7.3 Laminar Boundary Layer Similarity Solutions

Laminar Boundary Layer: $Re_{crit} = 91{,}000$; $Re_{trans} = 5 \times 10^5 - 3 \times 10^6$.

Similarity solutions (2D, steady, incompressible): method of reducing PDE to ODE by appropriate similarity transformation; also, because of transformation at least one coordinate lacks origin such that the solution collapses to same form at all length or time scales.

$$u_x + v_y = 0$$
$$uu_x + vu_y = UU_x + \nu u_{yy}$$

BCs: $u(x,0) = v(x,0) = 0$ no slip
$u(x,\infty) = U(x)$ matching outer flow
+ inlet condition

For Similarity $\frac{u(x,y)}{U(x)} = F\left(\frac{y}{g(x)}\right)$ expect $g(x)$ related to $\delta(x)$

Or in terms of stream function ψ : $u = \psi_y$ $v = -\psi_x$

For similarity $\psi = U(x)g(x)f(\eta)$ $\eta = y/g(x)$

$$u = \psi_y = Uf' \quad v = -\psi_x = -(U_x g f + U g_x f - U g_x \eta f')$$

BC:

$$u(x,0) = 0 \Rightarrow U(x)f'(0) = 0 \Rightarrow f'(0) = 0$$
$$v(x,0) = 0 \Rightarrow U_x(x)g(x)f(0) + U(x)g_x(x)f(0) - U(x)g_x(x)\times 0 \times f'(0) = 0$$
$$\Rightarrow \left(U_x(x)g(x) + U(x)g_x(x)\right)f(0) = 0$$
$$\Rightarrow f(0) = 0$$
$$u(x,\infty) = U(x) \Rightarrow U(x)f'(\infty) = U(x) \Rightarrow f'(\infty) = 1$$

Write boundary layer equations in terms of ψ

$$\psi_y\psi_{yx} - \psi_x\psi_{yy} = UU_x + \vartheta\psi_{yyy}$$

Substitute

$$\psi_{yy} = Uf''/g$$

$$\psi_{yyy} = Uf'''/g^2$$

$$\psi_{xy} = U_x f' - Uf''\eta g_x / g$$

Assemble them together:

$$(Uf')\left(U_x f' - \cancel{Uf''\eta\frac{g_x}{g}}\right) - \left(U_x g f + U g_x f - \cancel{U g_x \eta f'}\right)\left(Uf''/g\right) = UU_x + \nu\left(U f'''/g^2\right)$$

$$UU_x f'^2 - UU_x f f'' - (U^2 g_x/g) f f'' = UU_x + \nu\frac{U}{g^2} f'''$$

$$UU_x f'^2 - \frac{U}{g}(Ug)_x f f'' = UU_x + \nu\frac{U}{g^2} f'''$$

$$f''' + \underbrace{\frac{g}{\nu}(Ug)_x}_{C_1} f f'' + \underbrace{\frac{g^2}{\nu}U_x}_{C_2}\left(1 - f'^2\right) = 0$$

Where for similarity C_1 and C_2 are constant or function η only

- i.e. for a chosen pair of C_1 and C_2 → $U(x)$, $g(x)$ can be found, i.e., potential flow is NOT known a priori.
- Then solution of $f''' + C_1 ff'' + C_2\left(1 - f'^2\right) = 0$ gives $f(\eta)$ → $u(x, y)$, $\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_w = \frac{\mu U f''(0)}{g}$, δ, δ*, θ, H, C_f, C_D

The Blasius Solution for Flat-Plate Flow

U=constant→ $U_x = 0 \rightarrow C_2 = 0$

Then $C_1 = \frac{U}{\nu} g g_x \neq function(x)$

$\frac{d}{dx}(g^2) = \frac{2C_1\nu}{U}$ ⇒ $g(x) = [2C_1 \nu x/U]^{1/2}$

Let $C_1 = 1$, then $g(x) = \sqrt{\frac{2\nu x}{U}}$ ⇒ $\eta = y\sqrt{\frac{U}{2\nu x}} = \frac{y}{\sqrt{\frac{2\nu x}{U}}} \propto \frac{y}{\delta}$

