Chapters 1 Preliminary Concepts & 2 Fundamental Equations of Compressible Viscous Flow

(2) Fluid Properties

(1) **_Kinematic_**: linear ($\underline{V}$) angular ($\underline{\omega}/2$) velocity, rate of strain (ε_{ij}), vorticity ($\underline{\omega}$), and acceleration ($\underline{a}$).

(2) **_Transport_**: viscosity (μ), thermal conductivity (k), and mass diffusivity (D).

(3) **_Thermodynamic_**: pressure (p), density (ρ), temperature (T), internal energy ($\hat{u}$), enthalpy   ($h = \hat{u} + p/\rho$), entropy (s), specific heat (C_v, C_p, $\gamma = C_p/ C_v$, etc).

(4) **_Miscellaneous_**: surface tension (σ), vapor pressure (p_v), etc.

(1) Kinematic Properties:

Kinematics refers to the description of the flow pattern without consideration of forces and moments, whereas dynamics refers to descriptions of $\underline{F}$ and $\underline{M}$.

Lagrangian vs. Eulerian description of velocity and acceleration:

      (a)   Lagrangian approach focuses on tracking individual fixed particles.

$$\underline{V}_\rho = \frac{d\underline{r}_\rho}{dt}$$

$$\underline{a}_\rho = \frac{d\underline{V}_\rho}{dt} = \frac{d^2\underline{r}_\rho}{dt^2}$$
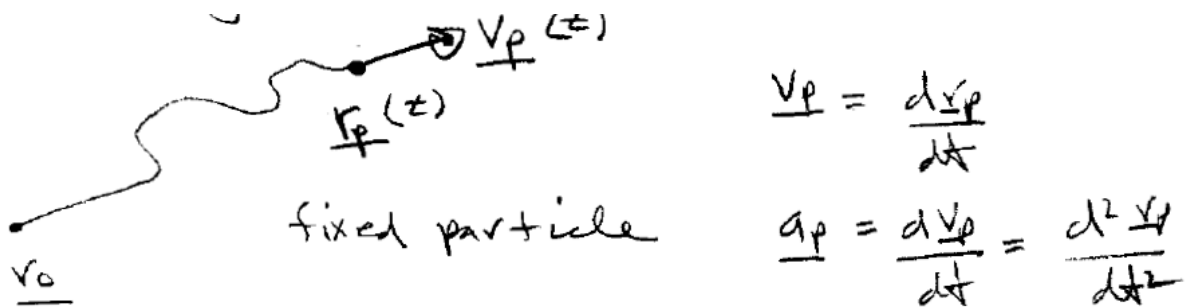
(b)   Eulerian approach focuses on fixed points in space.

$$\underline{V}(\underline{x},t) = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$$

fixed point $\underline{x} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$
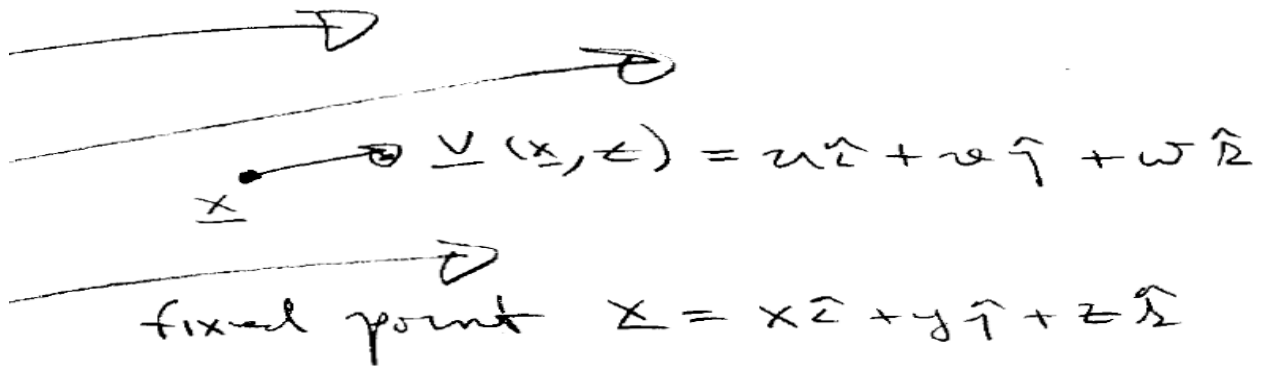
(u,v,w) = $\underline{V}(\underline{x},t)$ are velocity components in (x,y,z) directions.

$$d\underline{V}(\underline{x},t) = \frac{\partial \underline{V}}{\partial t} dt + \frac{\partial \underline{V}}{\partial x_i} dx_i$$

