Chapter 7 Dimensional Analysis and Modeling

The Need for Dimensional Analysis

Dimensional analysis is a process of formulating fluid mechanics problems in terms of nondimensional variables and parameters.

1. Reduction in Variables:

	F = functional form
If $F(A_1, A_2, \ldots, A_n) = 0$,	A_i = dimensional variables
Then $f(\Pi_1, \Pi_2, \ldots \Pi_{r<n}) = 0$	Π_j = nondimensional parameters
Thereby reduces number of experiments and/or simulations required to determine f vs. F	$= \Pi_j (A_i)$ i.e., Π_j consists of nondimensional groupings of A_i's

2. Helps in understanding physics

3. Useful in data analysis and modeling

4. Fundamental to concept of similarity and model testing

   Enables scaling for different physical dimensions and fluid properties

Dimensions and Equations

Basic dimensions: F, L, and t or M, L, and t
F and M related by $F = Ma = MLT^{-2}$

Buckingham Π Theorem

In a physical problem including n dimensional variables in which there are m dimensions, the variables can be arranged into $r = n - \hat{m}$ independent nondimensional parameters Π_r (where usually $\hat{m} = m$).

$F(A_1, A_2, \ldots, A_n) = 0$

$f(\Pi_1, \Pi_2, \ldots \Pi_r) = 0$

A_i's = dimensional variables required to formulate problem ($i = 1, n$)

Π_j's = nondimensional parameters consisting of groupings of A_i's ($j = 1, r$)

F, f represents functional relationships between A_n's and Π_r's, respectively

$\hat{m}$ = rank of dimensional matrix
= m (i.e., number of dimensions) usually

Dimensional Analysis

Methods for determining Π_i's

1. Functional Relationship Method

Identify functional relationships $F(A_i)$ and $f(\Pi_j)$by first determining A_i's and then evaluating Π_j's

a. Inspection — intuition
b. Step-by-step Method — text
c. Exponent Method — class

2. Nondimensionalize governing differential equations and initial and boundary conditions

Select appropriate quantities for nondimensionalizing the GDE, IC, and BC e.g. for M, L, and t

Put GDE, IC, and BC in nondimensional form

Identify Π_j's

Exponent Method for Determining Π_j's

1) determine the n essential quantities
2) select $\hat{m}$ of the A quantities, with different dimensions, that contain among them the $\hat{m}$ dimensions, and use them as repeating variables together with one of the other A quantities to determine each Π.

For example let A_1, A_2, and A_3 contain M, L, and t (not necessarily in each one, but collectively); then the Π_j parameters are formed as follows:

$$\left.\begin{aligned}
\Pi_1 &= A_1^{x_1} A_2^{y_1} A_3^{z_1} A_4 \\
\Pi_2 &= A_1^{x_2} A_2^{y_2} A_3^{z_2} A_5 \\
\Pi_{n-m} &= A_1^{x_{n-m}} A_2^{y_{n-m}} A_3^{z_{n-m}} A_n
\end{aligned}\right\}$$

Determine exponents such that Π_i's are dimensionless

3 equations and 3 unknowns for each Π_i

In these equations the exponents are determined so that each Π is dimensionless. This is accomplished by substituting the dimensions for each of the A_i in the equations and equating the sum of the exponents of M, L, and t each to zero. This produces three equations in three unknowns (x, y, t) for each Π parameter.

In using the above method, the designation of $\hat{m} = m$ as the number of basic dimensions needed to express the n variables dimensionally is not always correct. The correct value for $\hat{m}$ is the rank of the dimensional matrix, i.e., the next smaller square subgroup with a nonzero determinant.

