Chapter 6 Differential Analysis of Fluid Flow

Inviscid flow: Euler's equations of motion

Flow fields in which the shearing stresses are zero are said to be inviscid, nonviscous, or frictionless. for fluids in which there are no shearing stresses the normal stress at a point is independent of direction:

$$-p = \sigma_{xx} = \sigma_{yy} = \sigma_{zz}$$

For an inviscid flow in which all the shearing stresses are zero, and the normal stresses are replaced by −p, the Navier-Stokes Equations reduce to Euler's equations

$$\rho g - \nabla p = \rho \left[\frac{\partial \mathbf{V}}{\partial t} + \left(\mathbf{V} \cdot \nabla \right) \mathbf{V} \right]$$

In Cartesian coordinates:

$$\rho g_x - \frac{\partial p}{\partial x} = \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right)$$

$$\rho g_y - \frac{\partial p}{\partial y} = \rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right)$$

$$\rho g_z - \frac{\partial p}{\partial z} = \rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right)$$

<u>The Bernoulli equation derived from Euler's equations</u>

The Bernoulli equation can also be derived, starting from Euler's equations. For inviscid, incompressible fluids, we end up with the same equation

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = const$$

It is often convenient to write the Bernoulli equation between two points (1) and (2) along a streamline and to express the equation in the "head" form by dividing each term by g so that

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+z_2$$

The Bernoulli equation is restricted to the following:

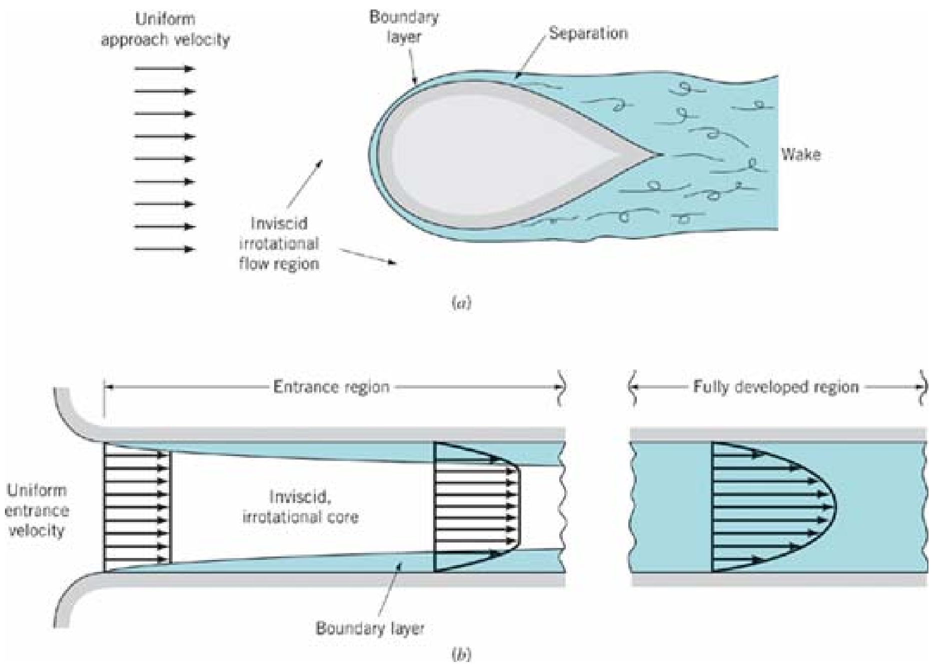
- inviscid flow
- steady flow
- incompressible flow
- flow along a streamline

<u>The Irrotational Flow and corresponding Bernoulli equation</u>

If we make one additional assumption—that the flow is irrotational $\nabla\times\mathbf{V}=0$ —the analysis of inviscid flow problems is further simplified. The Bernoulli equation has exactly the same form at that for inviscid flows:

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+z_2$$

but it can now be applied between any two points in the flow field, not limited to applications along a streamline.