Note $\frac{\delta}{x} = \frac{5}{\sqrt{Re_x}}$, i.e., $\delta = \frac{5x}{\sqrt{\frac{Ux}{\nu}}} = 5\sqrt{\frac{\nu x}{U}}$

$$\psi = U[2\nu x/U]^{1/2} f\left(y\sqrt{\frac{U}{2\nu x}}\right) = \sqrt{2\nu U x}\, f(\eta)$$

$$u = \psi_y = Uf'$$

$$v = -\psi_x = U g_x(\eta f' - f) = [U\nu/2x]^{1/2}(\eta f' - f) = \frac{U(\eta f' - f)}{\sqrt{2Re_x}}$$

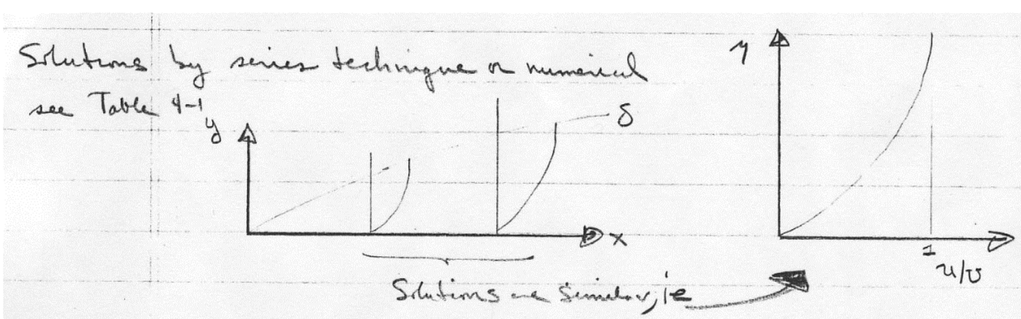
⇒

$$f''' + ff'' = 0$$

$$f(0) = f'(0) = 0,\ f'(\infty) = 1$$

Blasius equations for Flat Plate Boundary Layer

Solutions by series or numerical methods



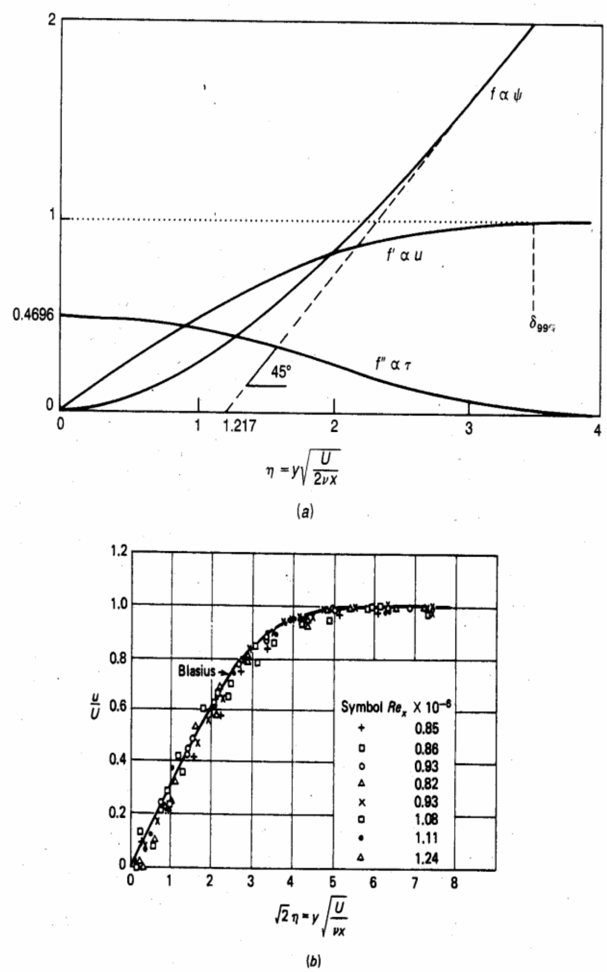


FIGURE 4-6
The Blasius solution for the flat-plate boundary layer: (*a*) numerical solution of Eq. (4-45); (*b*) comparison of $f' = u/U$ with experiments by Liepmann (1943).