Dimensional matrix =

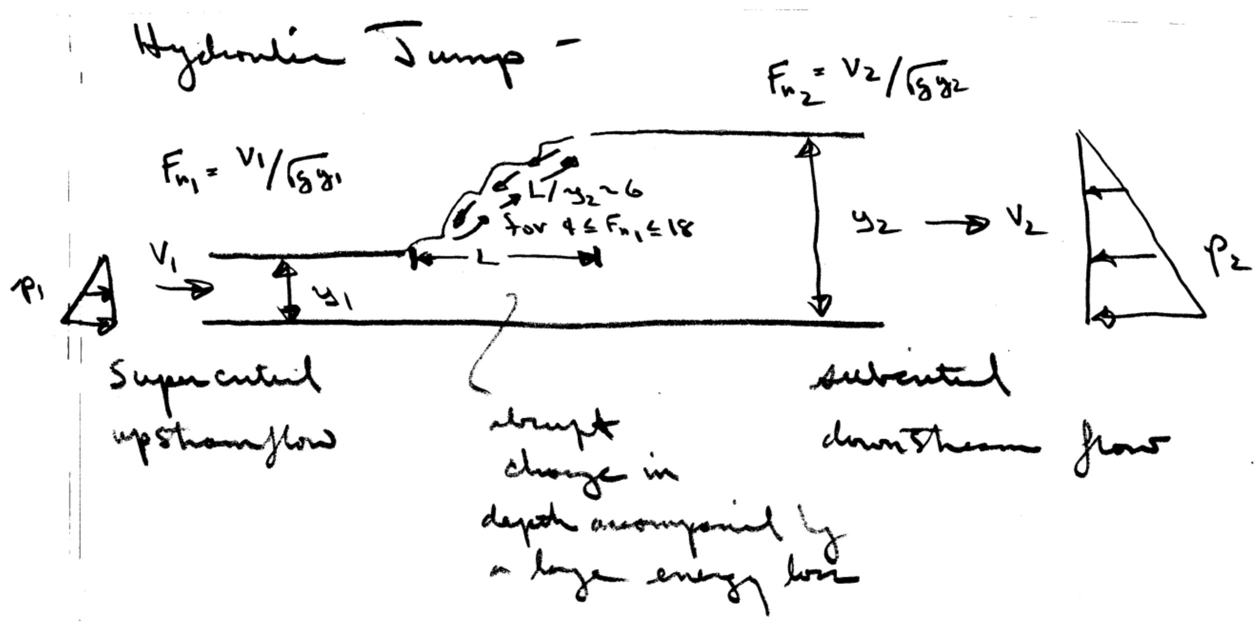
$$
\begin{array}{cc}
 & \begin{array}{ccc} A_1 & \cdots\cdots & A_n \end{array} \\
\begin{array}{c} M \\ L \\ t \\ \\ \\ \\ \\ \\ \end{array} &
\begin{bmatrix}
a_{11} & \cdots\cdots & a_{1n} \\
 & & \\
a_{31} & \cdots\cdots & a_{3n} \\
o & \cdots\cdots & o \\
\vdots & & \vdots \\
\vdots & & \vdots \\
\vdots & & \vdots \\
o & \cdots\cdots & o
\end{bmatrix}
\end{array}
$$

a_{ij} = exponent of M, L, or t in A_i

n x n matrix

Rank of dimensional matrix equals size of next smaller sub-group with nonzero determinant

Example: Hydraulic jump (see section 15.2)



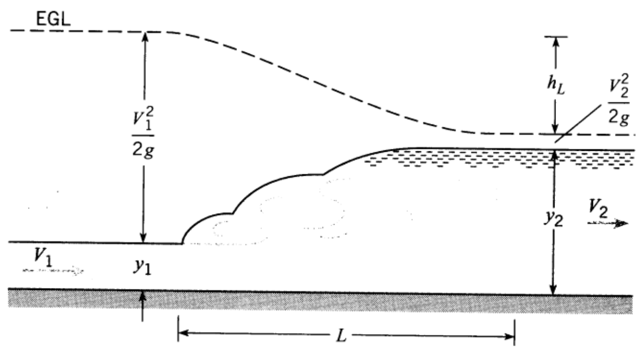


FIGURE 15.17
Definition sketch for the hydraulic jump.