However, dx_i and dt are not independent since derivative is assumed to follow a fluid particle i.e.

$$dx_i = u_i dt$$

$$\frac{d\underline{V}(\underline{x},t)}{dt} = \frac{\partial \underline{V}}{\partial t} + \frac{\partial \underline{V}}{\partial x_i} u_i$$

In fluid mechanics special notation is used to define substantial/material derivative, which follows a fluid particle:

$$\frac{D\underline{V}}{Dt} = \frac{\partial \underline{V}}{\partial t} + \frac{\partial \underline{V}}{\partial x} u + \frac{\partial \underline{V}}{\partial y} v + \frac{\partial \underline{V}}{\partial z} w$$

$$\frac{D\underline{V}}{Dt} = \frac{\partial \underline{V}}{\partial t} + (\underline{V} \cdot \nabla)\underline{V} \;,\; \nabla = gradient = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{V} \cdot \nabla$$ = substantial/material derivative = derivative following motion
of particle

$$\frac{D\underline{V}}{Dt} = \text{Lagrangian time rate of change of velocity}$$

$$\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} = \text{local \& convective acceleration in terms of Eulerian derivatives}$$

$$\underline{a} = a_x i + a_y j + a_z k$$

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

Note: $\underline{V} \cdot \nabla \underline{V} = \nabla \frac{V^2}{2} - \underline{V} \times (\nabla \times \underline{V})$ vector identity, i.e., for irrotational flow convective acceleration becomes familiar KE term in the Bernoulli equation.

The Eulerian approach is more convenient since we are seldom interested in simultaneous time history of individual fluid particles, but rather time history of fluid motion (and $\underline{F}$, $\underline{M}$) in fixed regions in space (control volumes). However, three fundamental laws of fluid mechanics (i.e. conservation of mass, momentum, and energy) are formulated for systems (i.e. particles) and not control volumes (i.e. regions) and therefore must be converted from system to CV: Reynolds Transport Theorem.

$\underline{V}(\underline{x},t)$ is a vector field: Vector operators *divergence* and *curl* lead to other kinematics properties:

$$\nabla \cdot \underline{V} = divergence\ \underline{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\left.\begin{array}{l} M = \rho \forall \quad (\forall = volume\ of\ fluid\ particle) \\ dM = \rho\,d\forall + \forall\,d\rho = 0 \\ -\dfrac{d\forall}{\forall} = \dfrac{d\rho}{\rho} \end{array}\right\} \Rightarrow \frac{1}{\rho}\frac{D\rho}{Dt} = -\frac{1}{\forall}\frac{D\forall}{Dt} \quad (1)$$

Continuity: $\dfrac{D\rho}{Dt} + \rho\nabla \cdot \underline{V} = 0 \Rightarrow \nabla \cdot \underline{V} = -\dfrac{1}{\rho}\dfrac{D\rho}{Dt} \quad (2)$

(1) and (2): $\quad \dfrac{1}{\forall}\dfrac{D\forall}{Dt} = \nabla \cdot \underline{V} = -\dfrac{1}{\rho}\dfrac{D\rho}{Dt}$

rate of change $\forall$ per unit $\forall$ = - rate of change ρ per unit ρ

For incompressible fluids, ρ = constant

$$\nabla \cdot \underline{V} = 0 \qquad \text{i.e. fluid particles have constant } \forall \text{, but not necessarily shape}$$

$$\nabla \times \underline{V} = curl \ \underline{V} = \underline{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

= vorticity = 2 * angular velocity of fluid particle

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \hat{i} - \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \right) \hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \hat{k}$$
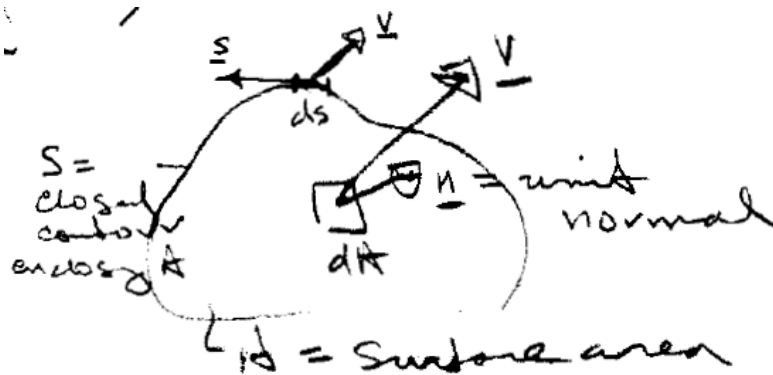
For irrotational flow $\nabla \times \underline{V} = 0$

i.e. $\underline{V} = \nabla \phi = u\hat{i} + v\hat{j} + w\hat{k} = \dfrac{\partial \phi}{\partial x} \hat{i} + \dfrac{\partial \phi}{\partial y} \hat{j} + \dfrac{\partial \phi}{\partial z} \hat{k}$

and for ρ = constant,

$$\nabla . \underline{V} = \nabla \cdot \nabla \phi = \nabla^2 \phi = 0 \rightarrow \quad \text{Potential Flow Theory}$$