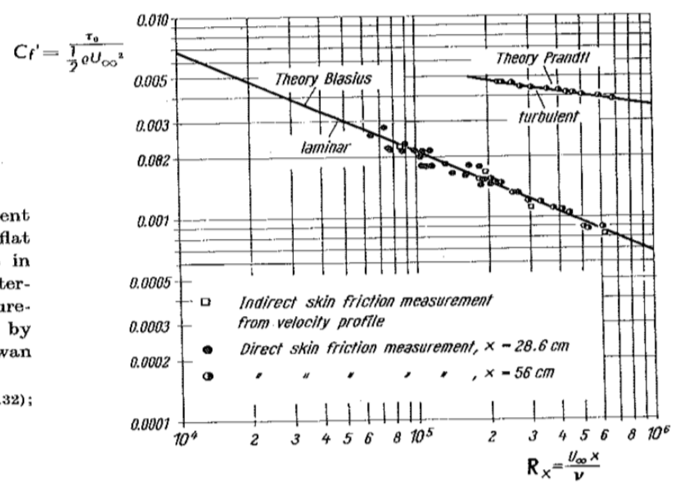


Fig. 7.10. Local coefficient of skin friction on a flat plate at zero incidence in incompressible flow, determined from direct measurement of shearing stress by Liepmann and Dhawan [6, 18]

Theory: laminar from eqn. (7.32); turbulent from eqn. (21.12)

TABLE 4-3

Highly resolved numerical solution of the Blasius equ[illegible]on for flow over a flat plate, Eq. (4-60)

η	$\eta/\eta_{99\%}$	$f(\eta)$	$f'(\eta)$	$f''(\eta)$	$f'''(\eta)$
0.0	0.00000	0.00000	0.00000	0.46960	0.00000
0.0	0.05761	0.00939	0.09391	0.46931	−0.00441
0.4	0.11521	0.03755	0.18761	0.46725	−0.01755
0.6	0.17282	0.08439	0.28058	0.46173	−0.03896
0.8	0.23042	0.14967	0.37196	0.45119	−0.06753
1.0	0.28803	0.23299	0.46063	0.43438	−0.10121
1.2	0.34563	0.33366	0.54525	0.41057	−0.13699
1.4	0.40324	0.45072	0.62439	0.37969	−0.17114
1.6	0.46084	0.58296	0.69670	0.34249	−0.19966
1.8	0.51845	0.72887	0.76106	0.30045	−0.21899
2.0	0.57606	0.88680	0.81669	0.25567	−0.22673
2.2	0.63366	1.05495	0.86330	0.21058	−0.22215
2.4	0.69127	1.23153	0.90107	0.16756	−0.20636
2.6	0.74887	1.41482	0.93060	0.12861	−0.18196
2.8	0.80648	1.60328	0.95288	0.09511	−0.15249
3.0	0.86408	1.79557	0.96905	0.06771	−0.12158
3.2	0.92169	1.99058	0.98036	0.04637	−0.09230
3.3	0.95049	2.08883	0.98456	0.03781	−0.07899
3.4	0.97929	2.18747	0.98797	0.03054	−0.06679
3.47†	1.00000	2.25856	0.99000	0.02603	−0.05878
3.5	1.00810	2.28641	0.99071	0.02441	−0.05582
3.6	1.03690	2.38559	0.99289	0.01933	−0.04611
3.8	1.09451	2.58450	0.99594	0.01176	−0.03039
4.0	1.15211	2.78389	0.99777	0.00687	−0.01914
4.2	1.20972	2.98356	0.99882	0.00386	−0.01152
4.4	1.26732	3.18338	0.99940	0.00208	−0.00663
4.6	1.32493	3.38330	0.99970	0.00108	−0.00366
4.8	1.38253	3.58325	0.99986	0.00054	−0.00193
5.0	1.44014	3.78323	0.99994	0.00026	−0.00098
5.2	1.49774	3.98323	0.99997	0.00012	−0.00047
5.4	1.55535	4.18322	0.99999	0.00005	−0.00022
5.6	1.61296	4.38322	1.00000	0.00002	−0.00010
5.8	1.67056	4.58322	1.00000	0.00001	−0.00004
6.0	1.72817	4.78322	1.00000	0.00000	−0.00002

