Chapter 7.2 Flat Plate CV Analysis and 3D BL Equations

Introduction:

Boundary layer flows: External flows around streamlined bodies at high Re such that viscous (no-slip and shear stress) effects confined close to the body surfaces and its wake but are nearly inviscid far from the body.

Applications of BL theory: *aerodynamics* (airplanes, rockets, projectiles), *hydrodynamics* (ships, submarines, torpedoes), *transportation* (automobiles, trucks, cycles), *wind engineering* (buildings, bridges, water towers), and *ocean engineering* (buoys, breakwaters, cables).

Historical perspective:

1. BL equations: 2D and axisymmetric similarity solutions

2. Momentum integral methods: success 2D and failure 3D due crossflow modeling

3. 3D BL differential codes

4. Separation: viscous/inviscid interaction and thick BL and partially parabolic equations

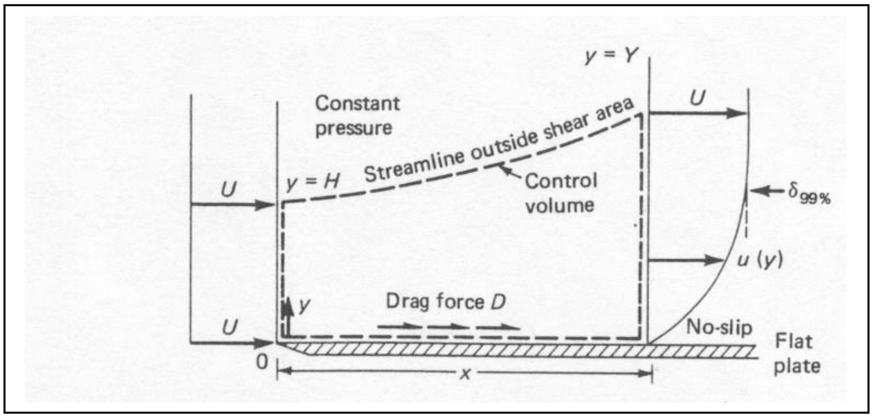
5. CFD: RANS, URANS, LES, Hybrid-RANS/LES, DNS

6. Multi fidelity, ML&AI

7. Fluid-structure interaction, multi-disciplinary

Flat-Plate Momentum Integral Analysis & Laminar approximate solution

Consider flow of a viscous fluid at high Re past a flat plate, i.e., flat plate fixed in a uniform stream of velocity $U\hat{i}$: 2D steady constant property flow, fixed CV, inlet U = constant, outlet u = u(y), no slip y = 0, no shear stress along outer streamline, i.e., at y = H at inlet and y = δ at outer boundary, thickness t = 0 such that p = constant.



Boundary-layer thickness arbitrarily defined by y = $\delta_{99\%}$ (where, $\delta_{99\%}$ is the value of y at u = 0.99U). Streamlines outside $\delta_{99\%}$ will deflect an amount δ^* (**the displacement thickness**). Thus, the streamlines move outward from $y = H$ at $x = 0$ to $y = Y = \delta = H + \delta^*$ at $x = x_1$.

Conservation of mass:

$$\int_{CS} \rho \underline{V} \bullet \underline{n} dA = \mathbf{0} = -\int_0^H \rho U b dy + \int_0^{H+\delta^*} \rho u b dy \qquad \text{b = span width}$$

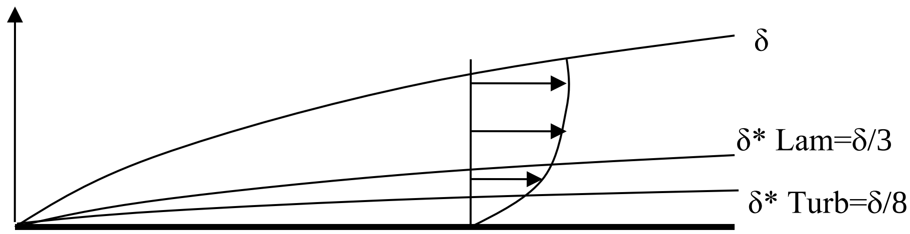
Which simplifies to:

$$UH = \int_0^Y u dy = \int_0^Y (U + u - U) dy = UY + \int_0^Y (u - U) dy$$

Substituting $Y = H + \delta^*$ results in the definition of displacement thickness:

$$\delta^* = \int_0^Y \left(1 - \frac{u}{U}\right) dy$$

δ^* which is only a function of x being an important measure of effect of BL on external flow. To see this more clearly, consider an alternate derivation based on an equivalent discharge/flow rate argument:



$$\underbrace{\int_{\delta^*}^{\delta} U dy}_{\text{Inviscid flow about } \delta^* \text{ body}} = \int_0^{\delta} u dy \qquad \text{Per unit span}$$