Other useful kinematic properties include volume and mass flow-rate $(Q, \dot{m})$, average velocity $(\overline{V})$, and circulation (Γ)

$Q = \int_A \underline{V} \cdot \underline{n}\, dA$ where Q = volume of fluid per unit time through A (flux

of Vn through A bounded by S:"flux" generally used to mean surface integral of variable)

$\dot{m} = \int_A \rho \underline{V} \cdot \underline{n}\, dA$ where $\dot{m}$ = mass of fluid per unit time

through A

$\overline{V} = Q/A$ where $\overline{V}$ = average velocity through A

$A = \int_A dA$ where A = surface area

$\Gamma = \oint_S \underline{V} \cdot \underline{ds} = \int_A \nabla \times \underline{V} \cdot \underline{dA}$ (Stokes theorem - relates line and area integrals)

line integral for tangential velocity component $= \int_A \underline{\omega} \cdot \underline{n}\, dA$ = flux (surface integral) of normal vorticity component

Kutta-Joukowski Theorem: lift (L) per unit span for an arbitrary 2D cylinder in uniform stream U with density ρ is $L = \rho U \Gamma$, with direction of L perpendicular to U.

(2) Transport Properties

There is a close analogy between momentum, heat, and mass transport; therefore, coefficient of viscosity (μ), thermal conductivity (k), and mass diffusivity (D) are referred to as transport properties.

<u>Heat Flux</u>:

Fourier's Law:
$$\underline{q} = -k\nabla T \quad \left[\frac{J}{m^2 s}\right]$$

(rate of heat flux is proportional to the temperature gradient per unit area; flux is from higher to lower T)

$$k\left[\frac{W}{mK}\right] \quad = \quad \text{f(x,y,z)} \qquad \rightarrow \quad \text{solid}$$

$$= \quad \text{constant} \qquad \rightarrow \quad \text{liquid \{isotropic\}}$$

<u>Mass Flux</u>:

Fick's Law:
$$\underline{q} = -D\nabla C \quad \left[\frac{kg}{m^2 s}\right]$$

(rate of mass flux is proportional to concentration (C) gradient per unit area; flux is from higher to lower C)

$$D \quad \left[\frac{m^2}{s}\right]$$

Momentum Flux:

Newtonian Fluid: $\qquad \tau = \mu \dfrac{du}{dy} \left[\dfrac{N}{m^2} \right] \qquad$ 1D flow

(rate of momentum flux/shear stress is proportional to the velocity gradient per unit area, which tends to smooth out the velocity profile)

$$\mu \left[\frac{Ns}{m^2} = \frac{kg}{ms} \right]$$

For 3D flow, the shear/rate of strain relationship is more complex, as will be shown later in the derivation of the momentum equation.

$$\tau_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \delta_{ij} \lambda \nabla \cdot \underline{V}$$

Where $u_i = (u, v, w)$, $x_i = (x, y, z)$

λ= 2nd coefficient of viscosity

For heat and mass, transported quantities are scalars and flux is a vector; whereas for momentum, transported quantity is a vector and flux is a tensor. Also, all three laws are phenomenological (i.e. based on empirical evidence: experience and experiments).

Non-Newtonian fluids follow nonlinear shear/rate of strain relationships

$$\tau \ \alpha \ \varepsilon_{ij}{}^n \qquad n < 1 \ \text{pseudoplastic}$$

$$n = 1 \ \text{Newtonian}$$

$$n > 1 \ \text{dilatant}$$

μ (and k) are also thermodynamic properties:

$\qquad \mu = \mu$(gas or liquid, T, p)

For both gases and liquids, μ increases with p, but Δ μ is small and usually neglected. For gases μ increases with T, whereas for liquids μ decreases with T. For gases, momentum transport and μ are roughly proportional to √T similarly as per random thermal speed. For liquids, shear stress is due to intermolecular cohesive forces more than thermal molecular motions, which decrease with T.