Various regions of flow: (a) around bodies;
(b) through channels

The Velocity Potential

For an irrotational flow:

$$\nabla \times \mathbf{V} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k} = 0$$

So we have

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

It follows that in this case the velocity components can be expressed in terms of a scalar function ϕ(x, y, z, t), called velocity potential, as

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z}$$

In vector form:

$$\mathbf{V} = \nabla \phi$$

The velocity potential is a consequence of the irrotationality of the flow field, whereas the stream function is a consequence of conservation of mass. It is to be noted, however, that the velocity potential can be defined for a general three-dimensional flow, whereas the stream function is restricted to two-dimensional flows.

For an incompressible flow we know from the conservation of mass:

$$\nabla \cdot \mathbf{V} = 0$$

and therefore for incompressible, irrotational flow, it follows that

$$\nabla^2 \phi = 0$$

The velocity potential satisfies the Laplace equation.

In Cartesian coordinates:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

In cylindrical coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2}\frac{\partial \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

<u>Some Basic, Plane Potential Flows</u>

For potential flow, basic solutions can be simply added to obtain more complicated solutions because of the major advantage of Laplace equation that it is a linear PDE. For simplicity, only plane (two-dimensional) flows will be considered. Since we can define a stream function for plane flow,

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

If we now impose the condition of irrotationality, it follows

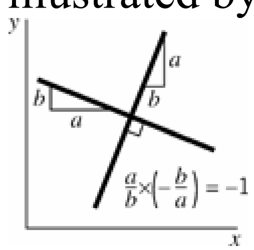
$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

and in terms of the stream function

$$\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right)$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Thus, for a plane irrotational flow we can use either the velocity potential or the stream function—both must satisfy Laplace's equation in two dimensions. It is apparent from these results that the velocity potential and the stream function are somehow related. It can be shown that lines of constant ϕ (called equipotential lines) are orthogonal to lines of constant ψ (streamlines) at all points where they intersect. Recall that two lines are orthogonal if the product of their slopes is −1, as illustrated by this figure



Along streamlines ψ=const:

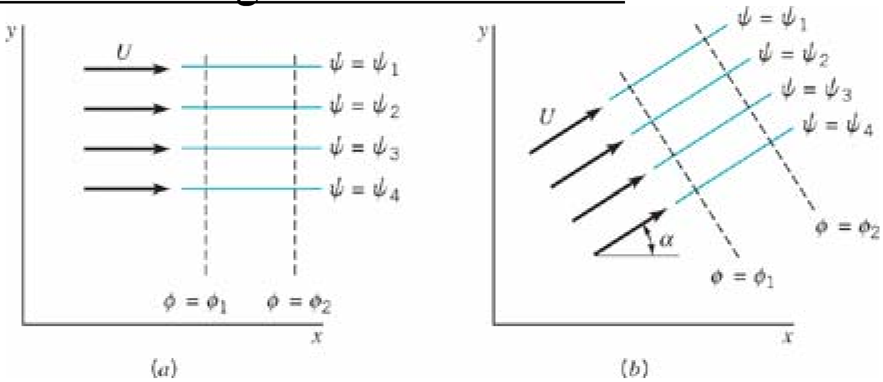
$$\left.\frac{dy}{dx}\right|_{along \quad \psi=const} = \frac{v}{u}$$