†Actual value to 16 significant digits: $\eta_\delta = 3.471886880405967$.

$$\frac{u}{U} = 0.99 \text{ when } \eta = 3.5 \rightarrow \frac{\delta}{x} = \frac{5}{\sqrt{Re_x}} \qquad Re_x = \frac{Ux}{\nu}$$

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy = \int_0^\infty (1 - f')d\eta \sqrt{\frac{2\vartheta x}{U}} \rightarrow \boxed{\frac{\delta^*}{x} = \frac{1.7208}{\sqrt{Re_x}}}$$

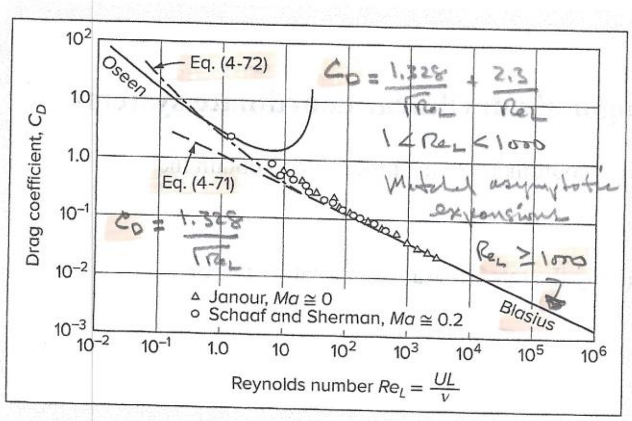
$$\theta = \int_0^\infty \left(1 - \frac{u}{U}\right)\frac{u}{U} dy = \int_0^\infty (1 - f')f' \sqrt{\frac{2\vartheta x}{U}} d\eta \rightarrow \boxed{\frac{\theta}{x} = \frac{0.664}{\sqrt{Re_x}}}$$

$$\boxed{\frac{\delta^*}{\theta} = H = shape\ parameter\ 2.59}$$

$$\tau_w = \mu \frac{\partial u}{\partial y}\Big|_w = \frac{\mu U f''(0)}{\sqrt{\frac{2\nu x}{U}}} \rightarrow \boxed{C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.664}{\sqrt{Re_x}} = \frac{\theta}{x}}$$

$$\boxed{C_D = \frac{D}{\frac{1}{2}\rho U^2 L} = \int_0^L C_f \frac{dx}{L} = \frac{1.328}{\sqrt{Re_L}}} \qquad Re_L = \frac{UL}{\nu}$$

$$\frac{v}{U} = \frac{\eta f' - f}{\sqrt{2Re_x}} \ll 1 \quad \text{for} \quad Re_x \gg 1$$



Stokes flow: Oseen Improvement, i.e., linearized convection

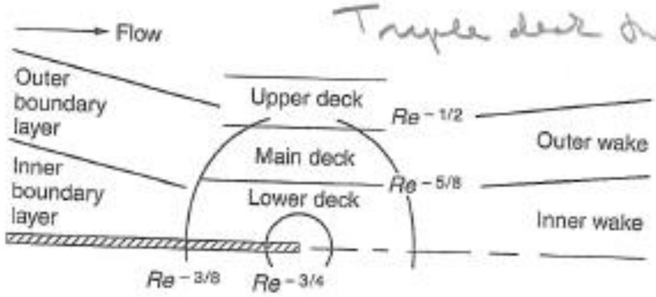
$$C_D = \frac{4\pi}{Re_L[1 - \Gamma + \ln(16/Re_L)]}$$