Flowrate between δ^* and δ of inviscid flow=actual flowrate, i.e., inviscid flow rate about displacement body = equivalent viscous flow rate about actual body

$$\int_0^{\delta} U dy - \int_0^{\delta^*} U dy = \int_0^{\delta} u dy \Rightarrow \delta^* = \int_0^{\delta}\left(1 - \frac{u}{U}\right) dy$$

w/o BL - displacement effect=actual discharge

For 3D flow, in addition it must also be explicitly required that δ^* is a stream surface of the inviscid flow continued from outside of the BL.

<u>Conservation of x-momentum</u>:

$$\sum F_x = -D = \int_{CS} \rho u \underline{V} \bullet \underline{n} dA = -\int_0^H \rho U(Ubdy) + \int_0^Y \rho u(ubdy)$$

$Drag = D = \rho U^2 Hb - \int_0^Y \rho u^2 \, bdy$ = Fluid force on plate = - Plate force on CV (fluid)

Using continuity: $H = \int_0^Y \frac{u}{U} dy$

$$D(x) = \rho b U^2 \int_0^Y u/U dy - \int_0^Y u^2\, b dy = b \int_0^x \tau_w\, dx$$

$$\frac{D}{\rho b U^2} = \theta = \int_0^{Y=\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

θ is the **momentum thickness (function of x only)**, an important measure of the drag.

$$\frac{dD}{dx} = b\tau_w = \rho b U^2 \frac{d\theta}{dx}$$

$$\tau_w = \rho U^2 \frac{d\theta}{dx}$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2}$$

$$\frac{C_f}{2} = \frac{d\theta}{dx}$$

Special case 2D momentum integral equation for dp/dx = 0

$$C_D = \frac{1}{L}\int_0^L C_f(x) dx = \frac{1}{L}\int_0^L 2\frac{d\theta}{dx} dx = \frac{2}{L}\theta(L)$$

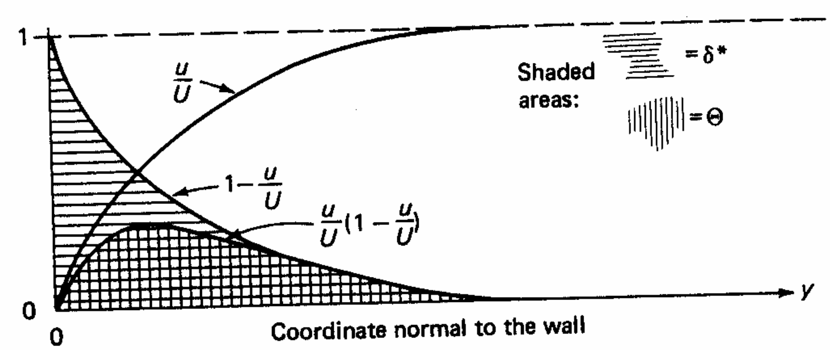


FIGURE 4-2
Momentum and displacement thicknesses.

Simple example solution momentum integral method flat plate BL with assumed velocity profile

Assume polynomial profile in y with degree determined number boundary conditions which can be satisfied

$$\frac{u}{U} = a_0 + a_1 y/\delta + a_2 y^2/\delta^2 \qquad u(x,0)=0 \Rightarrow a_0 = 0 \quad (1)$$

$$= 2y/\delta - (y/\delta)^2 \qquad u(x,\delta)=U \Rightarrow 1 = a_0 + a_1 + a_2 \quad (2)$$

$$\frac{\partial u}{\partial y}(x,\delta) = 0 \Rightarrow 0 = a_1 + 2a_2 \quad (3)$$