Kinematic viscosity:

$$\upsilon = \mu/\rho \left[\frac{m^2}{s} \right]$$ arises in equations as

diffusion coefficient   Fig. A.2 Textbook

Reynolds Number:

$$\mathrm{Re} = \frac{UL\rho}{\mu} = \frac{UL}{\upsilon}$$ U = velocity scale,

L = length scale

The Reynolds number is an important nondimensional parameter (ratio inertia/viscous forces) which characterizes fluid flow.
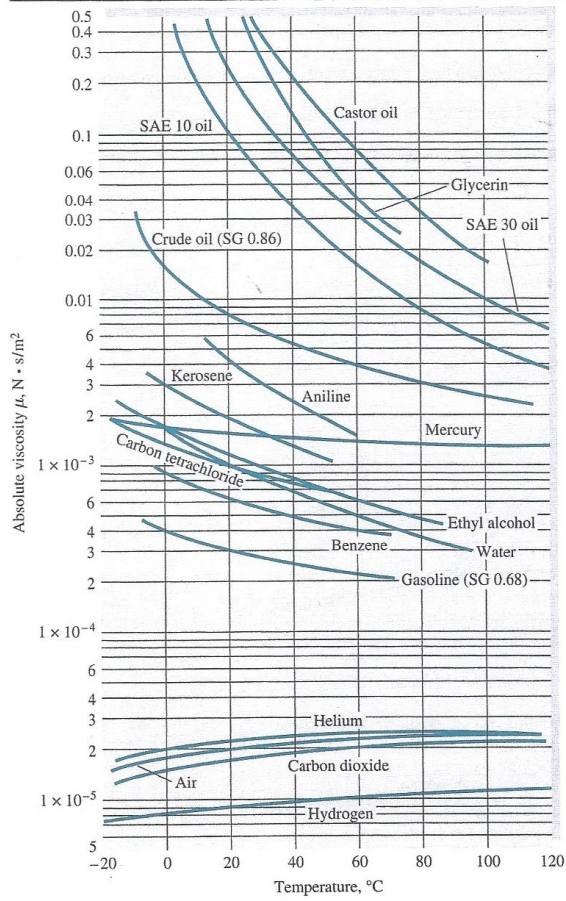
Appendix A
Physical Properties of Fluids

Fig. A.1 Absolute viscosity of common fluids at 1 atm.

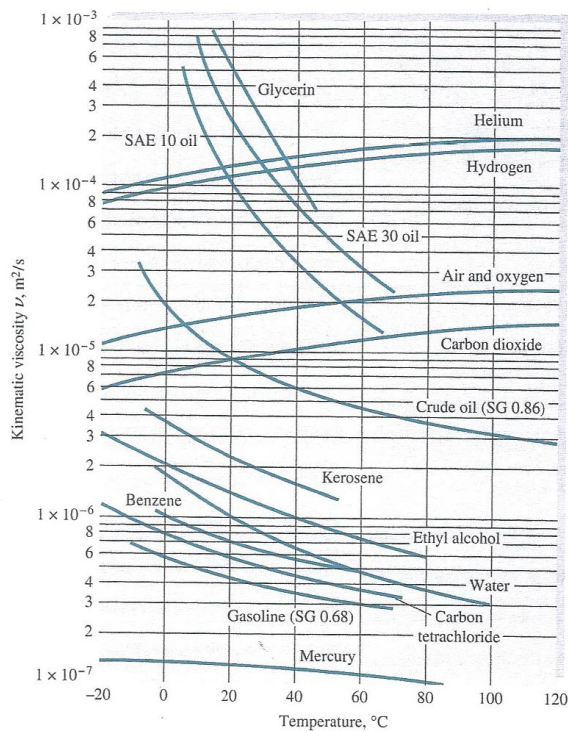**Fig. A.2** Kinematic viscosity of common fluids at 1 atm.

(3) Thermodynamic Properties

Classical Thermodynamics: the study of equilibrium states of matter, in which properties are assumed uniform in space and time.

Thermodynamic system = fixed mass separated from surroundings by boundary through which heat and work are exchanged (but not mass). Properties are state functions (i.e. depend on current state only and not path), whereas heat transfer and work are path functions.

A classical thermodynamic system is assumed static, whereas fluids are often in motion; however, if the relaxation time (time it takes material to adjust to a new state) is small compared to the time scale of fluid motion, an assumption is made that thermodynamic properties are point functions and that laws and state relations of static equilibrium thermodynamics are valid. In gases and liquids at normal pressure, relaxation time is very small; hence, only a few molecular collisions are needed for adjustment. Exceptions are rarefied gases, chemically reacting flows, sudden changes such as shock waves, etc.