$\Gamma = .577216$ = Euler constant

$Re_L \ll 1$

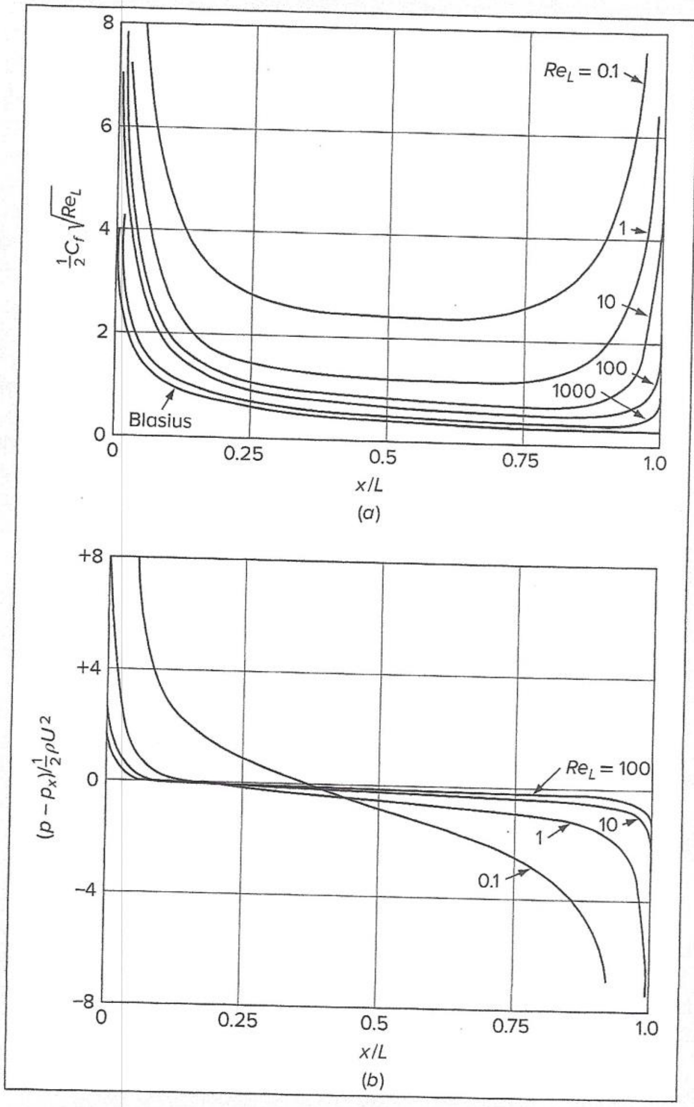
FIGURE 4-11
Theoretical and experimental drag of a flat plate.

Triple deck theory



$$C_D = \frac{1.328}{Re_L^{1/2}} + \frac{2.661}{Re_L^{7/8}} + O(Re_L^{-1})$$

FIGURE 4-40
Sketch of the triple-deck region at the trailing edge of a flat plate, merging into two-layer upstream and downstream regions. [*After Stewartson (1969) and Messiter (1970).*]



NS CFD $.1 < Re_L < 1000$
low Re large LE & TE effect
BL approximation $Re_L \geq 1000$

FIGURE 4-12
Numerical solution of the full Navier–Stokes equations for flat-plate flow at moderate Reynolds numbers: (*a*) local friction coefficient; (*b*) local surface pressure. [*After Dennis and Dunwoody (1966).*]

Falkner-Skan Wedge Flows

$$f''' + C_1 f f'' + C_2\left(1 - f'^2\right) = 0 \qquad f = f(\eta)$$

$$\eta = y/g(x)$$

$$u/U = f'(\eta)$$

$$f(0) = f'(0) = 0, \qquad f'(\infty) = 1$$

$$C_1 = \frac{g}{\upsilon}(Ug)_x \qquad C_2 = \frac{g^2}{\upsilon}U_x \qquad \text{(Blasius Solution: } C_2=0,\ C_1=1\text{)}$$

Similarity form of BL equations

Consider $(Ug^2)_x = 2Ugg_x + g^2U_x$

$$= 2Ugg_x + 2g^2U_x - g^2U_x = 2g(Ug_x + gU_x) - g^2U_x$$

$$= 2g(Ug)_x - g^2U_x$$

$$= 2\upsilon C_1 - \upsilon C_2$$

Hence $(Ug^2)_x = \upsilon(2C_1 - C_2)$

Choose $C_1=1$ and let $C_2 = C$

Integrate $Ug^2 = \upsilon(2 - C)x$ note $g^2 = \upsilon C/U_x$

Combine $\frac{U\upsilon C}{U_x} = \upsilon(2 - C)x$

Rearrange $\frac{\frac{dU}{dx}}{U} = \frac{C}{2-C}\frac{1}{x}$

Integrate $ln\, U = \frac{C}{2-C} ln\, x + \ln k$ where $\ln k$ = constant

$$ln\, U = ln\, x^{\frac{C}{2-C}} + ln\, k = \ln kx^{\frac{C}{2-C}}$$

$$U(x) = kx^{C/(2-C)}$$

$$g(x) = \left[\frac{\upsilon C}{U_x}\right]^{\frac{1}{2}} \text{ note } U_x = k\frac{C}{2-C}x^{\left(\frac{C}{2-C}-1\right)} = k\frac{C}{2-C}x^{\left(\frac{-2(1-C)}{2-C}\right)}$$