$u_y = \left(\frac{2}{\delta} - \frac{2y}{\delta^2}\right)U$

$u_{yy} = (-2/\delta^2)U$

$$\tau_w = \rho U^2 \frac{d\theta}{dx}$$

$$\theta = \int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy = \int_0^\delta \left[\frac{2y}{\delta} - \left(\frac{y}{\delta}\right)^2\right]\left[1 - \frac{2y}{\delta} + \left(\frac{y}{\delta}\right)^2\right]dy$$

$$= \delta \int_0^1 (2\eta - \eta^2)(1 - 2\eta + \eta^2)\, d\eta \qquad \eta = y/\delta \quad d\eta = \delta^{-1} dy$$

$$= 2/15\, \delta$$

$$\tau_w = \mu \left.\frac{du}{dy}\right|_{y=0} = \mu U \frac{\partial}{\partial y}\left[\frac{2y}{\delta} - (y/\delta)^2\right]_{y=0} = \frac{\mu U}{\delta}\frac{\partial}{\partial \eta}(2\eta - \eta^2)_{\eta=0}$$

$$= 2\mu U/\delta$$

$$= \rho U^2 \frac{d}{dx}\left(\frac{2}{15}\delta\right)$$

$$\frac{2\rho U^2}{15}\frac{d\delta}{dx} = \frac{2\mu U}{\delta} \Rightarrow \delta\, d\delta = 15\frac{\nu}{U}dx \Rightarrow \delta = \sqrt{30}\sqrt{\frac{\nu x}{U}}$$

$$\delta/x = 5.48\, Re_x^{-1/2}$$

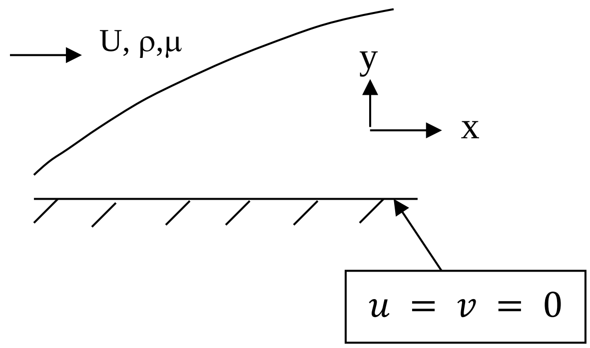
$$\tau_w / \tfrac{1}{2}\rho U^2 = c_f = .73\, Re_x^{-1/2} = \theta/x \qquad Re_x = Ux/\nu$$

$$\delta^*/x = 1.83\, Re_x^{-1/2} \qquad C_D = 1.46\, Re_L^{-1/2} = 2c_f(L)$$

10% error Blasius all $\propto Re_x^{-1/2}$

If $u/U = \frac{2y}{\delta}$ = linear only (1) & (2) can be satisfied & poorer results, whereas if 3rd or higher polynomial more accurate results. 2nd gives $u_{yy}(x,\delta) = -2U/\delta^2 \neq 0$ ie incorrect curvature

Boundary layer approximations, equations, and comments



2D NS, ρ=constant, neglect g (subscript indicates derivative)

$$u_x + v_y = 0$$
$$u_t + uu_x + vu_y = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu(u_{xx} + u_{yy})$$
$$v_t + uv_x + vv_y = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu(v_{xx} + v_{yy})$$

Introduce non-dimensional variables that includes scales such that all variables are of order magnitude O(1):

$$x^* = x / L$$
$$y^* = \frac{y}{L}\sqrt{\mathrm{Re}}$$
$$t^* = tU / L$$
$$u^* = u / U$$
$$v^* = \frac{\nu}{U}\sqrt{Re}$$
$$p^* = \frac{p - p_0}{\rho U^2}$$
$$\mathrm{Re} = UL / \nu$$

The NS equations become (drop *)

$$u_x + v_y = 0$$

$$u_t + uu_x + vu_y = -p_x + \underline{\frac{1}{Re}u_{xx}} + u_{yy}$$

$$\frac{1}{Re}\left(v_t + uv_x + vv_y\right) = -p_y + \underline{\frac{1}{Re^2}v_{xx}} + \underline{\frac{1}{Re}v_{yy}}$$

For large Re (BL assumptions) the underlined terms drop out and the BL equations are obtained.