For single-phase pure substances, only two properties are independent and all others follow through equations of state, which are determined experimentally or theoretically. Some mixtures, such as air, can also be considered a pure substance, whereas others such as salt water cannot and require additional numbers of independent properties, e.g., sea water requires three (salinity, T and p)

Pressure		p	$[N/m^2]$
Temperature	T		[K]
Density		ρ	$[kg/m^3]$
Internal Energy	û		$[Nm/kg] = [J/kg]$
Enthalpy		$h = û + p/\rho$	$[Nm/kg] = [J/kg]$

Entropy   s   [J/kg K]

$\rho = \rho(p,T)$ $\hat{u} = \hat{u}(p,T)$ $h = h(p,T)$ $s = s(p,T)$

Specific weight $\gamma = \rho g$ [N/m³]

$\rho_{air} = 1.205$ kg/m³ $\gamma_{air} = 11.8$ N/m³

$\rho_{water} = 1000$ kg/m³ $\gamma_{water} = 9790$ N/m³

$\rho_{mercury} = 13580$ kg/m³ $\gamma_{mercury} = 132{,}948$ N/m³

$$SG_{gas} = \frac{\rho_{gas}}{\rho_{air\,(20°C)}} = \frac{\rho_{gas}}{1.205\ kg/m^3}$$ $SG_{air} = 1$; $SG_{He} = 0.138$

$$SG_{liquid} = \frac{\rho_{liquid}}{\rho_{water(4°C)}} = \frac{\rho_{liquid}}{1000\,kg/m^3}$$ $SG_{water} = 1$; $SG_{Hg} = 13.6$

<u>Total stored energy per unit mass (e):</u>

$$e = \hat{u} + 1/2V^2 + gz$$

$\hat{u}$ = energy due to molecular activity and bonding forces (internal energy)

$1/2V^2$ = work required to change speed of mass from

0 to V per unit mass (kinematic energy)

gz = work required to move mass from 0 to

$$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k} \text{ against } \underline{g} = -g\hat{k} \text{ per unit mass}$$

$(-m\underline{g} \cdot \underline{r}/m)$ (potential energy)

(4) Miscellaneous Properties
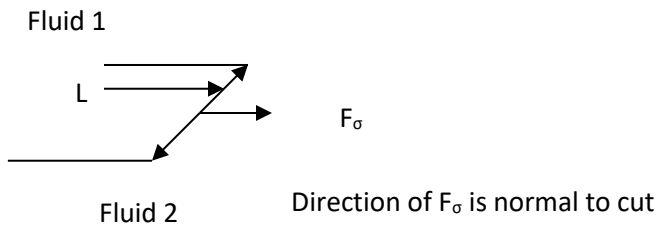
Surface Tension:

Two non-mixing liquids or liquids and gases form an interface across which there is a discontinuity in density. The interface behaves like a stretched membrane under tension. The tension originates due to strong intermolecular cohesive forces in the liquid that are unbalanced at the interface due to loss of neighbors, i.e., liquid molecules near the interface pull the molecules on the interface inward; resulting in contraction of the interface called surface tension per unit length.
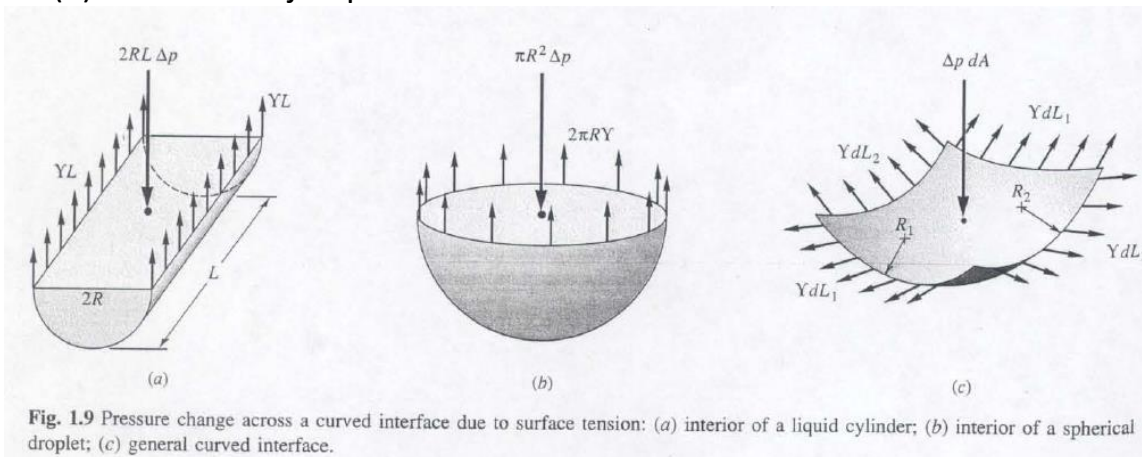
σ = coefficient of surface tension N/m