$$g(x) = \left[\frac{\upsilon C}{k\frac{C}{2-C}x^{\left(\frac{-2(1-C)}{2-C}\right)}}\right]^{\frac{1}{2}} = \left[\frac{\upsilon(2-C)}{k}x^{\left(\frac{-2(1-C)}{2-C}\right)}\right]^{\frac{1}{2}} = \sqrt{\frac{\upsilon(2-C)}{k}}x^{\frac{1-C}{2-C}}$$

using $a^{1/2}b^{1/2}=(ab)^{1/2}$ and $(a^m)^n=a^{mn}$

Alternatively, $U_x = \frac{C}{2-C}\,\mathrm{k}x^{(C/(2-C))}x^{-1} = \frac{C}{2-C}\,\mathrm{U}x^{-1}$

Such that $g(x) = \left[\frac{\upsilon C}{\frac{C}{2-C}\mathrm{U}x^{-1}}\right]^{\frac{1}{2}} = \left[\frac{\upsilon(2-C)x}{\mathrm{U}}\right]^{\frac{1}{2}}$

Change constant: $C = \beta = \frac{2m}{m+1}$ and $m = \frac{\beta}{2-\beta}$

$$U(x) = kx^m$$

$$\eta = \frac{y}{g} = y\sqrt{\frac{m+1}{2}\frac{U}{\nu x}}$$

$$f''' + ff'' + \beta\left(1 - f'^2\right) = 0$$

$$f(0) = f'(0) = 0 \text{ and } f'(\infty) = 1$$

Note:

$2\mathrm{m}/(\mathrm{m}+1) = \frac{2\beta}{2-\beta}/(\frac{\beta}{2-\beta}+1) = \frac{2\beta}{2-\beta}/(\frac{2}{(2-\beta)}) = \beta$

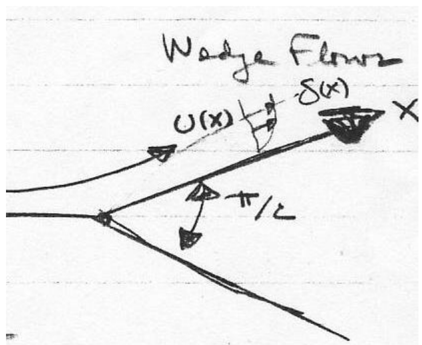
2-C = 2 -2m/(m+1) = 2/(m+1)

$g(x) = \left[\frac{\upsilon(2-C)x}{\mathrm{U}}\right]^{\frac{1}{2}} = \left[\frac{\upsilon 2x}{\mathrm{U(m+1)}}\right]^{\frac{1}{2}}$, i.e., $g(x)^{-1} = \left[\frac{m+1}{2}\frac{U}{\upsilon x}\right]^{\frac{1}{2}}$

Numerical solutions for $-0.19884 \le \beta \le 1.0$.

Separation ($\tau_w = 0$)

$$U(x) = kx^m$$



Solutions show many commonly observed characteristics of BL flow:

- The parameter β is a measure of the pressure gradient, dp/dx. For $\beta > 0$, $dp/dx < 0$and the pressure gradient is favorable. For $\beta < 0$, the $dp/dx > 0$ and the pressure gradient is adverse.
- Negative β solutions drop away from Blasius profiles as separation approached.
- Positive β solutions squeeze closer to wall due to flow acceleration.
- Accelerated flow: $\tau_{\max}$ near wall.
- Decelerated flow: $\tau_{\max}$ moves toward $\delta/2$