Therefore, y-momentum equation reduces to

$$p_y = 0$$

$$i.e., p = p(x,t)$$

$$\Rightarrow p_x = -\rho(U_t + UU_x)$$

External flow:
unsteady Euler equation or
steady Bernoulli equation
$p + \frac{1}{2}\rho U^2 = B$
$p_x = -\rho UU_x$

2D BL equations:

$$u_x + v_y = 0$$

$$u_t + uu_x + vu_y = (U_t + UU_x) + \nu u_{yy}$$

Note at y=0: $\frac{\partial p}{\partial x} = \nu u_{yy}$

$$x^* = x/L \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial x^*}\frac{\partial x^*}{\partial x} = L^{-1}\frac{\partial}{\partial x^*}$$

$$y^* = y/L\,\sqrt{Re} \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial y^*}\frac{\partial y^*}{\partial y} = L^{-1}\sqrt{Re}\frac{\partial}{\partial y^*}$$

$$t^* = t\,U/L \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial t^*}\frac{\partial t^*}{\partial t} = L^{-1}U\frac{\partial}{\partial t^*}$$

$$u^* = u/U \qquad u = u^*U$$

$$v^* = v/U\,\sqrt{Re} \qquad v = v^*U/\sqrt{Re}$$

$$p^* = p - p_0/\rho U^2 \qquad \Delta p = \rho U^2 p^*$$

$$Re = UL/\nu$$

$$u_x + v_y = 0 \qquad \frac{U}{L}\frac{\partial u^*}{\partial x^*} + \frac{\sqrt{Re}}{L}\frac{U}{\sqrt{Re}}\frac{\partial v^*}{\partial y^*} = 0$$

$$UL^{-1}\left(u^*_{x^*} + v^*_{y^*}\right) = 0$$

$$u_t + uu_x + vu_y = -p_x/\rho + \nu\left(u_{xx} + u_{yy}\right)$$

$$\frac{U^2}{L}\frac{\partial u^*}{\partial t^*} + u^*U\frac{U}{L}\frac{\partial u^*}{\partial x^*} + \frac{v^*U}{\sqrt{Re}}\frac{\sqrt{Re}}{L}\frac{\partial}{\partial y^*}u^*U = L^{-1}\frac{\partial}{\partial x^*}\left(U^2p^*\right)$$

$$U^2/L\left[\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right] = \qquad + \nu\left(L^{-2}\frac{\partial^2(u^*U)}{\partial x^{*2}} + L^{-2}Re\frac{\partial^2(u^*U)}{\partial y^{*2}}\right)$$

$$\frac{\rho U^2}{\rho L}\frac{\partial p^*}{\partial x^*} + \nu\left(\frac{U}{L^2}\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{U}{L^2}Re\frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = \frac{\partial p^*}{\partial x^*} + Re^{-1}\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}} \qquad \frac{U\nu}{L^2}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{UL}{\nu}\frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

$$v_t + uv_x + vv_y = -p_y/\rho + \nu\left(v_{xx} + v_{yy}\right) \qquad \frac{\nu}{UL}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{UL}{\nu}\frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

$$L^{-1}U\frac{\partial}{\partial t^*}v^*U/\sqrt{Re} + \frac{u^*U}{L}\frac{\partial}{\partial x^*}\frac{v^*U}{\sqrt{Re}} + \frac{v^*U}{\sqrt{Re}}\frac{\sqrt{Re}}{L}\frac{\partial}{\partial y^*}\frac{v^*U}{\sqrt{Re}} = \frac{\sqrt{Re}}{\rho L}\frac{\partial}{\partial y^*}\rho U^2p^*$$

$$+ \nu\left(L^{-2}\frac{\partial^2}{\partial x^{*2}}\frac{v^*U}{\sqrt{Re}} + L^{-2}Re\frac{\partial^2}{\partial y^{*2}}\frac{v^*U}{\sqrt{Re}}\right)$$

$$\frac{U^2}{L\sqrt{Re}}\left[\frac{\partial v^*}{\partial t^*} + u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*}\right] = \frac{\sqrt{Re}\,\rho U^2}{L\,\rho}\frac{\partial p^*}{\partial y^*} + \nu\left(\frac{U}{L^2\sqrt{Re}}\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{U}{L^2}\sqrt{Re}\frac{\partial^2 v^*}{\partial y^{*2}}\right)$$

$$\underbrace{\frac{\nu L}{\sqrt{Re}\,U^2}\left(\frac{U}{L^2\sqrt{Re}}\right)}_{Re^{-2}} \qquad \underbrace{\frac{U}{L^2}\sqrt{Re}}_{Re^{-1}}$$

$$Re^{-1}\left[\quad\right] = \frac{\partial p^*}{\partial y^*} + Re^{-2}\frac{\partial^2 v^*}{\partial x^{*2}} + Re^{-1}\frac{\partial^2 v^*}{\partial y^{*2}}$$

