega}$ $\quad \nu = \mu/\rho$ Vorticity diffusion eq.

α: diffusivity of heat
ν: " of vorticity
$\Rightarrow$ ratios $Pr = \nu/\alpha$, $Sc = \nu/D$, $Le = D/\alpha$ as previously discussed

Prandtl, Schmidt, Lewis

$\underline{x}^* = \underline{x}/L \qquad \underline{V}^* = \underline{V}/V_0 \qquad p^* = (p-p_0)/\rho_0 V_0^2$

$\Rightarrow \nabla^* = \nabla L$

$t^* = \frac{t V_0}{L}$ or ωt or t/τ

$T^* = (T-T_0)/(T_w - T_0) \qquad \mu^* = \mu/\mu_0$ etc.

Now drop * in presentation!

Continuity: $\frac{D\rho}{Dt} + \nabla \cdot (\rho \underline{V}) = 0$ no parameters enter

Energy: $\rho c_p \frac{DT}{Dt} = Ec \frac{Dp}{Dt} + \frac{1}{Re Pr} \nabla \cdot (k \nabla T) + \frac{Ec}{Re} \Phi$

$Re = \rho_0 V_0 L/\mu_0 \qquad Pr = \mu_0 c_{p0}/k_0 \qquad Ec = \frac{V_0^2}{c_{p0}(T_w - T_0)}$

Prandtl # Eckert #

High-speed flow: all three are important

Low-speed flow ($Ma \leq .3$): pressure + dissipation small (ie, $\underbrace{Re\,Pr}_{\text{Peclet Number}}$ important)

$$\boxed{\begin{array}{l} v^2 << c_p \Delta T \\ KE << \text{heat transfer} \\ \frac{Dp}{Dt} + \Phi = O(v^2) \end{array}}$$

Note: ρ^*, μ^*, etc. $= f(p^*, T^*)$

$\therefore$ for complete similarity not only must Re, Pr etc be similar, but also the state relations

Lastly, we consider the momentum equation

$$\rho \frac{D\underline{v}}{Dt} = -\nabla p + \rho \underline{g} + \nabla \cdot \tau'_{ij}$$

$$\tau'_{ij} = \mu (u_{i,j} + u_{j,i}) \quad \text{neglecting } \lambda$$

gravity term depends on type of flow:

High-speed flow: $\rho g \sim 0$

$$\Rightarrow \quad \rho \frac{D\underline{v}}{Dt} = -\nabla p + \frac{1}{Re} \nabla \cdot \tau'_{ij}$$

single parameter (Pr + Ec through energy equation)

Low-speed flow: $\rho = \rho_0 (1 - \beta \Delta T)$ ie $\rho = \rho(T)$

$\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p$ = coeff. of thermal expansion

different high speed flow due to definition P of $\underline{g}$ term w/ Gr

$$\Rightarrow \quad \frac{D\underline{v}}{Dt} = -\nabla P - \frac{Gr}{Re^2} \beta T \underline{g} + \frac{1}{Re} \nabla \cdot \tau'_{ij}$$

$$P = \frac{p + \rho_0 g z - p_0}{\rho_0 V_0^2} = \text{piezometric pressure coeff.}$$

$$Gr = g \beta_0 \rho_0^2 L^3 (T_w - T_0) / \mu_0^2 = \text{Grashof number}$$

measure of buoyancy (or free-convection) effects

Two Cases

① $\frac{Gr}{Re^2} = \frac{\beta \Delta T}{Fr}$ $\quad \beta \Delta T \sim 1$ $\quad$ forced motion (ie $U \neq 0$)

determines importance of buoyancy $\quad Fr = V_0^2 / gL$

② $\frac{Gr}{Re^2} = Gr$ $\quad U_0 = \frac{\mu_0}{\rho_0 L} \Rightarrow Re = 1$ $\quad$ free convection (ie $U = 0$ & motion due to ΔT)

$= 1$

2.9 Dimensionless Parameters in Viscous Flow

Equations involve: 9 fluid properties $\rho\ \mu\ \lambda\ k\ C_p\ C_v\ l\ \beta\ \sigma$ (mean free path, coeff. thermal exp., surface tension)

4 reference quantities V_0, p_0, T_0, L_0

1 wall heat flux

1 body force g

15 flow parameters

$$\underbrace{\underline{V} \text{ or } p \text{ or } T}_{} = f(\underline{x}, t, 15 \text{ flow parameters})$$

5 dependent variables in 4 basic dimensions M, L, t, T

Buckingham π theorem $\Rightarrow$ 11 dimensionless quantities

1	2	3	4	5	6	7	8	9	10	11
Re	Pr	Fr	Ec	γ	Gr	Nu	Kn ($= l/L$)	Ca	We	λ/μ
		FSBC				wall heat transfer	slip flow		FSBC	

These quantities enter through the equations and boundary conditions and are very important with regard to dimensional analysis and similarity

Other possibilities include: $Ma = \frac{V_0}{a}$, $St = \frac{\omega L}{V_0}$

In order to nondimensionalize the gde + bc, the following reference quantities are used:

velocity	V_0 (freestream) (forced convection) or $\frac{\mu_0}{\rho_0 L}$ free convection
length	L
time	L/V_0 or $1/\omega$ or τ = important time scale
properties	$p_0, \rho_0, T_0, \mu_0, k_0$, etc.

Non dimensional boundary conditions:

obviously depend on type of flow

free-stream: $\underline{V}=1$, $p=0$, $T=0$ no new parameters

solid wall: $\underline{V}=0$ $T=1$ or $k T_n = Nu = \frac{q_w L}{k_0 (T_w - T_0)}$

not valid for large Ma and low Re where there may be slip velocity + temperature jump and b.c. also involve Kn, Pr, $\gamma = c_p/c_v$

Nusselt number

Knudsen number $= \ell / L$

free-surface (inviscid): $\frac{DF}{Dt}=0$ no new parameters

$$p = C + \frac{1}{Fr}\eta - \frac{1}{We}\left(R_x^{-1} + R_y^{-1}\right)$$

vapor pressure

$C = p_a - p_v / \rho_0 V_0^2$ = Cavitation number ($C<1 \Rightarrow$ cavitation important)

$Fr = V_0^2 / gL$ = Froude number (usually $V_0/\sqrt{gL}$)

$We = \rho_0 V_0^2 L / \tau$ = Weber number

$\Rightarrow$ Other important parameters will arise when we consider Turbulent flows: roughness + free-stream turbulence