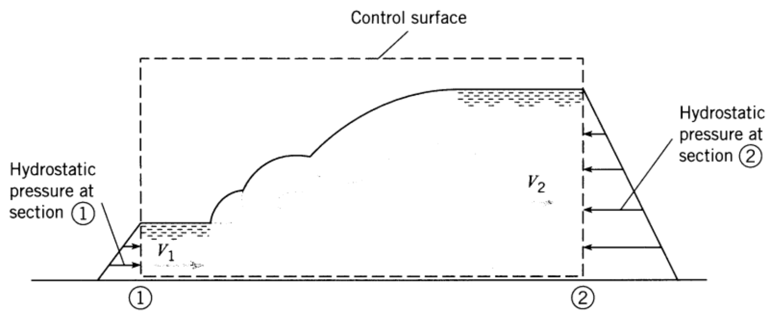
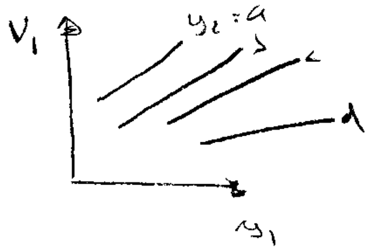


FIGURE 15.18
Control-volume analysis for the hydraulic jump.

Say we assume that

$$V_1 = V_1(\rho, g, \mu, y_1, y_2) \quad \text{or } V_2 = V_1 y_1/y_2$$

Dimensional analysis is a procedure whereby the functional relationship can be expressed in terms of r nondimensional parameters in which r < n = number of variables. Such a reduction is significant since in an experimental or numerical investigation a reduced number of experiments or calculations is extremely beneficial



1) ρ, g fixed; vary μ
2) ρ, μ fixed; vary g
3) μ, g fixed; vary ρ

Represents many, many experiments

In general: $F(A_1, A_2, \ldots, A_n) = 0$ dimensional form

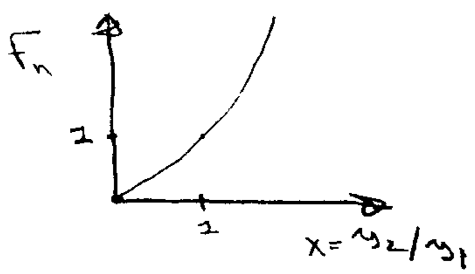
$f(\Pi_1, \Pi_2, \ldots \Pi_r) = 0$ nondimensional form with reduced # of variables

or $\Pi_1 = \Pi_1 (\Pi_2, \ldots, \Pi_r)$

It can be shown that

$$F_r = \frac{V_1}{\sqrt{gy_1}} = F_r\left(\frac{y_2}{y_1}\right)$$

neglect μ (ρ drops out as will be shown)
thus only need one experiment to determine the functional relationship



$$F_r^2 = \frac{1}{2}\left(x - x^2\right)$$

$$F_r = \left[\frac{1}{2}x(1+x)\right]^{1/2}$$

x	F_r
0	0
½	.61
1	1
2	1.7
5	3.9

For this particular application we can determine the functional relationship through the use of a control volume analysis: (neglecting μ and bottom friction)
x-momentum equation: $\sum F_x = \sum V_x \rho \underline{V} \cdot \underline{A}$

$$\gamma\frac{y_1^2}{2} - \gamma\frac{y_2^2}{2} = V_1\rho\left(-V_1 y_1\right) + V_2\rho\left(V_2 y_2\right)$$

$$\frac{\gamma}{2}\left(y_1^2 - y_2^2\right) = \frac{\gamma}{g}\left(V_2^2 y_2 - V_1^2 y_1\right)$$