Along equipotential lines ϕ = const

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = udx + vdy = 0$$

$$\left.\frac{dy}{dx}\right|_{along\ \ \phi=const} = -\frac{u}{v}$$

<u>Uniform flow at angle α with the x axis</u>

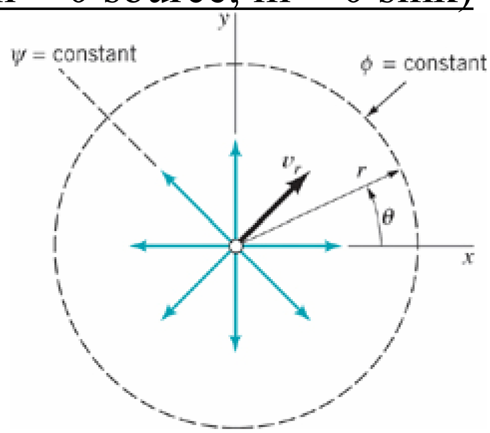


Velocity potential: $\phi = U\left(x\cos\alpha + y\sin\alpha\right)$

Stream function: $\psi = U\left(y\cos\alpha - x\sin\alpha\right)$

Velocity components: $u = U\cos\alpha, \quad v = U\sin\alpha$

<u>Source or sink (m > 0 source; m < 0 sink)</u>

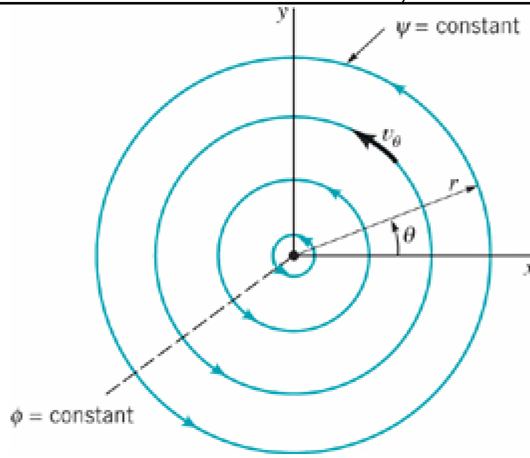


Velocity potential: $\phi = \dfrac{m}{2\pi}\ln r$

Stream function: $\psi = \dfrac{m}{2\pi}\theta$

Velocity components: $v_r = \dfrac{m}{2\pi r}, \quad v_\theta = 0$

Free vortex ($\Gamma > 0$ counterclockwise; $\Gamma < 0$ clockwise)

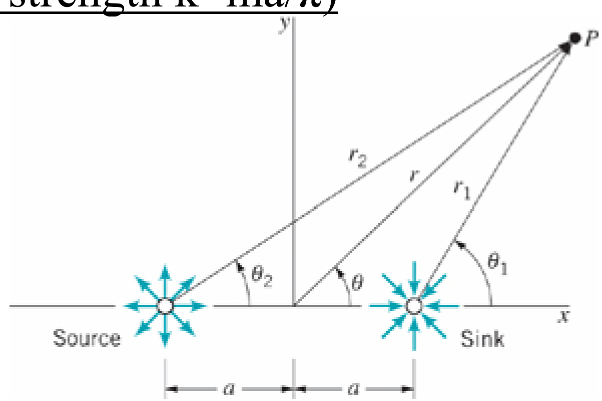


Velocity potential: $\phi = \dfrac{\Gamma}{2\pi}\theta$

Stream function: $\psi = -\dfrac{\Gamma}{2\pi}\ln r$

Velocity components: $v_r = 0, \quad v_\theta = \dfrac{\Gamma}{2\pi r}$

Doublet (with strength k=ma/π)



Velocity potential: $\phi = \dfrac{K\cos\theta}{r}$

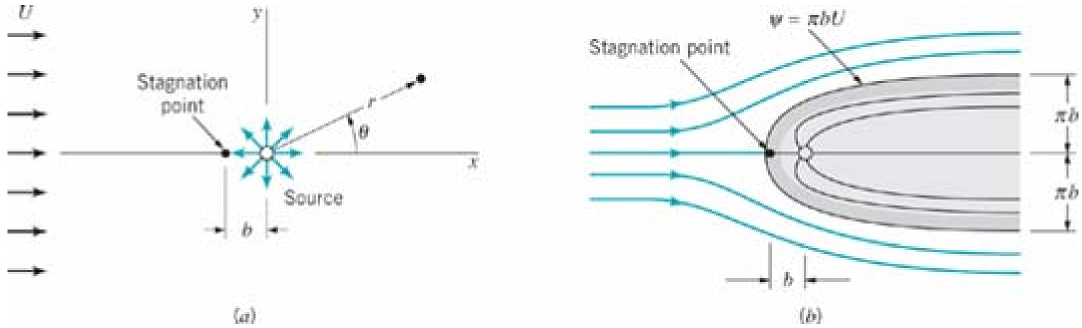
Stream function: $\psi = \dfrac{K\sin\theta}{r}$

Velocity components: $v_r = -\dfrac{K\cos\theta}{r^2}, \quad v_\theta = -\dfrac{K\sin\theta}{r^2}$

Superposition of Basic, Plane Potential Flows

Source in a Uniform Stream—Half-Body

Flow around a half-body is obtained by the addition of a source to a uniform flow.



The flow around a half-body: (a) superposition of a source and a uniform flow; (b) replacement of streamline $\psi = \pi bU$ with solid boundary to form half-body.