Line force = F_σ = σL where L = length of cut through interface

σ = f (two fluids, T)

Fluid 1

L

F_σ

Fluid 2   Direction of F_σ is normal to cut

Effects of surface tension (γ_f not considered):

    (1)   Pressure jumps across curved interfaces

Fig. 1.9 Pressure change across a curved interface due to surface tension: (a) interior of a liquid cylinder; (b) interior of a spherical droplet; (c) general curved interface.

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(a) Cylindrical interface

Force Balance:

$2\sigma L = 2(p_i - p_o)RL$

$\Delta p = \sigma/R$

$p_i > p_o$, i.e. pressure is larger on concave vs.
convex side of interface

(b) Spherical interface (droplet)

$2\pi R\sigma = \pi R^2 \Delta p$ → $\Delta p = 2\sigma/R$

(c) Bubble

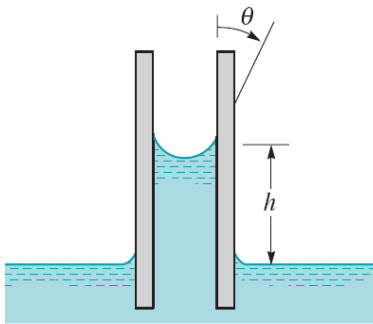$\pi R\sigma + 2\pi R\sigma = \pi R^2 \Delta p$ → $\Delta p = 4\sigma/R$

(d) General interface
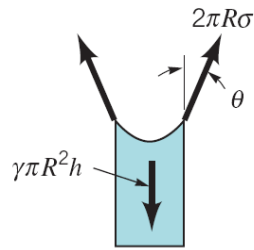
$\Delta p = \sigma(R_1^{-1} + R_2^{-1})$

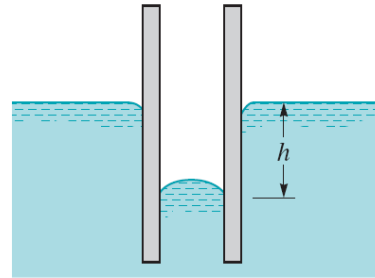$R_{1,2}$ = principle radii of curvature

(2) Contact Angle

When the surface of a solid intersects the interface the contact angle can either be wetting ($\theta < 90^0$) or non-wetting ($\theta > 90^0$). θ depends on both the two fluids and the solid surface properties. For clean glass intersecting an air-water interface $\theta=0$ (wetting) and an air-mercury interface $\theta=135$ (nonwetting).
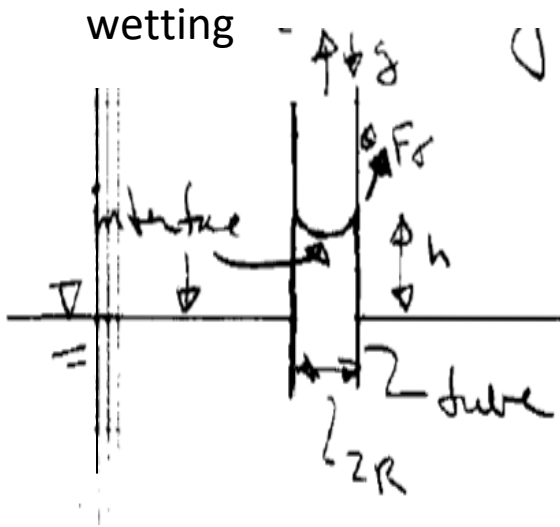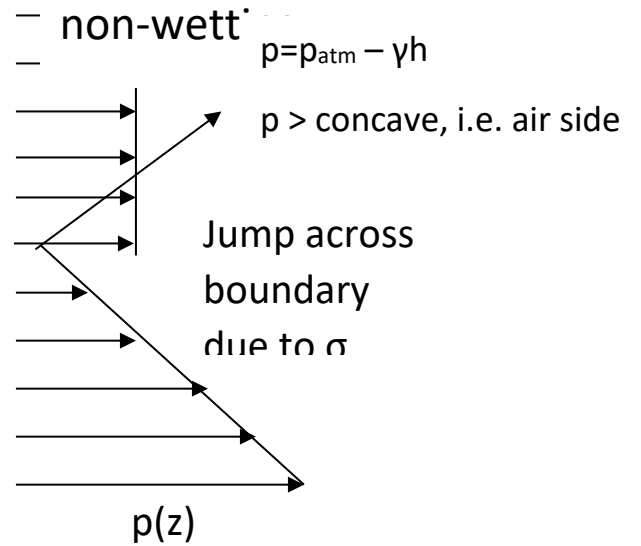
(a)   (b)   (c)