Note: each term in equation must have some units: principle of dimensional homogeneity, i.e., in this case, force per unit width N/m

continuity equation: $V_1 y_1 = V_2 y_2$

$$V_2 = \frac{V_1 y_1}{y_2}$$

$$\underbrace{\frac{\gamma y_1^2}{2}\left[1 - \left(\frac{y_2}{y_1}\right)^2\right]}_{\text{pressure forces due to gravity}} = \underbrace{V_1^2\frac{\gamma}{g}y_1\left(\frac{y_1}{y_2} - 1\right)}_{\text{inertial forces}}$$

pressure forces = inertial forces
due to gravity

now divide equation by $\dfrac{\left(1 - \dfrac{y_2}{y_1}\right)y_1^3}{g y_2}$

$$\frac{V_1^2}{g y_1} = \frac{1}{2}\frac{y_2}{y_1}\left(1 + \frac{y_2}{y_1}\right) \longleftarrow \text{dimensionless equation}$$

ratio of inertia forces/gravity forces = (Froude number)^2

note: $F_r = F_r(y_2/y_1)$ do not need to know both y_2 and y_1, only ratio to get F_r

Also, shows in an experiment it is not necessary to vary γ, y_1, y_2, V_1, and V_2, but only F_r and y_2/y_1

Next, can get an estimate of h_L from the energy equation (along free surface from $1 \rightarrow 2$)

$$\frac{V_1^2}{2g} + y_1 = \frac{V_2^2}{2g} + y_2 + h_L$$

$$h_L = \frac{(y_2 - y_1)^3}{4y_1 y_2}$$

$\neq f(\mu)$ due to assumptions made in deriving 1-D steady flow energy equations

Exponent method to determine Π_j's for Hydraulic jump

$F(g, V_1, y_1, y_2, \rho, \mu) = 0$ $\qquad n = 6$

$$\frac{L}{T^2} \quad \frac{L}{T} \quad L \quad L \quad \frac{M}{L^3} \quad \frac{M}{LT}$$

use V_1, y_1, ρ as repeating variables

Assume $\hat{m}$ = m to avoid evaluating rank of 6 x 6 dimensional matrix

$m = 3 \;\Rightarrow\; r = n - m = 3$

$\Pi_1 = V_1^{x1}\, y_1^{y1}\, \rho^{z1}\, \mu$
$\quad = (LT^{-1})^{x1}\, (L)^{y1}\, (ML^{-3})^{z1}\, ML^{-1}T^{-1}$

L	$x_1 + y_1 - 3z_1 - 1 = 0$	$y_1 = 3z_1 + 1 - x_1 = -1$
T	$-x_1 \qquad\qquad - 1 = 0$	$x_1 = -1$
M	$z_1 \qquad\qquad + 1 = 0$	$z_1 = -1$

$$\Pi_1 = \frac{\mu}{\rho y_1 V_1} \quad \text{or} \quad \Pi_1^{-1} = \frac{\rho y_1 V_1}{\mu} = \text{Reynolds number} = \text{Re}$$

$\Pi_2 = V_1^{x2}\, y_1^{y2}\, \rho^{z2}\, g$
$\quad = (LT^{-1})^{x2}\, (L)^{y2}\, (ML^{-3})^{z2}\, LT^{-2}$

L $\quad x_2 + y_2 - 3z_2 + 1 = 0 \qquad y_2 = -1 - x_2 = 1$
T $\quad -x_2 \qquad\qquad - 2 = 0 \qquad x_2 = -2$
M $\quad z_2 = 0$

$$\Pi_2 = V_1^{-2} y_1 g = \frac{g y_1}{V_1^2} \qquad \Pi_2^{-1/2} = \frac{V_1}{\sqrt{g y_1}} = \text{Froude number} = \text{Fr}$$

$\Pi_3 = V_1^{x3}\, y_1^{y3}\, \rho^{z3}\, y_2$
$\quad = (LT^{-1})^{x3}\, (L)^{y3}\, (ML^{-3})^{z3}\, L$