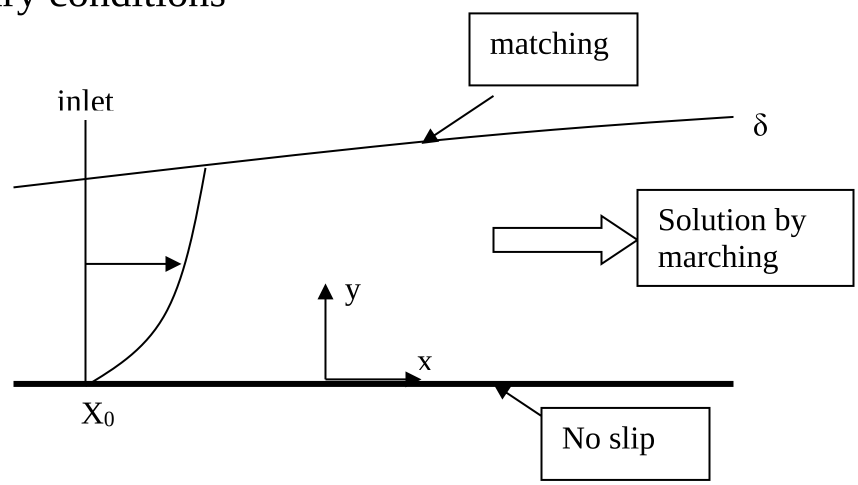
Note:

(1) U(x,t) and $\frac{\partial p(x,t)}{\partial x}$ impressed on BL by the external flow.

(2) $\frac{\partial^2}{\partial x^2} = 0$: i.e. longitudinal (stream-wise) diffusion is neglected.

(3) Due to (2), the equations are parabolic in x. Physically, this means all downstream influences are lost other than that contained in external flow. A marching solution is possible.

(4) Boundary conditions

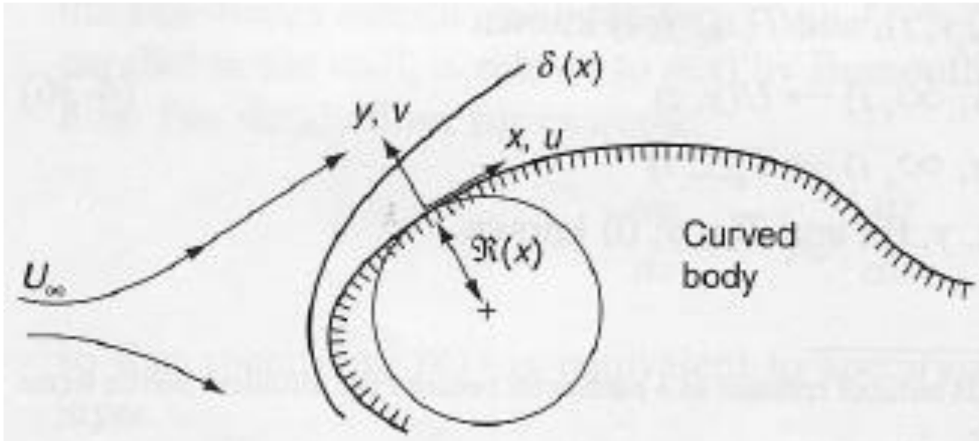


No slip: $u(x,0,t) = v(x,0,t) = 0$
Initial condition: $u(x,y,0)$ known.
Inlet condition: $u(x_0,y,t)$ given at x_0
Matching with outer flow: $u(x,\infty,t) = U(x,t)$

(5) When applying the boundary layer equations, one must keep in mind the restrictions imposed on them due to the basic BL assumptions.

→ not applicable for thick BL or separated flows (although they can be used to estimate occurrence of separation).