$2\pi R\sigma$

$\gamma\pi R^2 h$

θ

$2R$

h

Capilla $\theta < 90°$ $\theta > 90°$

wetting   non-wett·

$p = p_{atm} - \gamma h$

p > concave, i.e. air side

Jump across
boundary
due to σ

p(z)

Surface Tension Force = Weight of fluid

2πRσ cos θ = ρghπR²

$$h = \frac{2\sigma\cos\theta}{\gamma R}$$

h α R⁻¹ (i.e. larger h for smaller R)

Non-wetting drives liquid down tube

p_{atm}

p_{atm}

p_{atm}

p_{atm}

h

Pressure jump due to σ

p > concave, i.e. water side

$p = p_{atm} + \gamma h$

p(z)
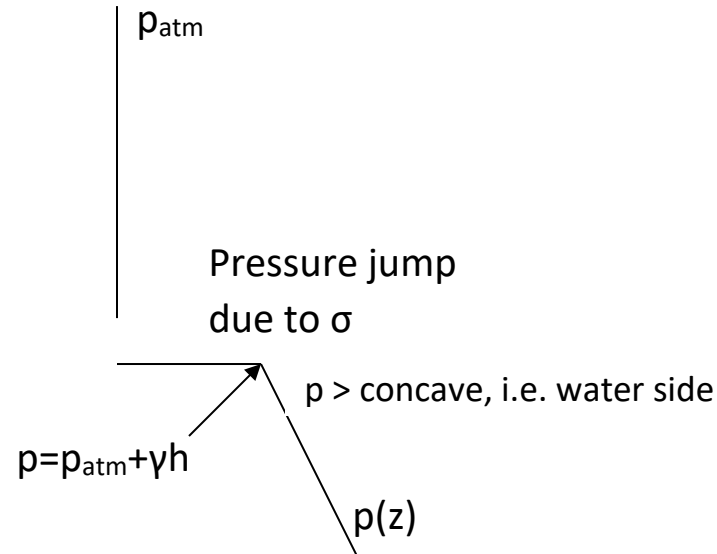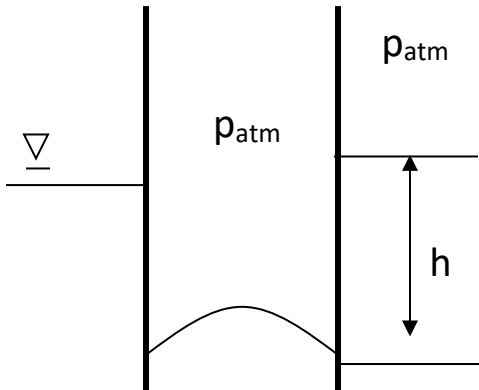
h > 0 = wetting, h < 0 = non-wetting

(b)   Parallel plates

For two parallel plates 2R apart with depth b:

Surface Tension Force = Weight of fluid

2bσ cos θ = ρgh2Rb $\Rightarrow h = \dfrac{\sigma \cos\theta}{\gamma R}$

(c)   Pressure jump

$$\frac{dp}{dz} = -\gamma \Rightarrow p = -\gamma z + c = -\gamma z + p_{atm} \Rightarrow \Delta p_{at\ z=h} = p_{z=h} - p_{atm}$$

$$= -\gamma h \quad \text{wetting}$$

$$= \gamma h \quad \text{non-wetting}$$

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For general interface:

h>0 (wetting):

$$\Delta p = \sigma(R_1^{-1} + R_2^{-1}) = -\gamma h < 0 \Rightarrow p_{water} < p_{air} \Rightarrow \text{concave}$$

shape

h<0 (non-wetting):

$$\Delta p = \sigma(R_1^{-1} + R_2^{-1}) = -\gamma h > 0 \Rightarrow p_{water} > p_{air} \Rightarrow \text{convex}$$

shape

(3) Transformation liquid jet into droplets

(4) Binding of wetted granular material such as sand

(5) Capillary waves

Like stretched membrane (string) waves, surface tension acts as restoring force resulting in interfacial waves called capillary waves.