L $\quad x_3 + y_3 - 3z_3 + 1 = 0 \qquad y_3 = -1$
T $\quad -x_3 = 0$
M $\quad -3z_3 = 0$

$$\Pi_3 = \frac{y_2}{y_1} \qquad \Pi_3^{-1} = \frac{y_1}{y_2} = \text{depth ratio}$$

$f(\Pi_1, \Pi_2, \Pi_3) = 0$
or, $\quad \Pi_2 = \Pi_2(\Pi_1, \Pi_3)$
i.e., $\quad F_r = F_r(Re, y_2/y_1)$

if we neglect μ then Re drops out

$$F_r = \frac{V_1}{\sqrt{g y_1}} = f\left(\frac{y_2}{y_1}\right)$$

Note that dimensional analysis does not provide the actual functional relationship. Recall that previously we used control volume analysis to derive

$$\frac{V_1^2}{gy_1} = \frac{1}{2}\frac{y_2}{y_1}\left(1 + \frac{y_2}{y_1}\right)$$

the actual relationship between F vs. y_2/y_1

$$F = F(Re, F_r, y_1/y_2)$$

or $F_r = F_r(Re, y_1/y_2)$

dimensional matrix:

$$\begin{array}{c} \\ M \\ L \\ t \\ \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccc} g & V_1 & y_1 & y_2 & \rho & \mu \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 3 & -1 \\ -2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Size of next smaller subgroup with nonzero determinant = 3 = rank of matrix

Common Dimensionless Parameters for Fluid Flow Problems

Most common physical quantities of importance in fluid flow problems are: (without heat transfer)

1	2	3	4	5	6	7	8
V,	ρ,	g,	μ,	σ,	K,	Δp,	L
velocity	density	gravity	viscosity	surface tension	compressibility	pressure change	length

$$n = 8 \qquad m = 3 \qquad \Rightarrow \quad 5 \text{ dimensionless parameters}$$

$F_i = Ma = \rho L^3 \frac{V^2}{L}$
$f_i = \rho V^2/L$

1) Reynolds number $= \dfrac{\rho V L}{\mu} = \dfrac{\text{inertia forces}}{\text{viscous forces}} \qquad \dfrac{\rho V^2/L}{\mu V/L^2} \qquad$ Re

$F_v = \tau L^2 = \nu V L$
$f_v = \mu V/L^2$

R_{crit} distinguishes among flow regions: laminar or turbulent value varies depending upon flow situation

2) Froude number $= \dfrac{V}{\sqrt{gL}} = \dfrac{\text{inertia forces}}{\text{gravity force}} \qquad \dfrac{\rho V^2/L}{\gamma} \qquad$ Fr

$F_g = \gamma L^3$
$f_g = \gamma$

important parameter in free-surface flows

3) Weber number $= \dfrac{\rho V^2 L}{\sigma} = \dfrac{\text{inertia force}}{\text{surface tension force}} \qquad \dfrac{\rho V^2/L}{\sigma/L^2} \qquad$ We

$F_\sigma = \sigma L$
$f_\sigma = \sigma/L^2$

important parameter at gas-liquid or liquid-liquid interfaces and when these surfaces are in contact with a boundary

4) Mach number $= \dfrac{V}{\sqrt{k/\rho}} = \dfrac{V}{a} = \sqrt{\dfrac{\text{inertia force}}{\text{compressibility force}}} \qquad$ Ma

speed of sound in liquid

$F_c = \rho a^2 L^2$
$f_\sigma = \rho a^2/L$

Paramount importance in high speed flow ($V \geq c$)

5) Pressure Coefficient $= \dfrac{\Delta p}{\rho V^2} = \dfrac{\text{pressure force}}{\text{inertia force}} \qquad \dfrac{\Delta p/L}{\rho V^2/L} \qquad$ C_p

(Euler Number)

$F_p = \Delta p L^2$
$f_p = \Delta p/L$

Nondimensionalization of the Basic Equation

It is very useful and instructive to nondimensionalize the basic equations and boundary conditions. Consider the situation for ρ and μ constant and for flow with a free surface