Invisid flow past wedges and corners

$U(x) = Kx^m$ exact solution potential flow past wedge or corner shapes

Plane polar coordinates $\nabla^2 \chi = 0$, $\nabla \times \underline{u} = 0$

$\frac{1}{r}\frac{\partial}{\partial r}(r\chi_r) + \frac{1}{r^2}\chi_{\theta\theta} = 0$ $v_r = \frac{1}{r}\chi_\theta$ $v_\theta = -\chi_r$

$\chi(r,\theta) = C r^{m+1} \sin[(m+1)\theta]$ $\beta = \frac{2m}{m+1}$

$m = \beta/(2-\beta)$

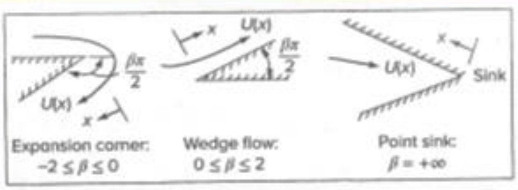
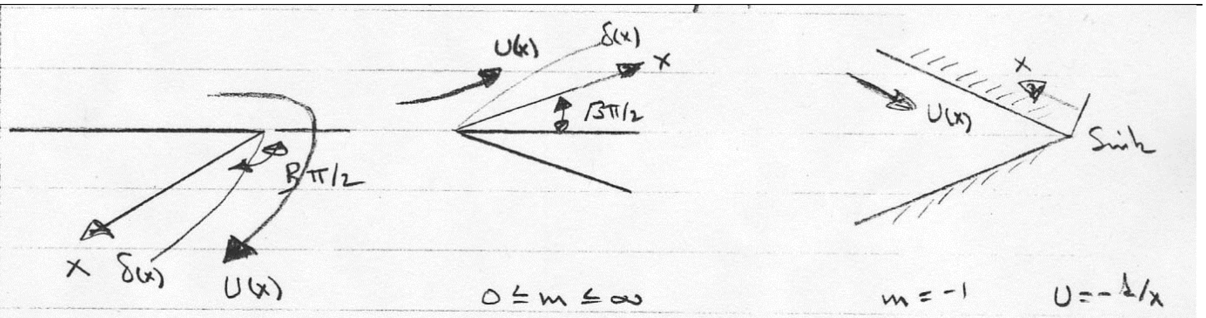


FIGURE 4-15
Some examples of Falkner–Skan potential flows.

This expression yields certain radial streamlines that can be interpreted as the "walls" of a wedge or a corner, as shown in Fig. 4-15, depending on the value of $\beta = 2m/(m+1)$. The velocity along these walls has the form $U = Kx^m$, which represents the freestream driving the boundary layer on the wall, with $x = 0$ at the tip of the wedge. The most prominent cases are:

- $-2 \le \beta \le 0$, $-\frac{1}{2} \le m \le 0$: flow around an expansion corner of turning angle $\beta\pi/2$
- $\beta = 0$, $m = 0$: the flat plate
- $0 \le \beta \le +2$, $0 \le m \le \infty$: flow against a wedge of half-angle $\beta\pi/2$
- $\beta = 1$, $m = 1$: the plane stagnation point (180° wedge)
- $\beta = +4$, $m = -2$: doublet flow near a plane wall
- $\beta = +5$, $m = -\frac{5}{3}$: doublet flow near a 90° corner
- $\beta = +\infty$, $m = -1$: flow toward a point sink [i.e., the boundary-layer version of the Jeffery–Hamel flow in a convergent wedge (Sec. 3-8)].

These are *similar* flows, i.e., for a given β, the velocity profiles all look alike when scaled by $U(x)$ and $\delta(x)$. They may also be used, with modest success, to predict the behavior of nonsimilar flows.



$-\frac{1}{2} \le m \le 0$

$-2 \le \beta \le 0$

Expansion Corner

$\beta = -.19884$ (Separation)

$0 \le m \le \infty$

$0 \le \beta \le 2$

Wedge Flow

$m = 0$, $\beta = 0$ } Flat Plate

$m = 1$, $\beta = 1$ } Stagnation Point

$m = -1$ $U = -A/x$

$\beta = \infty$ $\eta = \frac{y}{x}\sqrt{\frac{U}{\nu}}$

Sink : Simple Sol involving tanh

Sec. 4.3?