Cavitation:

When the pressure in a liquid falls below the vapor pressure, it will evaporate (i.e. become a gas). If due to temperature changes alone, the process is called boiling, whereas if due to liquid velocity, the process is called cavitation.

$$Ca = \frac{p_a - p_v}{1/2 \rho U^2}$$

Ca = Cavitation #
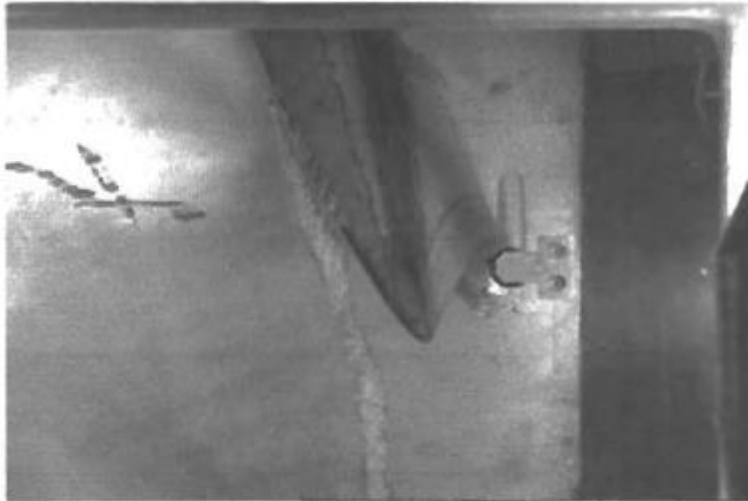
p_v = vapor pressure

p_a = ambient pressure

U = characteristic velocity

If the local pressure coefficient Cp ($Cp = \dfrac{p - p_a}{1/2\rho U^2}$) falls below the cavitation number Ca, the liquid will cavitate.
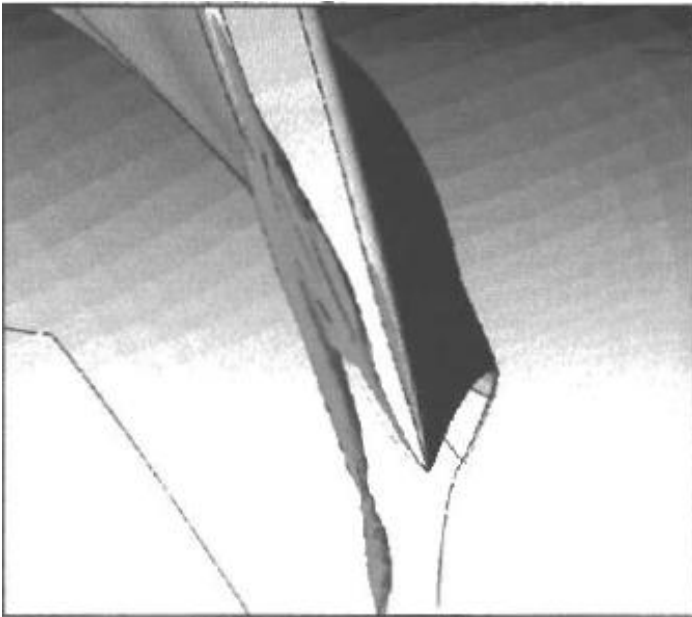
$$Ca = f \text{ (liquid/properties, T)}$$

Effects of cavitation:

    (1)   erosion
    (2)   vibration
    (3)   noise

(a) Experiment

(b) Computation
Cavitation comparison with experiment

Flow Classification:

    (1)   Spatial dimensions: 1D, 2D, 3D

    (2)   Steady or unsteady: $\dfrac{\partial}{\partial t} = 0$ or $\dfrac{\partial}{\partial t} \neq 0$

    (3)   Compressible ($\rho \neq$ constant) or incompressible (ρ = constant)

    (4)   Inviscid or Viscous: $\mu = 0$ or $\mu \neq 0$.

    (5)   Rotational or Irrotational: $\underline{\omega} \neq 0$ or $\underline{\omega} = 0$.

    (6)   Inviscid/Irrotational: potential flow

    (7)   Viscous, laminar or turbulent: Re_{trans}

    (8)   Viscous, low Re: Stokes flow

    (9)   Viscous, high Re external flow: boundary layer

    (10)   Etc.

Depending on flow classification, different approximations can be made to exact governing differential equations resulting in different forms of approximate equations and analysis techniques.

Flow Analysis Techniques:

"Reality"

Fluids Eng. Systems   Components   Idealized

EFD
$$U_D = \sqrt{B^2 + P^2}$$

Mathematical Physics Problem Formulation
$$U_M$$

AFD
$$U_M$$

CFD
$$U_S = \sqrt{U_{SM}^2 + U_{SN}^2}$$