Continuity: $\nabla \cdot \underline{V} = 0$

Momentum: $\rho \frac{D\underline{V}}{Dt} = -\nabla(p + \gamma z) + \mu \nabla^2 \underline{V}$

ρg = specific weight

Boundary Conditions:

1) fixed solid surface: $\underline{V} = 0$
2) inlet or outlet: $\underline{V} = \underline{V}_o$ $p = p_o$
3) free surface: $w = \frac{\partial \eta}{\partial t}$ $p = p_a - \gamma\left(R_x^{-1} + R_y^{-1}\right)$

(z = η) surface tension

All variables are now nondimensionalized in terms of ρ and

U = reference velocity

L = reference length

$$\underline{V}^* = \frac{\underline{V}}{U} \qquad t^* = \frac{tU}{L}$$

$$\underline{x}^* = \frac{\underline{x}}{L} \qquad p^* = \frac{p + \rho g z}{\rho U^2}$$

All equations can be put in nondimensional form by making the substitution

$$\underline{V} = \underline{V}^* U$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t^*}\frac{\partial t^*}{\partial t} = \frac{U}{L}\frac{\partial}{\partial t^*}$$

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$= \frac{\partial}{\partial x^*}\frac{\partial x^*}{\partial x}\hat{i} + \frac{\partial}{\partial y^*}\frac{\partial y^*}{\partial y}\hat{j} + \frac{\partial}{\partial z^*}\frac{\partial z^*}{\partial z}\hat{k}$$

$$= \frac{1}{L}\nabla^*$$

and $\frac{\partial u}{\partial x} = \frac{1}{L}\frac{\partial}{x^*}\left(Uu^*\right) = \frac{U}{L}\frac{\partial u^*}{\partial x^*}$ etc.

Result: $\nabla^* \cdot \underline{V}^* = 0$

$$\frac{D\underline{V}^*}{Dt} = -\nabla^* p^* + \underbrace{\frac{\mu}{\rho V L}}_{Re^{-1}} \nabla^{*2}\underline{V}^*$$

1) $\underline{V}^* = 0$

2) $\underline{V}^* = \frac{\underline{V}_o}{U}$ $\quad p* = \frac{p_o}{\rho V^2}$

3) $w^* = \frac{\partial \eta^*}{\partial t^*}$ $\quad p^* = \frac{p_o}{\rho V^2} + \frac{gL}{U^2} z^* + \frac{\gamma}{\rho V^2 L}\left(R_x^{*-1} + R_y^{*-1}\right)$

pressure coefficient

Fr^-2 → Fr^{-2} We → We^{-1} $V = U$

Similarity and Model Testing

Flow conditions for a model test are completely similar if all relevant dimensionless parameters have the same corresponding values for model and prototype

$$\Pi_{i\ model} = \Pi_{i\ prototype} \qquad i = 1, r = n - \hat{m}\ (m)$$

Enables extrapolation from model to full scale

However, complete similarity usually not possible

Therefore, often it is necessary to use Re, or Fr, or Ma scaling, i.e., select most important Π and accommodate others as best possible

Types of Similarity:

1) Geometric Similarity (similar length scales):
   A model and prototype are geometrically similar if and only if all body dimensions in all three coordinates have the same linear-scale ratios

$$\alpha = L_m/L_p \qquad (\alpha < 1)$$

1/10 or 1/50

2) Kinematic Similarity (similar length and time scales):
   The motions of two systems are kinematically similar if homologous (same relative position) particles lie at homologous points at homologous times

3) Dynamic Similarity (similar length, time and force (or mass) scales):
in addition to the requirements for kinematic similarity the model and prototype forces must be in a constant ratio

