Appendix D
Equations of Motion in Cylindrical Coordinates

The equations of motion of an incompressible newtonian fluid with constant μ, k, and c_p are given here in cylindrical coordinates (r, θ, z), which are related to cartesian coordinates (x, y, z) as in Fig. 4.2:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z \tag{D.1}$$

The velocity components are υ_r, υ_θ, and υ_z. Here are the equations:
Continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\upsilon_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(\upsilon_\theta) + \frac{\partial}{\partial z}(\upsilon_z) = 0 \tag{D.2}$$

Convective time derivative:

$$\mathbf{V}\cdot\nabla = \upsilon_r\frac{\partial}{\partial r} + \frac{1}{r}\upsilon_\theta\frac{\partial}{\partial\theta} + \upsilon_z\frac{\partial}{\partial z} \tag{D.3}$$

Laplacian operator:

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2} \tag{D.4}$$

The r-momentum equation:

$$\frac{\partial\upsilon_r}{\partial t} + (\mathbf{V}\cdot\nabla)\upsilon_r - \frac{1}{r}\upsilon_\theta^2 = -\frac{1}{\rho}\frac{\partial p}{\partial r} + g_r + \nu\left(\nabla^2\upsilon_r - \frac{\upsilon_r}{r^2} - \frac{2}{r^2}\frac{\partial\upsilon_\theta}{\partial\theta}\right) \tag{D.5}$$

The θ-momentum equation:

$$\frac{\partial\upsilon_\theta}{\partial t} + (\mathbf{V}\cdot\nabla)\upsilon_\theta + \frac{1}{r}\upsilon_r\upsilon_\theta = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + g_\theta + \nu\left(\nabla^2\upsilon_\theta - \frac{\upsilon_\theta}{r^2} + \frac{2}{r^2}\frac{\partial\upsilon_r}{\partial\theta}\right) \tag{D.6}$$

The z-momentum equation:

$$\frac{\partial v_z}{\partial t} + (\mathbf{V} \cdot \nabla) v_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g_z + \nu \nabla^2 v_z \tag{D.7}$$

The energy equation:

$$\rho c_p \left[\frac{\partial T}{\partial t} + (\mathbf{V} \cdot \nabla) T \right] = k \nabla^2 T + \mu[2(\varepsilon_{rr}^2 + \varepsilon_{\theta\theta}^2 + \varepsilon_{zz}^2) + \varepsilon_{\theta z}^2 + \varepsilon_{rz}^2 + \varepsilon_{r\theta}^2] \tag{D.8}$$

where

$$\varepsilon_{rr} = \frac{\partial v_r}{\partial r} \qquad \varepsilon_{\theta\theta} = \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right)$$
$$\varepsilon_{zz} = \frac{\partial v_z}{\partial z} \qquad \varepsilon_{\theta z} = \frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \tag{D.9}$$
$$\varepsilon_{rz} = \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \qquad \varepsilon_{r\theta} = \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right) + \frac{\partial v_\theta}{\partial r}$$

Viscous stress components:

$$\tau_{rr} = 2\mu\varepsilon_{rr} \qquad \tau_{\theta\theta} = 2\mu\varepsilon_{\theta\theta} \qquad \tau_{zz} = 2\mu\varepsilon_{zz}$$
$$\tau_{r\theta} = \mu\varepsilon_{r\theta} \qquad \tau_{\theta z} = \mu\varepsilon_{\theta z} \qquad \tau_{rz} = \mu\varepsilon_{rz} \tag{D.10}$$

Angular velocity components:

$$2\omega_r = \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}$$
$$2\omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \tag{D.11}$$
$$2\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$