BL Sol of Jeffery-Hamel flow

Also, $\beta = 4$, $m = -2$: doublet flow near a plane wall

$\beta = 5$, $m = -5/3$: doublet near a 90° corner

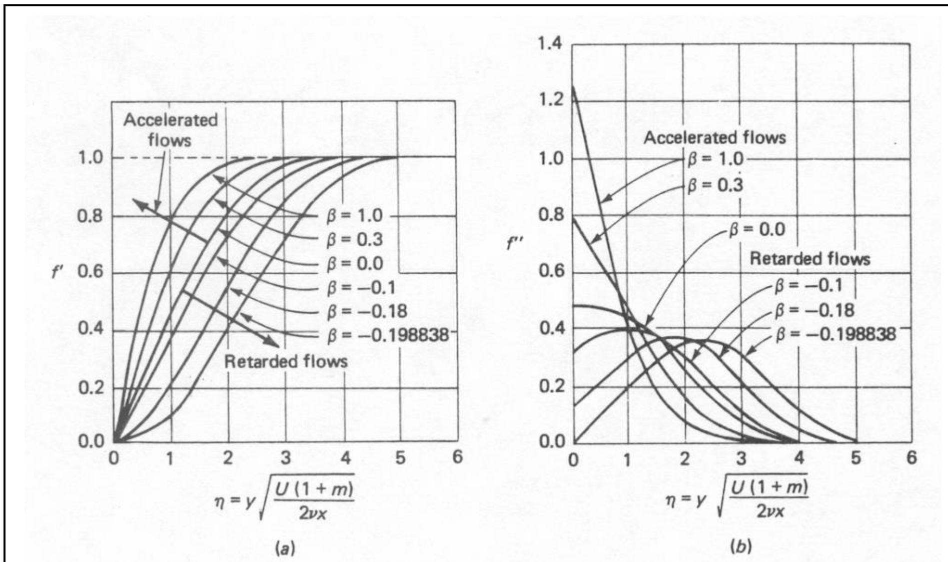


FIGURE 4-11
(*a*) Velocity profiles and (*b*) shear-stress profiles for the Falkner-Skan equation.

$u = U(x) f'$ where $U(x) = k x^m$
& velocity at edge of BL

$\beta = \frac{2m}{m+1}$ $\quad \eta = y/g(x) = y\left[\frac{m+1}{2}\frac{U(x)}{\nu x}\right]^{1/2}$

$\delta(x) = \left[\frac{2\nu}{(m+1)k}\right] x^{(1-m)/2}$ $\quad = y / \underbrace{\left[\frac{2}{m+1}\frac{\nu x}{k x^m}\right]^{1/2}}_{\delta(x)}$

$\delta^*(x) = \left(\frac{2}{m+1}\right)^{1/2}\left(\frac{\nu x}{U}\right)^{1/2}\int_0^\infty (1-f')\,d\eta$ $\quad \left(\frac{2\nu}{(m+1)k}\right) x^{(1-m)/2}$

$\tau_w(x) = \mu\left(\frac{m+1}{2}\right)^{1/2}\left(\frac{U^3}{\nu x}\right)^{1/2} f''(0)$ $\quad \mu\left(\frac{(m+1)k^3}{2\nu}\right) x^{(3m-1)/2}$

$\delta \propto x^{(1-m)/2}$ $\quad \delta^* \propto x^{(1-m)/2}$ $\quad \tau_w \propto x^{(3m-1)/2}$

Special Cases:

$U(x) =$ constant 1. Blasius: $m = 0$ $\quad \delta \propto x^{1/2}$ $\quad \delta^* \propto x^{1/2}$ $\quad \tau_w \propto x^{-1/2}$

$U(x) = kx$ 2. Stagnation Point: $m = 1$ $\quad \delta, \delta^* =$ constant $\quad \tau \propto x$
linearly increasing U thinning of BL exactly balanced by viscous diffusion such that $\delta, \delta^* =$ constant

$U = kx^3$ 3. $m = 3$: $\delta \propto x^{-1}$ $\quad \delta^* \propto x^{-1}$ $\quad \tau_w =$ constant
acceleration outer flow tendency increase τ_w exactly balanced by viscous diffusion spread momentum away from surface i.e. reduce τ_w