Model Testing in Water (with a free surface)

$$F(D, L, V, g, \rho, \nu) = 0$$

$\Rightarrow$ n = 6 and m = 3 thus r = n – m = 3 pi terms

In a dimensionless form,

$$f(C_D, Fr, Re) = 0$$

or $C_D = f(Fr, Re)$

where

$$C_D = \frac{D}{\frac{1}{2}\rho V^2 L^2}$$

$$Fr = \frac{V}{\sqrt{gL}}$$

$$Re = \frac{VL}{\nu}$$

If $Fr_m = Fr_p$ or $\frac{V_m}{\sqrt{gL_m}} = \frac{V_p}{\sqrt{gL_p}}$

$V_m = \sqrt{\alpha} V_p$ Froude scaling and $Re_m = Re_p$ or $\frac{V_m L_m}{\nu_m} = \frac{V_p L_p}{\nu_p}$

$$\frac{\nu_m}{\nu_p} = \frac{V_m L_m}{V_p L_p} = \alpha^{3/2}$$

Then,

$$C_{D_m} = C_{D_p} \text{ or } \frac{D_m}{\rho_m V_m^2 L_m^2} = \frac{D_p}{\rho_p V_p^2 L_p^2}$$

However, impossible to achieve, since

if $\alpha = 1/10$, $\nu_m = 3.1\times10^{-8}\, m^2/s < 1.2\times10^{-7}\, m^2/s$

For mercury $\nu = 1.2\times10^{-7}\, m^2/s$

Alternatively one could maintain Re similarity and obtain

$$V_m = V_p/\alpha$$

But, if $\alpha = 1/10$, $V_m = 10V_p$,

High speed testing is difficult and expensive.

$$\frac{V_m^2}{g_m L_m} = \frac{V_p^2}{g_p L_p}$$

$$\frac{g_m}{g_p} = \frac{V_m^2}{V_p^2}\frac{L_p}{L_m}$$

$$\frac{g_m}{g_p} = \frac{V_m^2}{V_p^2}\frac{L_p}{L_m}$$

$$\frac{g_m}{g_p} = \frac{1}{\alpha^2} \times \frac{1}{\alpha} = \alpha^{-3}$$

$$g_m = \frac{g_p}{\alpha^3}$$

But if $\alpha = 1/10$, $g_m = 1000 g_p$
Impossible to achieve

<u>Model Testing in Air</u>

$$F(D, L, V, \rho, \nu, a) = 0$$

$\Rightarrow$ n = 6 and m = 3 thus r = n – m = 3 pi terms

In a dimensionless form,

$$f(CD, Re, Ma) = 0$$

or

$$C_D = f(Re, Ma)$$

where

$$C_D = \frac{D}{\frac{1}{2}\rho V^2 L^2}$$

$$Re = \frac{VL}{\nu}$$

$$Ma = \frac{V}{a}$$

If $$\frac{V_m L_m}{\nu_m} = \frac{V_p L_p}{\nu_p}$$

and $$\frac{V_m}{a_m} = \frac{V_p}{a_p}$$

Then,

$$C_{D_m} = C_{D_p} \text{ or } \frac{D_m}{\rho_m V_m^2 L_m^2} = \frac{D_p}{\rho_p V_p^2 L_p^2}$$

However, $$\frac{\nu_m}{\nu_p} = \frac{L_m}{L_p}\left[\overset{1}{\frac{a_m}{a_p}}\right] = \alpha$$

again not possible

Therefore, in wind tunnel testing Re scaling is also violated

Model Studies w/o free surface

$$c_p = \Delta p \Big/ \frac{1}{2}\rho V^2$$

High Re

Model Studies with free surface

See text

In hydraulics model studies, Fr scaling used, but lack of We similarity can cause problems. Therefore, often models are distorted, i.e. vertical scale is increased by 10 or more compared to horizontal scale

Ship model testing:

$C_T = f(Re, F_r) = C_w(F_r) + C_v(Re)$

V_m determined for F_r scaling

$C_{wm} = C_{Tm} - C_v$

$C_{Ts} = C_{wm} + C_v$

Based on flat plate of same surface area