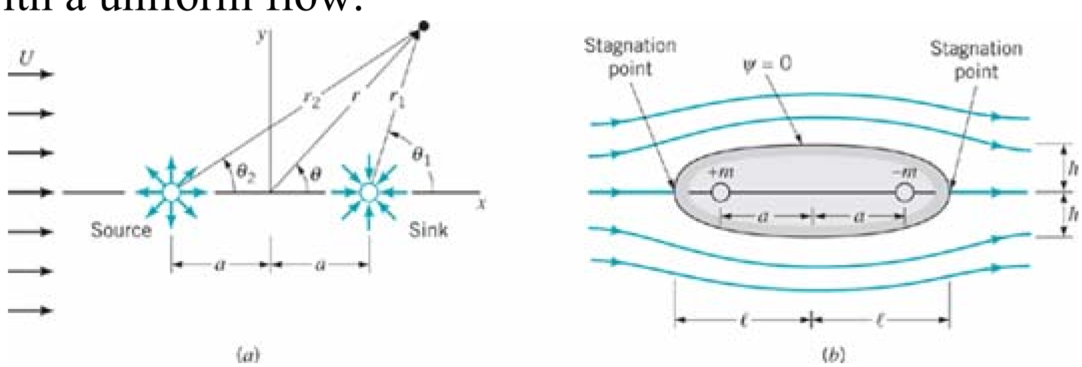
Velocity potential: $\phi = Ur\cos\theta + \dfrac{m}{2\pi}\ln r$

Stream function: $\psi = Ur\sin\theta + \dfrac{m}{2\pi}\theta$

Velocity components: $v_r = -\dfrac{m}{2\pi r}, \quad v_\theta = -U\sin\theta$

Rankine Ovals

Rankine ovals are formed by combining a source and sink with a uniform flow.



The flow around a Rankine oval: (a) superposition of source–sink pair and a uniform flow; (b) replacement of streamline ψ = 0 with solid boundary to form Rankine oval.

Velocity potential: $\phi = Ur\cos\theta - \dfrac{m}{2\pi}\left(\ln r_1 - \ln r_2\right)$

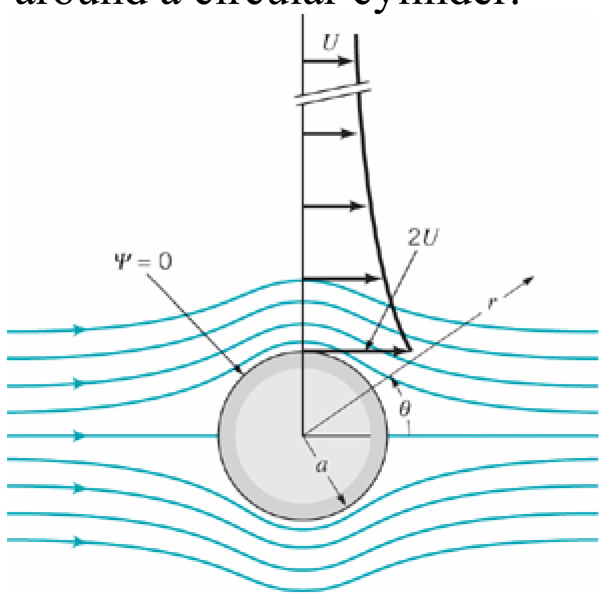
Stream function: $\psi = Ur\sin\theta - \dfrac{m}{2\pi}\tan^{-1}\left(\dfrac{2ar\sin\theta}{r^2 - a^2}\right)$

Body half length: $l = \left(\dfrac{ma}{\pi U} + a^2\right)^{1/2}$

Body half width: $h = \dfrac{h^2 - a^2}{2a}\tan\dfrac{2\pi Uh}{m}$

<u>Flow around a Circular Cylinder</u>

A doublet combined with a uniform flow can be used to represent flow around a circular cylinder.



The flow around a circular cylinder

Velocity potential: $\phi = Ur\cos\theta + \dfrac{K\cos\theta}{r}$

Stream function: $\psi = Ur\sin\theta - \frac{K\sin\theta}{r}$

Velocity components:

$$v_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta, \quad v_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$