(6) Curvilinear coordinates



Although BL equations have been written in Cartesian Coordinates, they apply to curved surfaces provided δ << R and x, y are curvilinear coordinates measured along and normal to the surface, respectively. In such a system under the BL assumptions:

$$p_y = \frac{\rho u^2}{R}$$

Assume *u* is a linear function of *y*: $u = Uy/\delta$

$$\frac{dp}{dy} = \frac{\rho U^2 y^2}{R\delta^2}$$

$$p(\delta) - p(0) \propto \frac{\rho U^2 \delta}{3R}$$

Or

$\frac{\Delta p}{\rho U^2} \propto \frac{\delta}{3R}$; therefore, we require $\delta << R$.

(7) Practical use of the BL theory

For a given body geometry:

(a) Inviscid theory gives p(x) → integration gives Lift and Drag = 0.

(b) BL theory gives → $\delta^*(x)$, $\tau_w(x)$, $\theta(x)$, etc. and predicts separation if any.

(c) If separation present then no further information → must use inviscid models, BL equation in inverse mode, or NS equations.

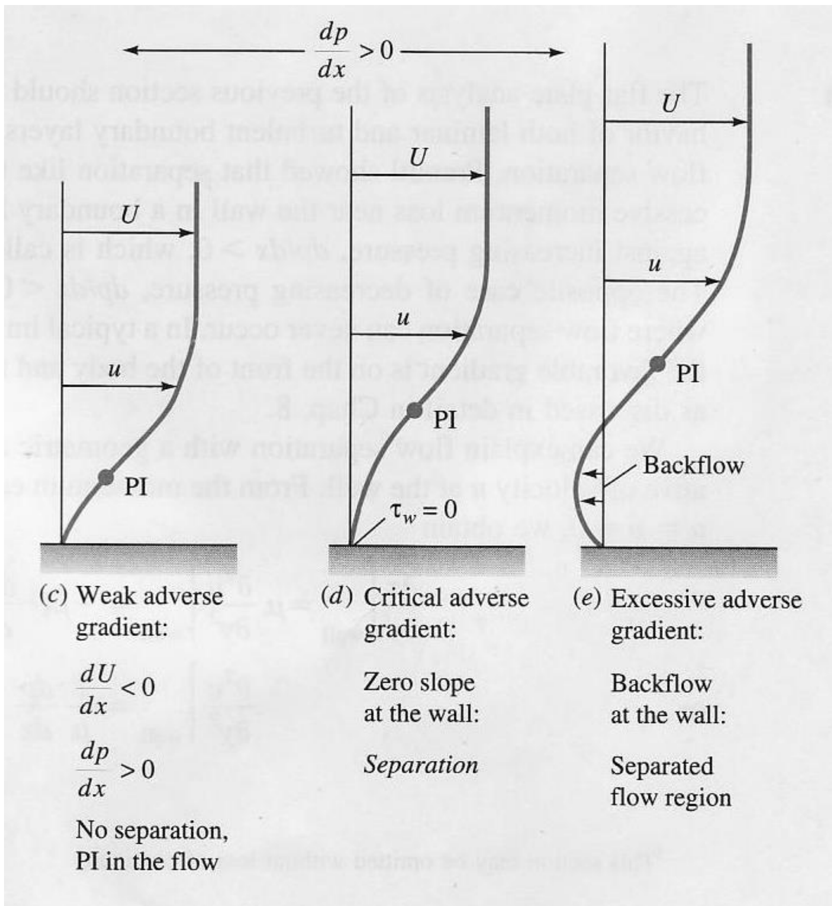
(d) If separation is absent, integration of $\tau_w(x)$ provides frictional resistance; displacement body (including δ^*) inviscid theory gives new p(x); and for displacement body drag go back to (2) for more accurate BL calculation including viscous – inviscid interaction.

(8) Separation and shear stress

At the wall,

$$u = v = 0 \rightarrow u_{yy} = \frac{1}{\mu} p_x \quad 2^{nd} \text{ derivative u depends on } p_x$$

1^{st} derivative u gives $\tau_w \rightarrow \tau_w = \mu u_y\big|_w$ $\tau_w = 0$ separation



Bernoulli: $p_x = - \rho U U_x$

Adverse pressure gradient $p_x > 0$ and $U_x < 0$:

H = shape parameter = $\frac{\delta^*}{\theta}$ depends shape velocity profile provides indicator for separation = 3.5 laminar = 2.4 turbulent flow